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LIPSCHITZ CONTINUITY OF SPECTRAL MEASURES

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A characterisation is given of all (finitely additive) spectral measures in a Banach space (and defined on an algebra of sets) which satisfy a Lipschitz condition. This also corrects (slightly) an analogous result in the more specialised setting of resolutions of the identity of scalar-type spectral operators (due to C.A. McCarthy).

We denote by $\mathcal{L}(X)$ the space of continuous linear operators on a Banach space X and use $\mathcal{L}_u(X)$ when it is to be considered equipped with the uniform operator topology. A set function $P : \Sigma \longrightarrow \mathcal{L}_u(X)$ is called *multiplicative* if $P(E \cap F) = P(E)P(F)$, for all $E, F \in \Sigma$. If, in addition, P is finitely additive and satisfies $P(\emptyset) = 0$ and $P(\Omega) = I$ (the identity operator on X), then P is called a *finitely additive spectral measure.* Here Σ is a σ-algebra of subsets of some set $\Omega \neq \emptyset$. A finitely additive spectral measure is simply called a *spectral measure* if it is countably additive for the strong operator topology. Such measures are natural extensions to the Banach space setting of resolutions of the identity of normal operators in Hilbert spaces. A result in [**2**, p.2082] (credited to C.A. Mc Carthy) states if Σ is the σ-algebra of Borel subsets of some subset of the complex plane, then *no* spectral measure $P : \Sigma \longrightarrow \mathcal{L}(X)$ can satisfy a Lipschitz condition, that is, there is no σ-additive measure $\mu : \Sigma \longrightarrow [0, \infty)$ and $M > 0$ such that $\|P(E)\| \leqslant M\mu(E)$ for all $E \in \Sigma$. The proof given there rests on the existence of Bade functionals, [**2**, p.2205], a relatively sophisticated result concerning spectral measures and more general Boolean algebras of projections.

The aim of this note is firstly to point out that McCarthy's result is not quite correct as it is formulated; there do exist spectral measures which satisfy a Lipschitz condition. These are completely characterised in Proposition 1. Moreover, there exist Banach spaces X in which *every* spectral measure satisfies a Lipschitz condition; see Proposition 2. Our second aim is to provide a more elementary proof of (the correct formulation of) McCarthy's result. It is based on some simple facts about projection operators; no use is made of Bade functionals.

It turns out that McCarthy's result remains valid in a slightly more general context. So, we assume henceforth that Σ is merely an *algebra* of subsets of some non-empty set Ω.

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DEFINITION. A finitely additive spectral measure $P : \Sigma \longrightarrow \mathcal{L}(X)$ is said to satisfy a *Lipschitz condition* if there exists a constant $M > 0$ and a bounded, finitely additive measure $\mu : \Sigma \longrightarrow [0, \infty)$ such that

$$\|P(E)\| \leqslant M\mu(E), \qquad E \in \Sigma. \tag{1}$$

Recall that a commuting family of projections $\mathcal{B} \subseteq \mathcal{L}(X)$ which contains 0 and I is called a *Boolean algebra of projections* if it is a Boolean algebra with respect to the partial order defined by $Q \leqslant R$ if and only if $QX \subseteq RX$. A projection $Q \in \mathcal{B}$ is an *atom* if it has the property that $B \in \{0, Q\}$ whenever $B \in \mathcal{B}$ satisfies $B \leqslant Q$.

Finally, a Banach space-valued vector measure $m : \Sigma \longrightarrow Y$ is said to be *strongly additive* [1, p.7] if the series $\sum_{n=1}^{\infty} m(E_n)$ converges in Y whenever $\{E_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence of sets from Σ.

For the definition of the *variation* of a Banach space-valued measure we refer to [1, p.2].

PROPOSITION 1. *Let X be a Banach space and $P : \Sigma \longrightarrow \mathcal{L}(X)$ be a finitely additive spectral measure defined on an algebra of sets Σ. The following statements are equivalent.*

(i) *P satisfies a Lipschitz condition.*

(ii) *The range $P(\Sigma)$ of P is a finite subset of $\mathcal{L}(X)$.*

(iii) *$P : \Sigma \longrightarrow \mathcal{L}_u(X)$ has finite variation.*

(iv) *P is strongly additive in $\mathcal{L}_u(X)$.*

(v) *There exists a finite partition $\{E_j\}_{j=1}^{k} \subseteq \Sigma$ of Ω such that each projection $P(E_j)$, $1 \leqslant j \leqslant k$, is an atom of the Boolean algebra $P(\Sigma)$ and*

$$P(F) = \sum_{j=1}^{k} P(F \cap E_j), \qquad F \in \Sigma. \tag{2}$$

There exist Banach spaces X with the property that *every* spectral measure (based on a σ-algebra) in $\mathcal{L}(X)$ *necessarily* has finite range, [3, 4]. This can be combined with Proposition 1 to yield the following result.

PROPOSITION 2. *Let X be a Grothendieck space with the Dunford-Pettis property or let X be a hereditarily indecomposable Banach space. Then every spectral measure in $\mathcal{L}(X)$ based on a σ-algebra satisfies a Lipschitz condition.*

The proof of Proposition 1 will require the following fact.

LEMMA 1. *Let X be a Banach space and $P : \Sigma \longrightarrow \mathcal{L}(X)$ be a finitely additive spectral measure defined on algebra of sets Σ. If $P(\Sigma) := \{P(E) : E \in \Sigma\}$ is an infinite subset of $\mathcal{L}(X)$, then there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise disjoint sets in Σ such that $P(E_n) \neq 0$ for each $n \geqslant 1$.*

Proof: Let $\mathcal{B} := P(\Sigma)$. Choose any $P_1 \in \mathcal{B}\backslash\{0, I\}$, in which case $P_1 = P(E_1)$ for some set $E_1 \in \Sigma$. Then $P_2 := P(E_2)$, where $E_2 := \Omega\backslash E_1$, satisfies $P_1 + P_2 = I$ and $P_2 \in \mathcal{B}\backslash\{0, I\}$. Since $P(F) = P(F \cap E_1) + P(F \cap E_2)$, for every $F \in \Sigma$, and $\mathcal{B}$ is infinite, it follows that P restricted to E_1 or P restricted to E_2 (or both) must have range an infinite subset of $\mathcal{L}(X)$. We can repeat the above argument and find disjoint sets from Σ, say F_1, F_2 in E_1 (or in E_2) with $F_1 \cup F_2 = E_1$ (or $F_1 \cup F_2 = E_2$) and $0 < P(F_j) < P_1$ (or $0 < P(F_j) < P_2$). Since $\mathcal{B}$ is infinite it again follows that P restricted to at least one of F_1 or F_2 or $\Omega\backslash(F_1 \cup F_2)$ must have range an infinite subset of $\mathcal{L}(X)$. Continuing by induction gives the desired conclusion. ∎

The proof of the following useful fact is trivial.

Lemma 2. *Let X be a Banach space. Then every non-zero projection $Q \in \mathcal{L}(X)$ satisfies $\|Q\| \geqslant 1$.*

Proof of Proposition 1: (i)$\Longrightarrow$(ii). Let $M > 0$ and $\mu : \Sigma \longrightarrow [0, \infty)$ be a bounded, finitely additive measure satisfying (1). Lemma 2 implies that $\mu(E) \geqslant M^{-1}$ whenever $P(E) \neq 0$ and hence, by finite additivity and boundedness of μ, there cannot exist an infinite sequence $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ of pairwise disjoint sets such that $P(E_n) \neq 0$ for all $n \geqslant 1$. Then Lemma 1 implies that $P(\Sigma)$ is a finite subset of $\mathcal{L}(X)$.

(ii)$\Longrightarrow$(i). By hypothesis $P(\Sigma)$ is a *finite* Boolean algebra. Accordingly, there exist *atoms* $P_1, \dots, P_n$ in $P(\Sigma)$ such that every element in $P(\Sigma)$ is a partial sum of $\{P_j\}_{j=1}^{n}$. Using the fact that $P_j P_k = 0 = P_k P_j$ whenever $k \neq j$, that $\sum_{j=1}^{n} P_j = I$, and that $P : \Sigma \longrightarrow P(\Sigma)$ is a Boolean algebra homomorphism of the Boolean algebra Σ onto the Boolean algebra of projections $P(\Sigma)$, it is routine to verify that there exists a partition of Ω into Σ-measurable sets $\{E_j\}_{j=1}^{n}$ such that $P(E_j) = P_j$, $1 \leqslant j \leqslant n$. Then (2) follows easily.

It is clear from (2) and the fact that $P(F \cap E_j) \in \{0, P_j\}$ for each $1 \leqslant j \leqslant n$, that $P(\Sigma)$ is a uniformly bounded subset of $\mathcal{L}(X)$. Let $E \in \Sigma$ and $\{F_r\}_{r=1}^{m}$ be any Σ-partition of E. Then

$$P(F_r) = \sum_{k=1}^{n} P(F_r \cap E_k), \qquad 1 \leqslant r \leqslant m. \tag{3}$$

Define sets $J_r := \{k : P(F_r \cap E_k) = P_k\}$, for $1 \leqslant r \leqslant m$. It turns out that $J_r \cap J_j = \emptyset$ whenever $j \neq r$. Accordingly, (3) implies that

$$\sum_{r=1}^{m} \|P(F_r)\| \leqslant \sum_{r=1}^{m} \sum_{k \in J_r} \|P_r\| \leqslant n \cdot \max_{1 \leqslant j \leqslant n} \|P_j\|.$$

If $|P|$ denotes the variation measure of $P : \Sigma \longrightarrow \mathcal{L}_u(X)$, then we have just shown that $|P|(E) \leqslant n \cdot \max_{1 \leqslant j \leqslant n} \|P_j\| < \infty$. But, $E \in \Sigma$ is arbitrary. So, $\mu := |P|$ is a bounded, finitely additive measure $\mu : \Sigma \longrightarrow [0, \infty)$ satisfying (1) with $M = 1$.

(i)$\Longrightarrow$(iii). It is clear from (1) and the definition of variation measure that $|P|(E) \leqslant M\mu(E)$, for $E \in \Sigma$. Hence, $P : \Sigma \longrightarrow \mathcal{L}_u(X)$ has finite variation.

(iii)$\Longrightarrow$(i). This is immediate by choosing $\mu := |P|$.

(iv)$\Longrightarrow$(ii). Let $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ be pairwise disjoint sets. By hypothesis the series $\sum_{n=1}^{\infty} P(E_n)$ converges in $\mathcal{L}_u(X)$. In particular, $\|P(E_n)\| \longrightarrow 0$ as $n \to \infty$. By Lemma 2 only finitely many of the projections $\{P(E_n)\}_{n=1}^{\infty}$ can be non-zero. It follows from Lemma 1 that $P(\Sigma)$ is a finite subset of $\mathcal{L}(X)$.

(ii)$\Longrightarrow$(iv). Since (ii)$\Longrightarrow$(i) it suffices to show that (i)$\Longrightarrow$(iv). Let $\mu : \Sigma \longrightarrow [0, \infty)$ be a bounded, finitely additive measure satisfying (1). Let $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ be pairwise disjoint sets. It follows from (1) that

$$\sum_{n=1}^{\infty} \|P(E_n)\| \leqslant M \sum_{n=1}^{\infty} \mu(E_n) = M \sup_N \sum_{n=1}^{N} \mu(E_n) = M \sup_N \mu\Big(\bigcup_{n=1}^{N} E_n\Big) < \infty.$$

Hence, $\sum_{n=1}^{\infty} P(E_n)$ is absolutely convergent in the complete space $\mathcal{L}_u(X)$ and so converges in $\mathcal{L}_u(X)$. This shows that P is strongly additive.

(ii)$\Longrightarrow$(v). This was established in the proof of (ii) $\Longrightarrow$(i).

(v)$\Longrightarrow$(ii). Since $P(F \cap E_j) \in \{0, P(E_j)\}$, for each $1 \leqslant j \leqslant k$, it is immediate from (2) that $P(\Sigma)$ is a finite subset of $\mathcal{L}(X)$. ∎

Remark. In McCarthy's original formulation of the Lipschitz condition the set function $P : \Sigma \longrightarrow \mathcal{L}(X)$ was actually a spectral measure defined on a *σ-algebra* Σ and the scalar measure $\mu : \Sigma \longrightarrow [0, \infty)$ satisfying (1) was required to be countably additive. Under these additional restrictions the equivalences in Proposition 1 are also equivalent to the requirement:

(vi) $P : \Sigma \longrightarrow \mathcal{L}_u(X)$ *is countably additive.*

Indeed, if (i) holds for a countably additive measure $\mu : \Sigma \longrightarrow [0, \infty)$ then it is clear from (1) that P is countably additive in $\mathcal{L}_u(X)$, that is, $P(E_n) \longrightarrow 0$ in $\mathcal{L}_u(X)$ whenever $\{E_n\}_{n=1}^{\infty} \subseteq \Sigma$ decreases to $\emptyset$. On the other hand, (vi)$\Longrightarrow$(iv) since every countably additive, Banach space-valued measure defined on a σ-algebra is clearly strongly additive.

In view of Proposition 1 (especially the equivalence (i)$\Longleftrightarrow$(ii)) we conclude with a correct version of McCarthy's original result.

Theorem 1. *Let $\mathcal{B}(K)$ be the Borel σ-algebra of some compact set $K \subseteq \mathbb{C}$ and $P : \mathcal{B}(K) \longrightarrow \mathcal{L}(X)$ be a spectral measure. Then there exists a σ-additive measure $\mu : \mathcal{B}(K) \longrightarrow [0, \infty)$ and $M > 0$ satisfying $\|P(E)\| \leqslant M\mu(E)$, for $E \in \mathcal{B}(K)$, if and only if the range $P(\mathcal{B}(K))$ of P is a finite subset of $\mathcal{L}(X)$.*

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