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AN EIGENVALUE PROBLEM FOR A NON-BOUNDED QUASILINEAR OPERATOR

JOSÉ CARMONA[1] AND ANTONIO SUÁREZ[2]

[1] *Departamento de Algebra y Análisis Matemático, Facultad de Ciencias, Cañada de San Urbano, Almería, Spain* (jcarmona@ual.es)
[2] *Departamento de Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Sevilla, Spain* (suarez@us.es)

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Abstract In this paper we study the eigenvalues associated with a positive eigenfunction of a quasilinear elliptic problem with an operator that is not necessarily bounded. For that, we use the bifurcation theory and obtain the existence of positive solutions for a range of values of the bifurcation parameter.

Keywords: bifurcation; positive solutions; non-bounded quasilinear operator; eigenvalue problems

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ and let $A(x,s)$ be a real symmetric matrix whose entries, $a_{ij} : \bar{\Omega} \times \mathbb{R}_0^+ \to \mathbb{R}$, are Carathéodory functions.

We assume that there exists a positive constant α satisfying, for every $(x,s,\xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$,

$$A(x,s)\xi \cdot \xi \geqslant \alpha|\xi|^2. \tag{A_1}$$

In this paper we analyse the nonlinear eigenvalue problem

$$\left.\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= \lambda u, \quad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned}\right\} \tag{P_λ}$$

where we say that λ is an eigenvalue for this problem if (P_λ) admits a positive and non-trivial solution, that is, if there exists $u \in H_0^1(\Omega)$, $u \geqslant 0$, $u \not\equiv 0$, such that $A(x,u)\nabla u \in (L^2(\Omega))^N$ and

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v = \lambda \int_\Omega uv, \quad \forall v \in H_0^1(\Omega).$$

In addition to interest itself in the study of (P_λ), this kind of equation has been used to model a species inhabiting Ω where its diffusion depends on the density of the species, which arises in more realistic models (see [**3**, **4**] and references therein).

Problem (P_λ) is well known when A does not depend on s, i.e. when $A(x,s) = B(x)$ with $B = (b_{ij})$ and $b_{ij} \in L^\infty(\Omega)$, $b_{ij} \geqslant b_0 > 0$ in Ω. In this case, there exists the principal eigenvalue, denoted by $\lambda_1(B)$, for the problem

$$\left.\begin{aligned} -\operatorname{div}(B(x)\nabla u) &= \lambda u, \quad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

being the unique eigenvalue with a positive eigenfunction (see, for example, [**6**]).

In [**2**], assuming that A satisfies (A_1) and

$$|A(x,s)| \leqslant \beta \quad \text{for each } (x,s) \in \Omega \times \mathbb{R}, \tag{A_2}$$

the author proved that for each $r > 0$, there exists $\lambda_r > 0$ and a positive solution $u_r \in H_0^1(\Omega)$, of (P_{λ_r}) such that $\|u_r\|_2 = r$. Moreover, defining

$$\lambda_0 := \lambda_1(A(x,0)),$$

he showed that if $r \to 0$, then $\lambda_r \to \lambda_0$ and u_r/r converges to a positive eigenfunction associated with λ_0 in $H_0^1(\Omega)$. Finally, if A also verifies

$$\lim_{s\to\infty} A(x,s) = A_\infty(x) \quad \text{uniformly in } x \in \Omega, \tag{A_3}$$

then $\lambda_r \to \lambda_\infty$ and u_r/r goes to a positive eigenfunction associated with λ_∞ in $H_0^1(\Omega)$ as $r \to \infty$, where

$$\lambda_\infty := \lambda_1(A_\infty(x)).$$

In [**5**], a slight modification of (P_λ) is analysed. Under conditions (A_1)–(A_3), $\lambda u + h(x)$ for some $0 \leqslant h \in L^2(\Omega)$ is considered instead of λu. But the arguments used to prove the existence of a solution leads to the trivial one in the case $h \equiv 0$.

In [**1**], assuming in addition the existence of an Osgood function $\omega : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$|A(x,s_1) - A(x,s_2)| \leqslant \omega(|s_1 - s_2|), \tag{A_4}$$

for every $(x,s_1),(x,s_2) \in \Omega \times \mathbb{R}$, using a bifurcation analysis, the authors study a more general problem

$$\begin{aligned} -\operatorname{div}(A(x,u)\nabla u) &= f(\lambda,x,s), \quad x \in \Omega, \\ u &= 0, \qquad\qquad\quad x \in \partial\Omega, \end{aligned}$$

for $f : \mathbb{R} \times \bar{\Omega} \times \mathbb{R} \mapsto \mathbb{R}$ and A satisfying (A_1)–(A_4). In the particular case $f(\lambda,x,s) = \lambda s$, from their results can be deduced the existence of an unbounded continuum (closed and connected subset) of positive solutions bifurcating from the trivial solution at $\lambda = \lambda_0$ and meeting with infinity at the value $\lambda = \lambda_\infty$. Thus, as a consequence, there exists a positive solution of (P_λ) for $\lambda \in (\lambda_0, \lambda_\infty)$ or $(\lambda_\infty, \lambda_0)$. In the following section we complete this study for A satisfying (A_1)–(A_4) by giving sufficient conditions for the uniqueness of a positive solution.

The main goal of this work (see § 3) is to analyse (P_λ) when A is not necessarily bounded and/or does not satisfy (A_3). In this case, we show that there exists an unbounded continuum of positive solutions bifurcating from the trivial one at $\lambda = \lambda_0$. If, in addition, there exists a continuous function $g : \mathbb{R}_0^+ \to \mathbb{R}$, with $\lim_{s\to+\infty} g(s) = +\infty$, satisfying, for every $(x, s, \xi) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$,

$$A(x, s)\xi \cdot \xi \geqslant g(s)|\xi|^2 \geqslant \alpha|\xi|^2, \qquad (A_\infty)$$

then the bifurcation from infinity at $\lambda = \lambda_\infty$ (which exists in the bounded case) 'disappears'. Specifically, there exists at least one positive solution u_λ for $\lambda \in (\lambda_0, \infty)$ and $\|u_\lambda\| \to \infty$ as $\lambda \to \infty$. However, if A is bounded in a subset of Ω, then again a bifurcation to infinity exists.

Throughout the paper we will use the following notation.

(i) $H_0^1(\Omega)$ and $E = C_0(\bar{\Omega})$ are the usual Sobolev space and the space of the continuous functions in $\bar{\Omega}$ vanishing on $\partial\Omega$ endowed with the norms $\|u\| = \|\nabla u\|_2$ and $\|u\|_0 = \sup_\Omega |u|$, respectively.

(ii) $\mathrm{cl}(D)$ denotes the closure of the set D.

(iii) $\mathcal{S}$ denotes the set

$$\mathcal{S} = \mathrm{cl}\{(\lambda, u) \in \mathbb{R} \times E : u \text{ is a solution for } (P_\lambda),\ u \geqslant 0,\ u \not\equiv 0\}.$$

Any continuum subset of $\mathcal{S}$ will be called a continuum of positive solutions of (P_λ), although it may contain the trivial solution $(\lambda, 0)$ for some value of $\lambda > 0$.

(iv) I will denote both the identity matrix and the identity operator.

(v) Given square matrices B_1, B_2 we say that $B_1 > 0$ (respectively, $B_1 \geqslant 0$) if the quadratic form induced by B_1 is definite positive (respectively, semidefinite positive). We say that $B_1 < B_2$ (respectively, $B_1 \leqslant B_2$) if $B_2 - B_1 > 0$ (respectively, $B_2 - B_1 \geqslant 0$).

(vi) The map $\mathrm{Proj}_\mathbb{R} : \mathbb{R} \times E \mapsto \mathbb{R}$ stands for the projection of the product space $\mathbb{R} \times E$ onto $\mathbb{R}$.

2. The case of bounded matrices A

In order to study problem (P_λ), let us recall that, for matrices A satisfying (A_1), (A_2), if $u \in H_0^1(\Omega)$ is a solution of (P_λ), then using the De Giorgi–Stampacchia Theorem (see [**9**, Théorème 7.3] and [**7**, Theorem I] or [**8**, Theorem 8.29]), $u \in C^{0,\gamma}(\bar{\Omega})$ for some $0 < \gamma < 1$. Moreover, if the coefficients of the matrix A satisfy

$$a_{ij} \in C^{1,\gamma'}(\bar{\Omega} \times \mathbb{R}) \quad \text{for some } 0 < \gamma' < 1, \qquad (2.1)$$

then by Theorem 15.17 in [**8**] we have that $u \in C_0^{2,\gamma\gamma'}(\bar{\Omega})$.

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We also recall that for every $(\lambda, u) \in \mathcal{S}$ with $u \in \mathcal{C}^1(\bar{\Omega})$ and $u \not\equiv 0$, using the Hopf maximum principle, we have that $u > 0$ in Ω and the normal exterior derivative $\partial u/\partial n_{\mathrm{e}}$ is negative in $\partial\Omega$.

The following lemma gives us necessary conditions in $\lambda \in \mathbb{R}$ for which (P_λ) admits a solution in some special cases.

Lemma 2.1. *Assume (A_1), (A_3) and that (P_λ) admits a positive solution. Then*

(1) *$\lambda_0 \leqslant \lambda$ (respectively, $<$, $\geqslant$, $>$) if for every $s \in \mathbb{R}^+$, $A(x,0) \leqslant A(x,s)$ (respectively, $<$, $\geqslant$, $>$); and*

(2) *$\lambda_\infty \geqslant \lambda$ (respectively, $>$, $\leqslant$, $<$) if for every $s \in \mathbb{R}^+$, $A_\infty(x) \geqslant A(x,s)$ (respectively, $>$, $\leqslant$, $<$).*

Proof. The result follows from the fact that for given symmetric matrices $B_1(x)$, $B_2(x)$ for which there exist $\lambda_1(B_1)$ and $\lambda_1(B_2)$, with $0 < B_1 \leqslant B_2$, then

$$\lambda_1(B_1) = \inf\left\{\int_\Omega B_1(x)\nabla u \cdot \nabla u,\ u \in H_0^1(\Omega),\ \|u\|_2 = 1\right\} \leqslant \lambda_1(B_2).$$

Thus, if $u \in H_0^1(\Omega)$ is a solution of (P_λ), we conclude by taking into account that $\lambda = \lambda_1(A(x,u))$. □

The main result of this section is the following.

Theorem 2.2. *Assume (A_1)–(A_4). We have that λ_0 and λ_∞ are the only bifurcation points from the trivial solution and from infinity, respectively, and there exists a continuum $\Sigma \subset \mathcal{S}$ of positive solutions meeting $(\lambda_0, 0)$ and (λ_∞, ∞); in particular, (P_λ) possesses a positive solution for every $\lambda \in (\lambda_0, \lambda_\infty)$ or $\lambda \in (\lambda_\infty, \lambda_0)$. Moreover,*

(i) *the bifurcation from λ_0 is subcritical (respectively, supercritical) if there exists $s_0 > 0$ such that*

$$A(x,s) < A(x,0) \quad \text{(respectively, } A(x,s) > A(x,0)), \quad \forall s \in (0, s_0),$$

(ii) *the bifurcation from λ_∞ is subcritical (respectively, supercritical) if*

$$A(x,s) < A_\infty(x) \quad \text{(respectively, } A(x,s) > A_\infty(x)), \quad \forall s \in \mathbb{R}^+.$$

Furthermore, we have the following.

(i) *If $A(x,0) < A(x,s) < A_\infty(x)$ for every $s \in \mathbb{R}^+$, then there exists a non-trivial solution for (P_λ) if, and only if, $\lambda \in (\lambda_0, \lambda_\infty)$; in particular, $\mathrm{Proj}_{\mathbb{R}}\Sigma = [\lambda_0, \lambda_\infty)$. If, in addition, $A(x,s)$ is increasing in s and it satisfies (2.1), the solution is unique.*

(ii) *If $A(x,0) > A(x,s) > A_\infty(x)$ for every $s \in \mathbb{R}^+$, then there exists a non-trivial solution for (P_λ) if, and only if, $\lambda \in (\lambda_\infty, \lambda_0)$; in particular, $\mathrm{Proj}_{\mathbb{R}}\Sigma = (\lambda_\infty, \lambda_0]$.*

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Proof. The existence of the continuum Σ of positive solutions follows by Theorem 5.1 in [**1**]; in particular, we have the existence of positive solutions for every λ in $(\lambda_0, \lambda_\infty)$ or in $(\lambda_\infty, \lambda_0)$.

The description $\mathrm{Proj}_{\mathbb{R}}\Sigma$, in the cases $A(x,0) < A(x,s) < A_\infty(x)$ or $A(x,0) < A(x,s) < A_\infty(x)$ for every $s \in \mathbb{R}^+$, follows directly from Lemma 2.1. Moreover, arguing as in that lemma we get the laterality of the bifurcations.

Now, assume that $A(x,s)$ is increasing in s and (2.1) is satisfied. In order to prove the uniqueness of the solution for (P_λ), let us suppose that there exists $\lambda \in (\lambda_0, \lambda_\infty)$ for which (P_λ) admits two solutions, $u_1, u_2 \in E$, with $u_1 \not\equiv u_2$. We claim that u_1, u_2 can be chosen such that $u_1 \leqslant u_2$. Indeed, this is a consequence of the existence of a sequence (λ_n, u_n) with $\lambda_n \to \lambda_0$ and $u_n \to 0$ in E. In fact, by regularity results, $u_n \to 0$ in $\mathcal{C}^1(\bar{\Omega})$. Thus, for $\lambda_n < \lambda$, u_n is a subsolution for (P_λ) and for large n, $u_n \leqslant \min\{u_1, u_2\}$. Then, by the subsolution and supersolution method, there exists $w \in E$, a solution of (P_λ) with

$$u_n \leqslant w \leqslant u_1, \qquad u_n \leqslant w \leqslant u_2.$$

This implies that $w \not\equiv u_1$ or $w \not\equiv u_2$, and the claim is proved by taking $u_1 = w$ and $u_2 = u_i$ for some $i = 1, 2$.

Now we take $v = u_2^2/u_1$ as a test function in the equation satisfied by u_1, and $v = u_2$ in that satisfied by u_2. Thus, subtracting both equalities we have

$$\begin{aligned} 0 &= \int_\Omega A(x,u_1)\nabla u_1 \cdot \nabla\left(\frac{u_2^2}{u_1}\right) - \int_\Omega A(x,u_2)\nabla u_2 \cdot \nabla u_2 \\ &= -\int_\Omega A(x,u_1)\left(\frac{u_2}{u_1}\nabla u_1 - \nabla u_2\right) \cdot \left(\frac{u_2}{u_1}\nabla u_1 - \nabla u_2\right) \\ &\qquad\qquad - \int_\Omega (A(x,u_2) - A(x,u_1))\nabla u_2 \cdot \nabla u_2 < 0. \end{aligned}$$

This contradiction gives the uniqueness. □

3. The case of unbounded matrices A

In this section, we study (P_λ) when A is not necessarily bounded and does not satisfy (A_3). We prove firstly that every solution of (P_λ) is bounded. More precisely we have the following lemma.

Lemma 3.1. *Let $A(x,s)$ satisfy (A_1) and let $u \in H_0^1(\Omega)$ be a solution of (P_λ), then $u \in E$. Moreover, there exist positive constants c_1, c_2, γ_1, γ_2 such that*

$$\|u\|_0^{\gamma_1} \leqslant c_1 + c_2\|u\|^{\gamma_2}. \tag{3.1}$$

Proof. Once we know that $u \in L^\infty(\Omega)$, and $\|u\|_\infty^{\gamma_1} \leqslant c_1 + c_2\|u\|^{\gamma_2}$ for some positive constants c_1, c_2, γ_1, γ_2, then the result follows directly from the De Giorgi–Stampacchia Theorem. Let us prove the $L^\infty(\Omega)$-estimate. We consider for every $k \in \mathbb{R}^+$ the function $G_k : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

$$G_k(s) = \begin{cases} 0, & 0 \leqslant s \leqslant k, \\ s-k, & s > k. \end{cases}$$

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Thus, we can take $v = G_k(u)$ as a test function in the weak equation satisfied by u and using (A_1) we have

$$\alpha\|\nabla G_k(u)\|_2^2 \leqslant \int_\Omega A(x,u)\nabla u \nabla G_k(u) \leqslant \lambda \int_{\Omega_k} u G_k(u), \tag{3.2}$$

where $\Omega_k \equiv \{x \in \Omega : u(x) > k\}$.

Using the Sobolev and Hölder inequalities, in the case $N > 2$, by (3.2) we have, for $u \in L^r(\Omega)$ with $r > 2^*/(2^* - 1)$ and for some positive constant c,

$$\|G_k(u)\|_{2^*}^2 \leqslant c\|u\|_r \|G_k(u)\|_{2^*} (\operatorname{meas} \Omega_k)^{(1-1/r-1/2^*)}. \tag{3.3}$$

Taking into account that, for every $h > k$, $G_k(u) \geqslant h - k$ in Ω_h, (3.3) implies that

$$(h-k)(\operatorname{meas} \Omega_h)^{1/2^*} \leqslant c\|u\|_r (\operatorname{meas} \Omega_k)^{(1-1/r-1/2^*)},$$

or, equivalently,

$$\operatorname{meas} \Omega_h \leqslant \frac{c\|u\|_r^{2^*} (\operatorname{meas} \Omega_k)^{2^*-1-2^*/r}}{(h-k)^{2^*}}. \tag{3.4}$$

We can now apply the Stampacchia Lemma [**9**, Lemma 4.1] to deduce that

(i) if $u \in L^r(\Omega)$ with $r > N/2$, then $u \in L^\infty(\Omega)$ and $\|u\|_\infty \leqslant c\|u\|_r$;

(ii) if $u \in L^r(\Omega)$ with $r = N/2$, then $u \in L^t(\Omega)$ for $t \in [1, \infty)$ and $\|u\|_t^t \leqslant c + c'\|u\|_r^t$; and

(iii) if $u \in L^r(\Omega)$ with $r < N/2$, then $u \in L^t(\Omega)$ for

$$t = \frac{2^* r}{(2-2^*)r + 2^*} - \delta$$

and $\delta > 0$ arbitrarily small—moreover, $\|u\|_t^t \leqslant c + c'\|u\|_r^{t+\delta}$.

Since $u \in L^{2^*}(\Omega)$ and $2^* > 2^*/(2^* - 1)$, we can argue as before for $r_0 = 2^*$. Thus, if $2^* > N/2$, we obtain the $L^\infty(\Omega)$ estimate by using item (i) above. In the case $2^* = N/2$ we use item (ii) in order to take $r_1 > N/2$ and again the $L^\infty(\Omega)$ estimate follows from item (i) above. Finally, in the case $2^* < N/2$ we can take

$$r_1 = \frac{2^* r_0}{(2-2^*)r_0 + 2^*} - \delta_1 > r_0.$$

As before, if $r_1 \geqslant N/2$, we easily conclude. In the other case we take

$$r_2 = \frac{2^* r_1}{(2-2^*)r_1 + 2^*} - \delta_2.$$

By an iterative argument we conclude after a finite number of steps. Indeed, in the other case, we have that r_n is bounded, where r_n is defined recurrently by

$$r_0 = 2^*,$$
$$r_{n+1} = \frac{2^* r_n}{(2-2^*)r_n + 2^*} - \delta_{n+1},$$

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where $\lim_{n\to\infty} \delta_n = 0$. Moreover, r_n is non-decreasing and so it converges to $r \in (2^*, N/2]$, which satisfies

$$r = \frac{2^* r}{(2-2^*)r + 2^*};$$

that is, $2^* = (2-2^*)r + 2^*$, which implies that $r = 0$, and this is a contradiction.

Observe that the estimate (3.1) follows, after this finite number of steps, from estimates in items (i)–(iii) and the Sobolev embedding.

Finally, in the case $N = 2$ we can choose $r > q/(q-2)$ for any $q > 2$ and argue as before with 2^* replaced by q. In this case we obtain the $L^\infty(\Omega)$ estimate directly by using item (i) above. □

In this section we assume, instead of (A_2), that for each $s_0 \in \mathbb{R}^+$ there exists $\beta(s_0)$ such that

$$|A(x,s)| \leqslant \beta(s_0), \tag{$\tilde{A}_2$}$$

for $(x,s) \in \bar{\Omega} \times [0, s_0]$.

We consider the truncated problems

$$\left.\begin{aligned} -\operatorname{div}(A(x, T_n(u))\nabla u) &= \lambda u, \quad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega, \end{aligned}\right\} \tag{$P_{\lambda,n}$}$$

$T_n(s)$ being the map defined, for each $n \in \mathbb{N}$, by

$$T_n(s) = \begin{cases} s, & 0 \leqslant s \leqslant n, \\ n, & s > n. \end{cases}$$

By Theorem 2.2, there exist Σ_n unbounded maximal continua of positive solutions such that $(\lambda_0, 0) \in \Sigma_n$ for each $n \in \mathbb{N}$. Now, we can prove the following theorem.

Theorem 3.2. *Suppose that A satisfies (A_1), (A_4) and $(\tilde{A}_2)$. Then there exists an unbounded continuum $\Sigma \subset \mathcal{S}$ such that $(\lambda_0, 0) \in \Sigma$.*

Proof. Firstly, we denote by Σ_k^n the connected component of $\Sigma_k \cap (\mathbb{R} \times \bar{B}_n(0))$ containing $(\lambda_0, 0)$. We claim that

$$\Sigma_k^n = \Sigma_n^n \quad \text{for } k \geqslant n. \tag{3.5}$$

Indeed, if $k \geqslant n$ and $(\lambda, u) \in \Sigma_k^n$, then u is a solution of $(P_{\lambda,n})$. Thus, Σ_k^n is a closed and connected subset of

$$\operatorname{cl}\{(\lambda, u) \in \mathbb{R} \times E : u \text{ is a non-trivial solution of } (P_{\lambda,n})\}$$

containing $(\lambda_0, 0)$. So, $\Sigma_k^n \subset \Sigma_n$, from which we deduce that $\Sigma_k^n \subset \Sigma_n^n$. We can reason similarly and obtain that $\Sigma_n^n \subset \Sigma_k \cap (\mathbb{R} \times \bar{B}_n(0))$; thus Σ_n^n and Σ_k^n are connected components of $\Sigma_k \cap (\mathbb{R} \times \bar{B}_n(0))$ containing $(\lambda_0, 0)$, which implies (3.5). So, we get

$$\Sigma_n^n = \lim_k \Sigma_k^n.$$

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Therefore, for each $n \in \mathbb{N}$ we have a continuum

$$\Sigma_n^n \subset \mathrm{cl}\{(\lambda, u) \in \mathbb{R} \times E : u \text{ is a non-trivial solution of } (P_\lambda)\}$$

containing $(\lambda_0, 0)$, and if $(\lambda, u) \in \Sigma_n^n$, then $\|u\|_0 \leqslant n$.

Now, we are going to prove that

$$\Sigma_n^n \subset \Sigma_{n+1}^{n+1} \quad \text{for each } n \in \mathbb{N}. \tag{3.6}$$

Indeed, observe that

$$\Sigma_n^n = \Sigma_{n+1}^n \subset \Sigma_{n+1} \cap (\mathbb{R} \times \bar{B}_n(0)) \subset \Sigma_{n+1} \cap (\mathbb{R} \times \bar{B}_{n+1}(0)),$$

so, since Σ_{n+1}^{n+1} is the connected component of $\Sigma_{n+1} \cap (\mathbb{R} \times \bar{B}_{n+1}(0))$ containing $(\lambda_0, 0)$ and Σ_n^n is a connected component of such a subset containing it, (3.6) follows.

Finally, we show that the set

$$\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n^n$$

satisfies the theorem. Firstly, observe that since Σ_n is unbounded, Σ is also unbounded. Indeed, since $\mathrm{Proj}_{\mathbb{R}} \Sigma_n$ is bounded, so there exists a connected subset of $\Sigma_n \cap (\mathbb{R} \times \bar{B}_n(0))$ containing $(\lambda_0, 0)$ and intersecting with $\mathbb{R} \times \partial \bar{B}_n(0)$ for each $n \in \mathbb{N}$; i.e. for each $n \in \mathbb{N}$ there exists $(\lambda_n, u_n) \in \Sigma_n^n$, with $\|u_n\|_0 = n$.

On the other hand, since Σ_n^n is connected and $(\lambda_0, 0) \in \Sigma_n^n$ for each $n \in \mathbb{N}$, it follows that Σ is connected.

Finally, we will prove that Σ is closed. Let $(\lambda, u) \in \bar{\Sigma}$. Since $\bar{\Sigma}$ is connected, there exists a connected and bounded set $\Sigma' \subset \bar{\Sigma}$ containing $(\lambda_0, 0)$ and (λ, u). Thus, there exists $n \in \mathbb{N}$ such that

$$\Sigma' \subset \mathrm{cl}\{(\lambda, u) \in \mathbb{R} \times E : \|u\|_0 \leqslant n, u \text{ is a non-trivial solution of } (P_{\lambda,n})\}.$$

In particular, $\Sigma' \subset \Sigma_n \cap (\mathbb{R} \times \bar{B}_n(0))$, from which $\Sigma' \subset \Sigma_n^n$ and so $(\lambda, u) \in \Sigma_n^n \subset \Sigma$. □

Remark 3.3.

(i) We point out that the above result is true even in the case in which the limit of $A(x, s)$ does not exist as $s \to \infty$.

(ii) In the case where A is bounded in some subset of Ω, we can conclude that $\mathrm{Proj}_{\mathbb{R}} \Sigma$ is bounded. Indeed, assume that $|A(x, s)| \leqslant \gamma$ if $x \in B$, where B is a ball such that $B \subset \Omega$, then using the monotonicity of the principal eigenvalue with respect to the domain, we obtain

$$\lambda = \lambda_1(A(x, u)) \leqslant \lambda_1^B(A(x, u)) \leqslant \lambda_1^B(\gamma I) = \gamma \lambda_1^B(I).$$

(iii) In this case we can obtain a similar result to the main one in [**2**]. Indeed, for each $r > 0$ there exists $\lambda_r > 0$ and $u_r \in H_0^1(\Omega)$, a solution of (P_λ) with $\|u\|_0 = r$.

In the next result we show that when $A(x,s)$ tends to infinity as $s\to\infty$ in the sense of (A_∞), then the bifurcation at infinity disappears, in some sense $\lambda_\infty\to+\infty$ when $A(x,s)$ tends to infinity.

Theorem 3.4. *Assume that A satisfies (A_4), $(\tilde{A}_2)$ and (A_∞). Then there exists a continuum $\Sigma\subset\mathcal{S}$ such that $(\lambda_0,0)\in\Sigma$. Moreover, the interval $(\lambda_0,+\infty)\subset \mathrm{Proj}_{\mathbb{R}}\Sigma$ and*

$$\lim_{\substack{\lambda\to+\infty\\(\lambda,u_\lambda)\in\Sigma}}\|u_\lambda\|_0=+\infty.$$

Proof. The existence of the continuum unbounded Σ bifurcating from $(\lambda_0,0)$ follows by Theorem 3.2. Since $\lambda=\lambda_1(A(x,u))\geqslant\lambda_1(\alpha I)=\alpha\lambda_1(I)$, there do not exist positive solutions for λ small. So, it suffices to prove that bifurcation from infinity is not possible. In order to do this, we observe that problem (P_λ) can be written as

$$\left.\begin{aligned}-\operatorname{div}(B(x,u)g(u)\nabla u)&=\lambda u, && x\in\Omega,\\ u&=0, && x\in\partial\Omega,\end{aligned}\right\}\qquad (P_\lambda)$$

where g is given by hypothesis (A_∞) and

$$B(x,u):=\frac{A(x,u)}{g(u)}.$$

Moreover, if we perform the change of variable

$$w=\tilde{g}(u)=\int_0^u g(t)\,\mathrm{d}t,$$

problem (P_λ) is equivalent to

$$\left.\begin{aligned}-\operatorname{div}(C(x,w)\nabla w)&=\lambda f(w), && x\in\Omega,\\ w&=0, && x\in\partial\Omega,\end{aligned}\right\}\qquad (Q_\lambda)$$

where

$$C(x,w):=B(x,\tilde{g}^{-1}(w))\quad\text{and}\quad f(w):=\tilde{g}^{-1}(w).$$

Now we argue by contradiction, and assume that there exists a sequence of solutions (λ_n,u_n) of (P_{λ_n}) such that $\lambda_n\to\bar{\lambda}>0$ and $\|u_n\|_0\to\infty$. Then, by (3.1), we have that $\|u_n\|\to\infty$ and taking $w_n=\tilde{g}(u_n)$, it is clear that $\|w_n\|_0\to\infty$. In addition, since (A_∞) implies that $\alpha^2\|u_n\|^2\leqslant\|w_n\|^2$, we also have that $\|w_n\|\to\infty$. For the normalized sequence $z_n:=w_n/\|w_n\|$ we know the existence of $z\in H_0^1(\Omega)$, such that

$$z_n\to z\quad\text{strongly in } L^2(\Omega)\text{ and a.e. in }\Omega.$$

and so, taking $w_n/\|w_n\|^2$ as a test function in (Q_{λ_n}), we obtain that

$$\alpha\leqslant\int_\Omega C(x,w_n)\nabla z_n\cdot\nabla z_n=\lambda_n\int_\Omega\frac{f(w_n)}{\|w_n\|}z_n. \tag{3.7}$$

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Now, taking into account that

$$\frac{f(s)}{s} \to 0 \quad \text{as } s \to \infty$$

and that $f(s) \leqslant (1/\alpha)s$ for each $s \in \mathbb{R}^+$, we can argue as in Theorem 5.5 in [**1**] and conclude that

$$\int_\Omega \frac{f(w_n)}{\|w_n\|} z_n \to 0 \quad \text{as } n \to \infty.$$

Indeed, we can write for every $n \in \mathbb{N}$

$$\begin{aligned}\int_\Omega \frac{f(w_n)}{\|w_n\|} z_n &= \int_\Omega \frac{f(w_n)}{\|w_n\|}(z_n - z) + \int_\Omega \frac{f(w_n)}{\|w_n\|} z \\ &\leqslant \frac{1}{\alpha}\|z_n\|_2\|z_n - z\|_2 + \int_{\Omega_0} \frac{f(w_n)}{\|w_n\|} z,\end{aligned}$$

where $\Omega_0 = \{x \in \Omega : z(x) \neq 0\}$. Thus, we only have to prove that

$$\lim_{n\to\infty} \int_{\Omega_0} \frac{f(w_n)}{\|w_n\|} z = 0,$$

which is a direct consequence of the Lebesgue Theorem, since for a.e. $x \in \Omega_0$, $w_n(x) = z_n(x)\|w_n\| \to +\infty$ and then

$$\frac{f(w_n(x))}{\|w_n\|} z(x) \to 0 \quad \text{a.e. } x \in \Omega_0.$$

Thus, taking limits in (3.7), we have that $\alpha \leqslant 0$, which is a contradiction. □

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