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INTEGRAL EQUATION FORMULATION FOR SHOUT OPTIONS

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Abstract

We use an integral equation formulation approach to value shout options, which are exotic options giving an investor the ability to "shout" and lock in profits while retaining the right to benefit from potentially favourable movements in the underlying asset price. Mathematically, the valuation is a free boundary problem involving an optimal exercise boundary which marks the region between shouting and not shouting. We also find the behaviour of the optimal exercise boundary for one- and two-shout options close to expiry.

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1. Introduction

Over the past three decades, there have been tremendous advances in both the theory and practice of financial engineering, a field which has come of age since the early 1970s, when the Nobel prize winning Black–Scholes option pricing formula [6, 30] was published and the Chicago Board Options Exchange opened its doors as the first organized options exchange. A large part of the efforts of financial engineers has been directed at the pricing and hedging of derivatives securities, whose values are based on some other underlying asset, with options garnering the lion's share of the attention. Options are derivatives which grant the holder the right but not the obligation to carry out a specified transaction on the underlying security. Along with this growth in the theory of financial engineering has come a tremendous growth in the use and complexity of financial derivatives. This includes derivatives that are embedded in other contracts. Such complexities have caused a dramatic change in the financial industry, with numerous parties now using derivatives to accomplish objectives such

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as speculating, reallocating risk, augmenting market liquidity and arbitrage. To help accomplish these objectives, over the years, a number of so-called exotic options have been introduced in the financial markets to cater for the demands of the various issuers and investors. One such exotic option is the shout option [35], which is the topic of this study.

Before we define shout options, it is necessary to first define vanilla options, which come in two flavours: European and American. European options can be exercised only at expiry, which is specified in the contract. A European call option on a stock pays the holder the amount $\max(S_T - E, 0)$ at expiry, where S_T is the price of the underlying stock at expiry time $t = T$ and E is the strike price of the option, also specified in the contract, while a European put option will pay an amount $\max(E - S_T, 0)$ at expiry. Although many options are cash-settled, a call essentially gives the right to sell the stock at the price E, while a put gives the right to buy the stock at that price. It is comparatively straightforward to price European options using the Black–Scholes model [6, 30]. One drawback of European options for the holder is that the payoff is based solely on the price of the underlying stock at expiry, so that if a European option is deep in-the-money before expiry, the holder needs to either sell the option or else wait until expiry and hope that the intrinsic value does not decrease. American options partly address this drawback as they can be exercised at any time at or before expiry, with an American call option paying $S_t - E > 0$ when exercised, where S_t is the stock price at time t, and an American put option paying $E - S_t > 0$. A third class of option, known as the Bermudan or semi-American option, allows early exercise but only on a finite number of discrete dates. Clearly, an American option allows the holder the opportunity to lock in the profits at any time, but, in doing so, he must forfeit the right to benefit from any potentially favourable asset price movements.

Shout options offer investors a way to lock in the profits to date while still retaining the right to benefit from future upsides. To accomplish this with a one-shout call option, if a shout is made at time $t < T$, it locks in the profit $\max(S_t - E, 0)$ to be paid to the holder at expiry and it also resets the strike price to S_t, giving the holder a new at-the-money European call. At-the-money means that the strike price of the new option is set equal to S_t and this new option allows the holder to benefit from future upside. Similarly, upon shouting with a one-shout put option at time $t < T$, a locked-in payment of $\max(E - S_t, 0)$ is guaranteed to the holder at expiry and it also resets the strike price to S_t, giving the holder a new at-the-money European put. Some shout options offer the holder to shout more than once and the general rule is that if an n-shout option is exercised early at time t, the holder receives $\max(S_t - E, 0)$ for a call and $\max(E - S_t, 0)$ for a put (at expiry) along with a new at-the-money $(n - 1)$-shout option. Because they offer a way to lock in the profits to date while still retaining the right to benefit from future upsides, shouts are very attractive to investors and have recently gained popularity. Note that other financial instruments exist which have embedded "shout" features. These include strike reset options, which, upon shouting, simply reset the strike price to the current stock price. Dai et al. [11] developed a linear complementarity problem to analyse such options. Other examples of shout options include protective floor indexes and certain "segregated funds" sold by Canadian life insurance companies.

Returning to American options, the early exercise feature means that the holder of an American option should constantly decide either to exercise the option or retain it, with the holder aiming to maximize the present value of the payoff from the option. This in turn leads to a free boundary, known as the optimal exercise boundary, on which exercise will take place. Karatzas [21] was able to prove the existence of an optimal exercise policy for American options and showed that there was an optimal stopping time. Duffie [13] provided a good overview of Karatzas's work. Also useful are the studies by Friedman [15] and Tao [34] on the analyticity of the free boundary in Stefan problems, which are a class of physical problems involving melting and solidification and which are formulated in a manner very similar to American options.

In addition to the studies mentioned above, there has been a considerable body of research aimed both at pricing American options and locating the free boundary. For the numerical aspect of the problem, the reader is referred to the books by Kwok [26] and Wilmott [36]. On the theoretical side, two popular approaches have been Tao's method [34] and the integral equation approach. Tao's method involves applying asymptotics to the underlying partial differential equation (PDE) using the time remaining until expiry as a small parameter. Studies such as by Dewynne et al. [12] and Zhang and Li [38] have yielded the first few terms in the series for the value of the option and the location of the free boundary close to expiry. The integral equation approach as seen in the papers by Evans et al. [14], Jacka [19], Kim [22], Knessl [23], Kuske and Keller [25] and McKean [29] involves decoupling the location of the free boundary from the pricing of the option, leading to an integral equation for the location of the free boundary, which can subsequently be solved asymptotically or numerically. One of these integral equation methods is of particular interest to us, as we will be applying that method to shouts in the present study, and that was the method used by Kim [22] and Jacka [19]. This approach was originally developed for physical Stefan problems [24] and later applied to economics by McKean [29]. It was also applied to vanilla Americans with great success by Kim [22] and Jacka [19], who independently derived the same results. Kim used McKean's formula and then took the continuous limit of the Geske–Johnson formula [16] (which is a discrete approximation for American options), while Jacka applied probability theory to the optimal stopping problem. Carr [8] later used these results to show how to decompose the value of an American option into intrinsic value and time value. Evans et al. [14] and Kuske and Keller [25] got explicit expressions, valid near expiry for the optimal exercise boundaries of American calls and puts, using a Green's theorem approach. Chiarella et al. [10] have surveyed the Green's function approach to American options using the incomplete Fourier transform and demonstrated how the various representations are related. They also considered their economic interpretations. The relationship between the Green's function approach and the integral transform approach is discussed by Alobaidi et al. [3].

The approach by Jacka [19], Kim [22] and McKean [29] leads to an integral equation for the location of the free boundary, which was solved numerically by

Huang et al. [18] and by approximating the free boundary as a multipiece exponential function by Ju [20]. Our contribution is to extend the analysis of Jacka et al. to shout options.

Turning to shout options, much of the work done to date is numerical, although as with other options involving a free boundary and choice on the part of an investor, some standard numerical techniques such as the forward-looking Monte Carlo method present great difficulties because of the optimization component of shout options. However, other standard numerical methods, such as finite differences, have successfully been applied to shout options by Windcliff et al. [37]. One paper we would mention, in particular, is that by Boyle et al. [7], in which a Green's function approach was used. With this approach, it was assumed that early exercise could only occur on a limited number of fixed times $t < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$, so that the option was treated as Bermudan-style, and then the value of the option at time t_m was used to compute the value at time t_{m-1}, which in turn was used to compute the value at time t_{m-2} and so on. The value at time t_{m-1} was computed by using an integral involving the product of the Green's function with the value at time t_m, with this integral being evaluated numerically. Theoretical studies of shout options include the series solutions by Goard [17] (based on a technique by Tao [34]) and the partial Laplace transform study by Alobaidi et al. [4]. However, advantages of applying the integral equation approach, as is done in this paper, include:

(i) the decoupling of the option valuation problem from finding the free boundary and, very importantly,
(ii) its ability to handle multiple free boundaries corresponding to multiple shouts.

The rest of the paper is organized as follows. The bulk of our work will be on one-shout options and we will use the techniques developed by Jacka [19], Kim [22], Kolodner [24] and McKean [29] to write down expressions involving integrals for the value of one-shout calls and puts. We will then evaluate these expressions on the free boundary to arrive at integral equations for the location of the free boundary for a one-shout option and solve these integral equations close to expiry. We will also touch on the valuation of multiple shout options, presenting a series solution for the free boundary for the two-shout option close to expiry and speculating on the nature of the free boundaries for multiple shout options in general. The final section contains a discussion of our results.

2. Analysis

Our starting point is the Black–Scholes–Merton partial differential equation (PDE) [6, 30] governing the price $V(S,t)$ of an equity derivative

$$\left[\frac{\partial}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r-D)S\frac{\partial}{\partial S} - r\right]V = 0, \tag{2.1}$$

where S is the price of the underlying stock, σ the stock price volatility, r the risk-free interest rate and D the dividend yield on the stock. In our analysis, σ, r and D

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are assumed to be constant. Equation (2.1) governs American and European options as well as shout options. Indeed, long ago, Merton [31] pointed out that different securities may satisfy the same equation with the boundary and initial conditions differentiating the securities; an example of this is that shout options and American options have different early exercise payoffs.

To simplify the analysis, for an option which expires at time T, we will work in terms of the tenor or time remaining until expiry $\tau = T - t$ as follows:

$$\mathcal{L}V = \left[\frac{\partial}{\partial\tau} - \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} - (r - D)S\frac{\partial}{\partial S} + r\right]V = 0; \tag{2.2}$$

here we define the operator $\mathcal{L}$ to be used later. It is straightforward to price a European option which obeys condition (2.2): the value of a European option with a payoff at expiry of $V(S, 0)$ is given by

$$V^{(e)}(S, \tau) = \int_0^\infty V(Z, 0)G(S, Z, \tau)\, dZ, \tag{2.3}$$

where G is the Green's function

$$G(S, Z, \tau) = \frac{e^{-r\tau}}{Z\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{[\ln(S/Z) + r_2\tau]^2}{2\sigma^2\tau}\right)$$

and we have introduced the shorthand $r_2 = r - D - \sigma^2/2$. Equation (2.3) is simply the Green's function formulation of the linear boundary value problem for the parabolic PDE (2.2). Using this formula, the price of a European call with strike $E^{(0)}$, for which $V(S, 0) = \max(S - E^{(0)}, 0)$, is

$$V^{(e)}(S, \tau) = \frac{S e^{-D\tau}}{2} \operatorname{erfc}\left(-\frac{\ln(S/E^{(0)}) + r_1\tau}{\sigma\sqrt{2\tau}}\right) - \frac{E^{(0)}e^{-r\tau}}{2} \operatorname{erfc}\left(-\frac{\ln(S/E^{(0)}) + r_2\tau}{\sigma\sqrt{2\tau}}\right),$$

while that of a European put with strike $E^{(0)}$ and $V(S, 0) = \max(E^{(0)} - S, 0)$ is

$$V^{(e)}(S, \tau) = -\frac{S e^{-D\tau}}{2} \operatorname{erfc}\left(\frac{\ln(S/E^{(0)}) + r_1\tau}{\sigma\sqrt{2\tau}}\right) + \frac{E^{(0)}e^{-r\tau}}{2} \operatorname{erfc}\left(\frac{\ln(S/E^{(0)}) + r_2\tau}{\sigma\sqrt{2\tau}}\right).$$

Here "erfc" denotes the complementary error function and $r_1 = r - D + \sigma^2/2$. For the American-style options with early exercise features, it follows from an application of Green's theorem that if such an option obeys (2.2) where it is optimal to hold the option and the payoff at expiry is $V(S, 0)$ while that from immediate exercise is $P(S, \tau)$, then we can write the value of the option as the sum of the value of the corresponding European option $V^{(e)}(S, \tau)$ together with another term representing the premium from early exercise,

$$V(S, \tau) = V^{(e)}(S, \tau) + \int_0^\tau \int_0^\infty \mathcal{F}(Z, \zeta)G(S, Z, \tau - \zeta)\, dZ\, d\zeta. \tag{2.4}$$

In this equation, $\mathcal{F}(S, \tau)$ is equal to 0 where it is optimal to hold the option, while where exercise is optimal $\mathcal{F}(S, \tau)$ is the result of substituting the early exercise payoff $P(S, \tau)$ into the Black–Scholes–Merton PDE,

$$\mathcal{F}(S, \tau) = \mathcal{L}P, \tag{2.5}$$

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where the operator $\mathcal{L}$ was defined in (2.2). Similarly, shout options satisfy $\mathcal{L}V = \mathcal{F}(S,\tau)$, where $\mathcal{F} = 0$ when it is not optimal to shout; otherwise $\mathcal{F} = \mathcal{L}P$, where P is the payoff from shouting (see the paper by Dai et al. [11] for the related reset options). Hence, for shout options, we can use the formulae (2.4) and (2.5) recursively. We use the notation $V^{(n)}(S,\tau)$ for the value of a shout option with n shouting opportunities and $E^{(n)}$ for the strike price of an n-shout option. If held until expiry, an n-shout call will pay $\max(S - E^{(n)}, 0)$, while a put will pay $\max(E^{(n)} - S, 0)$. At the first shout, which will occur at the free boundary $S_f^{(n)}(\tau)$, we exchange this n-shout option for a lock-in payment at expiry of the difference between the current stock price S and the strike price $E^{(n)}$ together with a new at-the-money $(n-1)$-shout option $V^{(n-1)}(S,\tau)|_{E^{(n-1)}=S}$. The term "at-the-money" in this context means that the strike price $E^{(n-1)}$ of this new $(n-1)$-shout option is set equal to the stock price S at the time of exercise. It follows that the payoff from exercise for an n-shout call is

$$P^{(n)}(S,\tau) = (S - E^{(n)})e^{-r\tau} + V^{(n-1)}(S,\tau)|_{E^{(n-1)}=S}, \tag{2.6}$$

while for an n-shout put it is

$$P^{(n)}(S,\tau) = (E^{(n)} - S)e^{-r\tau} + V^{(n-1)}(S,\tau)|_{E^{(n-1)}=S}. \tag{2.7}$$

An option with zero shouts remaining is just a vanilla European, $V^{(0)}(S,\tau) = V^{(e)}(S,\tau)$. If we exercise a one-shout option, we receive at expiry the difference between the stock price and the initial strike price and we receive immediately a zero-shout option, which is simply an at-the-money European option. So, the payoff from shouting for a one-shout call is

$$\begin{aligned} P^{(1)}(S,\tau) &= (S - E^{(1)})e^{-r\tau} + V^{(0)}(S,\tau)|_{E^{(0)}=S} \\ &= (S - E^{(1)})e^{-r\tau} + \frac{S}{2}\Big[e^{-D\tau}\operatorname{erfc}\Big(-\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big) - e^{-r\tau}\operatorname{erfc}\Big(-\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\Big], \end{aligned} \tag{2.8}$$

while for a one-shout put it is

$$\begin{aligned} P^{(1)}(S,\tau) &= (E^{(1)} - S)e^{-r\tau} + V^{(0)}(S,\tau)|_{E^{(0)}=S} \\ &= (E^{(1)} - S)e^{-r\tau} - \frac{S}{2}\Big[e^{-D\tau}\operatorname{erfc}\Big(\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big) - e^{-r\tau}\operatorname{erfc}\Big(\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\Big]. \end{aligned} \tag{2.9}$$

Using (2.5), the forcing term in the formula (2.4) for a one-shout call is

$$\mathcal{F}^{(1)}(S,\tau) = -\frac{S}{2}e^{-r\tau}\Big[(r-D)\operatorname{erfc}\Big(\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big) - \frac{\sigma}{\sqrt{2\pi\tau}}\exp\Big(-\frac{r_2^2\tau}{2\sigma^2}\Big)\Big],$$

while that for a one-shout put is

$$\mathcal{F}^{(1)}(S,\tau) = \frac{S}{2}e^{-r\tau}\Big[(r-D)\operatorname{erfc}\Big(\frac{-r_2\tau}{\sigma\sqrt{2\tau}}\Big) + \frac{\sigma}{\sqrt{2\pi\tau}}\exp\Big(-\frac{r_2^2\tau}{2\sigma^2}\Big)\Big].$$

Applying formula (2.4) to a shout call where it is optimal to hold if $S < S_f^{(1)}(\tau)$, and exercise if $S \geq S_f^{(1)}(\tau)$, we find the value of a one-shout call:

$$\begin{aligned}
V^{(1)}(S,\tau) &= V^{(e)}(S,\tau) + \int_0^\tau \int_{S_f^{(1)}(\zeta)}^\infty \bigg[\bigg(\frac{-rZ}{2}e^{-r\zeta} + \frac{DZ}{2}e^{-r\zeta}\bigg)\operatorname{erfc}\bigg(\frac{r_2\zeta}{\sigma\sqrt{2\zeta}}\bigg) \\
&\quad + \frac{Z\sigma e^{-r\zeta}}{2\sqrt{2\pi\zeta}}\exp\bigg(-\frac{r_2^2\zeta}{2\sigma^2}\bigg)\bigg]G(S,Z,\tau-\zeta)\,dZ\,d\zeta \\
&= \frac{S\,e^{-D\tau}}{2}\operatorname{erfc}\bigg(-\frac{\ln(S/E^{(1)}) + r_1\tau}{\sigma\sqrt{2\tau}}\bigg) - \frac{Ee^{-r\tau}}{2}\operatorname{erfc}\bigg(-\frac{\ln(S/E^{(1)}) + r_2\tau}{\sigma\sqrt{2\tau}}\bigg) \\
&\quad + \int_0^\tau \bigg[\frac{-(r-D)}{2\sigma\sqrt{2\pi}\sqrt{\tau-\zeta}}e^{-r\tau}\operatorname{erfc}\bigg(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\bigg) + \frac{e^{-r\tau}}{4\pi\sqrt{\zeta}\sqrt{\tau-\zeta}}\exp\bigg(-\frac{r_2^2\zeta}{2\sigma^2}\bigg)\bigg] \\
&\quad \times \frac{\sigma S\sqrt{\pi(\tau-\zeta)}}{\sqrt{2}}e^{(r-D)(\tau-\zeta)}\operatorname{erfc}\bigg(\frac{\ln(S_f(\zeta)) - \ln(S) - r_1(\tau-\zeta)}{\sigma\sqrt{2(\tau-\zeta)}}\bigg)\,d\zeta.
\end{aligned} \tag{2.10}$$

However, if we apply (2.4) to a shout put where it is optimal to hold if $S < S_f^{(1)}(\tau)$ and shout if $S \geq S_f^{(1)}(\tau)$, we find the value of a one-shout put:

$$\begin{aligned}
V^{(1)}(S,\tau) &= V^{(e)}(S,\tau) + \int_0^\tau \int_0^{S_f^{(1)}(\zeta)} \bigg[\frac{(r-D)e^{-r\zeta}}{2}\operatorname{erfc}\bigg(-\frac{r_2\zeta}{\sigma\sqrt{2\zeta}}\bigg) + \frac{\sigma e^{-r\zeta}}{2\sqrt{2\pi\zeta}}e^{-r_2^2\zeta/2\sigma^2}\bigg] \\
&\quad \times \frac{e^{-r(\tau-\zeta)}}{\sigma\sqrt{2\pi(\tau-\zeta)}}\exp\bigg(-\frac{\{\ln(S) - \ln(Z) + r_2(\tau-\zeta)\}^2}{2\sigma^2(\tau-\zeta)}\bigg)\,dZ\,d\zeta \\
&= -\frac{S\,e^{-D\tau}}{2}\operatorname{erfc}\bigg(\frac{\ln(S/E^{(1)}) + r_1\tau}{\sigma\sqrt{2\tau}}\bigg) + \frac{Ee^{-r\tau}}{2}\operatorname{erfc}\bigg(\frac{\ln(S/E^{(1)}) + r_2\tau}{\sigma\sqrt{2\tau}}\bigg) \\
&\quad + \int_0^\tau \bigg[\frac{(r-D)e^{-r\tau}}{2\sigma\sqrt{2\pi(\tau-\zeta)}}\operatorname{erfc}\bigg(-\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\bigg) + \frac{e^{-r\tau}e^{-r_2^2\zeta/2\sigma^2}}{4\pi\sqrt{\zeta(\tau-\zeta)}}\bigg] \\
&\quad \times S\,e^{(r-D)(\tau-\zeta)}\frac{\sqrt{\pi}\sigma\sqrt{\tau-\zeta}}{\sqrt{2}}\operatorname{erfc}\bigg(-\frac{(\ln(S_f(\zeta)) - \ln(S) - r_1(\tau-\zeta))}{\sigma\sqrt{2(\tau-\zeta)}}\bigg)\,d\zeta.
\end{aligned} \tag{2.11}$$

An interesting point to note is that while vanilla Americans possess put–call symmetry [9, 28], the option to shout feature appears to destroy this symmetry.

Although a detailed examination of multiple shout options is beyond the scope of this paper because of the complexity of the algebra, we will touch on the free boundary of a two-shout option in a later section, for which we will require the payoff from early exercise for such an option. To write this, we note that in the next section we will write the free boundary for an n-shout option as $S_f^{(n)}(\tau) = E^{(n)}\exp(x_f^{(n)}(\tau))$. Using the general expression (2.6), we see that for a two-shout call this payoff is the excess of the stock

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price over the strike price, together with an at-the-money one-shout option,

$$\begin{aligned} P^{(2)}(S,\tau) &= (S - E^{(2)})e^{-r\tau} + V^{(1)}(S,\tau)|_{E^{(1)}=S} \\ &= (S - E^{(2)})e^{-r\tau}\frac{S\,e^{-D\tau}}{2}\operatorname{erfc}\Big(-\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big) - \frac{S\,e^{-r\tau}}{2}\operatorname{erfc}\Big(-\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big) \\ &\quad + \int_0^\tau\Big[\frac{-(r-D)}{2\sigma\sqrt{2\pi}\sqrt{\tau-\zeta}}e^{-r\tau}\operatorname{erfc}\Big(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\Big) + \frac{e^{-r\tau}}{4\pi\sqrt{\zeta}\sqrt{\tau-\zeta}}\exp\Big(-\frac{r_2^2\zeta}{2\sigma^2}\Big)\Big] \\ &\quad \times \frac{\sigma S\sqrt{\pi(\tau-\zeta)}}{\sqrt{2}}e^{(r-D)(\tau-\zeta)}\operatorname{erfc}\Big(\frac{x_f^{(1)}(\zeta) - r_1(\tau-\zeta)}{\sigma\sqrt{2(\tau-\zeta)}}\Big)d\zeta. \end{aligned} \tag{2.12}$$

Similarly, using the general expression (2.7), we see that for a two-shout put option, this payoff is the deficit of the stock price below the strike price, together with an at-the-money one-shout option,

$$\begin{aligned} P^{(2)}(S,\tau) &= (E^{(2)} - S)e^{-r\tau} + V^{(1)}(S,\tau)|_{E^{(1)}=S} \\ &= (E^{(2)} - S)e^{-r\tau} - \frac{S\,e^{-D\tau}}{2}\operatorname{erfc}\Big(\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big) + \frac{S\,e^{-r\tau}}{2}\operatorname{erfc}\Big(\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big) \\ &\quad + \int_0^\tau\Big[\frac{(r-D)e^{-r\tau}}{2\sigma\sqrt{2\pi(\tau-\zeta)}}\operatorname{erfc}\Big(-\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\Big) + \frac{e^{-r\tau}e^{-r_2^2\zeta/2\sigma^2}}{4\pi\sqrt{\zeta(\tau-\zeta)}}\Big] \\ &\quad \times S\,e^{(r-D)(\tau-\zeta)}\frac{\sqrt{\pi}\sigma\sqrt{\tau-\zeta}}{\sqrt{2}}\operatorname{erfc}\Big(\frac{-x_f^{(1)}(\zeta) + r_1(\tau-\zeta)}{\sigma\sqrt{2(\tau-\zeta)}}\Big)d\zeta. \end{aligned} \tag{2.13}$$

3. Integral equations for one-shout options

In this section, we obtain integral equations for the location of the free boundary $S = S_f^{(1)}(\tau)$ for a one-shout option. These equations are obtained by substituting the expressions for the one-shout call (2.10) and put (2.11) found in the previous section into the conditions at the free boundary. To simplify the analysis, we write $S_f^{(1)}(\tau) = E^{(1)}\exp(x_f^{(1)}(\tau))$. This is motivated by a similar change of variable $S = Ee^x$, used to reduce the Black–Scholes–Merton PDE [6] to a constant-coefficient equation. We note that the free boundary starts from $S_f^{(1)}(0) = E^{(1)}$ or, equivalently, $x_f^{(1)}(0) = 0$ at expiry. The conditions at the free boundary are that the option price and the delta of the option are continuous there, so that $V^{(1)} = P^{(1)}$ and $(\partial V^{(1)}/\partial S) = (\partial P^{(1)}/\partial S)$ at $S = S_f^{(1)}(\tau)$. The condition on the price is known as the value matching condition and that on the delta is known as the high contact or smooth pasting condition [32].

For the call option, using (2.10) and (2.8) for the condition $V^{(1)} = P^{(1)}$ at the free boundary $S = S_f^{(1)}(\tau)$ yields

$$(e^{x_f^{(1)}(\tau)} - 1)\exp(-r\tau) + \frac{e^{x_f^{(1)}(\tau)}}{2}\Big[e^{-D\tau}\operatorname{erfc}\Big(-\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big) - e^{-r\tau}\operatorname{erfc}\Big(-\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\Big]$$

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$$
\begin{aligned}
&-\frac{e^{x_f^{(1)}(\tau)}e^{-D\tau}}{2}\operatorname{erfc}\Big(-\frac{(x_f^{(1)}(\tau)+r_1\tau)}{\sigma\sqrt{2\tau}}\Big)+\frac{e^{-r\tau}}{2}\operatorname{erfc}\Big(-\frac{(x_f^{(1)}(\tau)+r_2\tau)}{\sigma\sqrt{2\tau}}\Big)\\
&=\int_0^{\tau}\Big[\frac{-(r-D)}{2\sigma\sqrt{2\pi}\sqrt{\tau-\zeta}}e^{-r\tau}\operatorname{erfc}\Big(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\Big)+\frac{e^{-r\tau}}{4\pi\sqrt{\zeta}\sqrt{\tau-\zeta}}\exp\Big(-\frac{r_2^2\zeta}{2\sigma^2}\Big)\Big]\\
&\quad\times\frac{\sigma\sqrt{\pi(\tau-\zeta)}}{\sqrt{2}}e^{(r-D)(\tau-\zeta)}e^{x_f^{(1)}(\tau)}\operatorname{erfc}\Big(\frac{x_f^{(1)}(\zeta)-x_f^{(1)}(\tau)-r_1(\tau-\zeta))}{\sigma\sqrt{2(\tau-\zeta)}}\Big)d\zeta, \qquad (3.1)
\end{aligned}
$$

while the condition $(\partial V^{(1)}/\partial S)=(\partial P^{(1)}/\partial S)$ gives

$$
\begin{aligned}
&e^{-r\tau}+\frac{e^{-D\tau}}{2}\Big[\operatorname{erfc}\Big(-\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big)-\operatorname{erfc}\Big(-\frac{x_f^{(1)}(\tau)+r_1\tau}{\sigma\sqrt{2\tau}}\Big)\Big]-\frac{e^{-r\tau}}{2}\operatorname{erfc}\Big(-\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\\
&\quad-\frac{e^{-D\tau}}{\sigma\sqrt{2\pi\tau}}\exp\Big(-\frac{[x_f^{(1)}(\tau)+r_1\tau]^2}{2\sigma^2\tau}\Big)+\frac{e^{-r\tau-x_f^{(1)}(\tau)}}{\sigma\sqrt{2\pi\tau}}\exp\Big(-\frac{[x_f^{(1)}(\tau)+r_2\tau]^2}{2\sigma^2\tau}\Big)\\
&=\int_0^{\tau}\Big[\frac{e^{-D(\tau-\zeta)}e^{-r\zeta}}{4}\operatorname{erfc}\Big(-\frac{x_f^{(1)}(\tau)-x_f^{(1)}(\zeta)+r_1(\tau-\zeta)}{\sigma\sqrt{2(\tau-\zeta)}}\Big)\Big\{-(r-D)\operatorname{erfc}\Big(\frac{r_2\zeta}{\sigma\sqrt{2\zeta}}\Big)\\
&\quad+\frac{\sigma}{\sqrt{2\pi\zeta}}\exp\Big(-\frac{r_2^2\zeta}{2\sigma^2}\Big)\Big\}+\frac{e^{-D(\tau-\zeta)}e^{-r\zeta}}{2\sigma\sqrt{2\pi(\tau-\zeta)}}\exp\Big(-\frac{[x_f^{(1)}(\tau)-x_f^{(1)}(\zeta)+r_1(\tau-\zeta)]^2}{2\sigma^2(\tau-\zeta)}\Big)\\
&\quad\times\Big\{-(r-D)\operatorname{erfc}\Big(\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\Big)+\frac{e^{-r_2^2\zeta/2\sigma^2}\sigma}{\sqrt{2\pi\zeta}}\Big\}\Big]d\zeta. \qquad (3.2)
\end{aligned}
$$

The two equations (3.1) and (3.2) constitute a pair of integral equations for the location of the free boundary $x_f^{(1)}(\tau)$ for a one-shout call option. They are very similar to their counterparts for an American call option, but both contain additional terms due to the presence of the additional payoff at the free boundary for the shout option.

Similarly, for the put option, using (2.11) and (2.9) for the condition $V^{(1)}=P^{(1)}$ at the free boundary $S=S_f^{(1)}(\tau)$ yields

$$
\begin{aligned}
&(1-e^{x_f^{(1)}(\tau)})e^{-r\tau}-\frac{e^{x_f^{(1)}(\tau)}}{2}\Big[e^{-D\tau}\operatorname{erfc}\Big(\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big)-e^{-r\tau}\operatorname{erfc}\Big(\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\Big]\\
&\quad+\frac{e^{x_f^{(1)}(\tau)}e^{-D\tau}}{2}\operatorname{erfc}\Big(\frac{x_f^{(1)}(\tau)+r_1\tau}{\sigma\sqrt{2\tau}}\Big)-\frac{e^{-r\tau}}{2}\operatorname{erfc}\Big(\frac{x_f^{(1)}(\tau)+r_2\tau}{\sigma\sqrt{2\tau}}\Big)\\
&=\int_0^{\tau}\Big[\frac{(r-D)e^{-r\tau}}{2\sigma\sqrt{2\pi(\tau-\zeta)}}\operatorname{erfc}\Big(-\frac{r_2\sqrt{\zeta}}{\sigma\sqrt{2}}\Big)+\frac{e^{-r\tau}e^{-r_2^2\zeta/2\sigma^2}}{4\pi\sqrt{\zeta(\tau-\zeta)}}\Big]\\
&\quad\times e^{x_f^{(1)}(\tau)}e^{(r-D)(\tau-\zeta)}\frac{\sqrt{\pi}\sigma\sqrt{\tau-\zeta}}{\sqrt{2}}\operatorname{erfc}\Big(-\frac{(x_f^{(1)}(\zeta)-x_f^{(1)}(\tau)-r_1(\tau-\zeta))}{\sigma\sqrt{2(\tau-\zeta)}}\Big)d\zeta,
\end{aligned}
$$
$$(3.3)$$

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while the condition $\partial V^{(1)}/\partial S = \partial P^{(1)}/\partial S$ gives

$$
\begin{aligned}
&-\frac{e^{-D\tau}}{2}\Big[\operatorname{erfc}\Big(\frac{r_1\tau}{\sigma\sqrt{2\tau}}\Big)-\operatorname{erfc}\Big(\frac{x_f^{(1)}(\tau)+r_1\tau}{\sigma\sqrt{2\tau}}\Big)\Big]-\frac{e^{-r\tau}}{2}\operatorname{erfc}\Big(-\frac{r_2\tau}{\sigma\sqrt{2\tau}}\Big)\\
&\qquad-\frac{e^{-D\tau}}{\sigma\sqrt{2\pi\tau}}\exp\Big(-\frac{[x_f^{(1)}(\tau)+r_1\tau]^2}{2\sigma^2\tau}\Big)+\frac{e^{-r\tau-x_f^{(1)}(\tau)}}{\sigma\sqrt{2\pi\tau}}\exp\Big(-\frac{[x_f^{(1)}(\tau)+r_2\tau]^2}{2\sigma^2\tau}\Big)\\
&\quad=\int_0^\tau e^{-r\zeta-D(\tau-\zeta)}\Big[\frac{(r-D)}{4\sqrt{2\pi(\tau-\zeta)}}\operatorname{erfc}\Big(-\frac{r_2\zeta}{\sigma\sqrt{2\zeta}}\Big)+\frac{e^{-r_2^2\zeta/2\sigma^2}}{8\pi\sqrt{\zeta}\sqrt{\tau-\zeta}}\Big]\\
&\qquad\times\Big[\sigma\sqrt{2\pi}\sqrt{\tau-\zeta}\operatorname{erfc}\Big(-\frac{(x_f^{(1)}(\zeta)-x_f^{(1)}(\tau)-r_1(\tau-\zeta))}{\sigma\sqrt{2}\sqrt{\tau-\zeta}}\Big)\\
&\qquad-2\exp\Big(-\frac{[x_f^{(1)}(\tau)-x_f^{(1)}(\zeta)+r_1(\tau-\zeta)]^2}{2\sigma^2(\tau-\zeta)}\Big)\Big]\,d\zeta. \qquad (3.4)
\end{aligned}
$$

The two equations (3.3) and (3.4) constitute a pair of integral equations for the location of the free boundary $x_f^{(1)}(\tau)$ for a one-shout put option.

4. Solution of the integral equations for the call option close to expiry

In this section, we solve the integral equations (3.1) and (3.2) for the one-shout call option close to expiry to find expressions for the location of the free boundary $x_f^{(1)}(\tau) = \ln(S_f^{(1)}(\tau)/E^{(1)})$ in the limit $\tau \to 0$. We will assume a solution of the form

$$x_f^{(1)}(\tau) \sim \sum_{n=1}^{\infty} x_n^{(1)}\tau^{n/2}, \qquad (4.1)$$

which is motivated both by a previous work on this problem (see, for example, the paper by Goard [17]) and the classic work of Tao [33, 34] on Stefan problems in general. The results for the one-shout put option are provided in Appendix A.

We now substitute the series (4.1) for $x_f^{(1)}(\tau)$ into (3.1)–(3.2) and expand, collect and equate powers of τ. To evaluate the integrals on the right-hand sides of (3.1)–(3.2), we make the change of variable $\zeta = \tau\eta$, which enables us to pull the τ-dependence outside of the integrals when we expand. All calculations are performed using the mathematics package Maple [27]. From equation (3.1) at $O(\tau^{1/2})$ we find

$$\frac{x_1^{(1)}}{2}\operatorname{erfc}\Big(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)+\frac{\sigma}{\sqrt{2\pi}}\Big\{1-\exp\Big(-\frac{x_1^{(1)2}}{2\sigma^2}\Big)\Big\}=\frac{\sigma}{4\sqrt{2\pi}}\int_0^1\frac{1}{\sqrt{\eta}}\operatorname{erfc}\Big(-\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta, \qquad (4.2)$$

while from equation (3.2) at $O(\tau^0)$ we find

$$\frac{1}{2}\operatorname{erfc}\Big(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)=\frac{1}{4\pi}\int_0^1\frac{1}{\sqrt{\eta(1-\eta)}}\exp\Big(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\,d\eta. \qquad (4.3)$$

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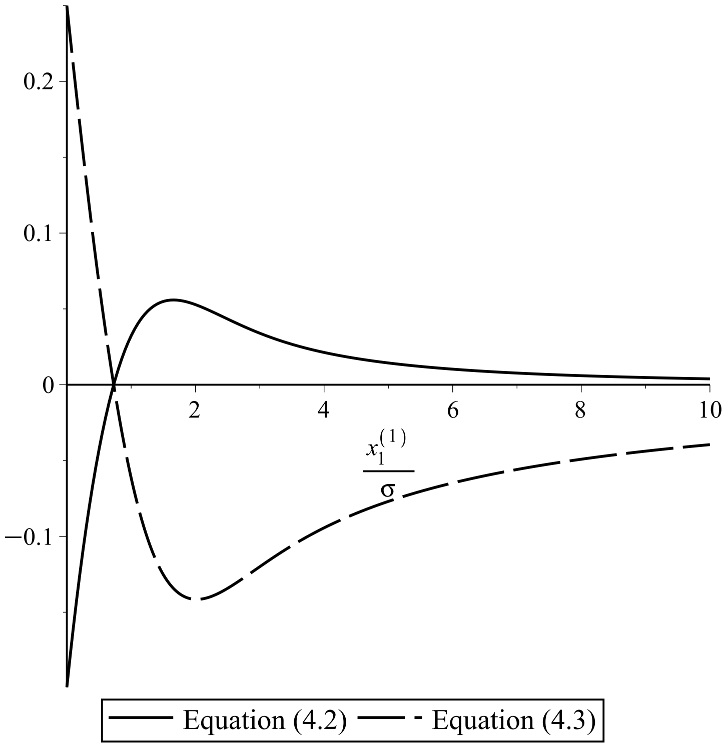

FIGURE 1. Plot of equations (4.2) (solid line) and (4.3) (dashed line) showing their numerical root.

These two equations (4.2) and (4.3) have a numerical root $x_1^{(1)} = 0.728600109\sigma$ (found by using the package Maple [27]) (see Figure 1, which is a Maple plot of the left-hand side (LHS) minus the right-hand side (RHS) of (4.2) and the LHS–RHS of (4.3)).

For American calls (see the paper by Zhang and Li [38]) when $D > r$, the free boundary starts from the strike price as it does here and the leading order term for small τ has the form $x_f(\tau) \sim \tau^{1/2}\sqrt{-\ln\tau}$, which is much steeper than the series (4.1) found here for the one-shout call, while, for the American call with $D < r$, although the series is of the form (4.1), the free boundary starts from rE/D, so that in either case a one-shout call is more likely to be exercised early than an American call.

Continuing with our expansion, at the next order, from (3.1) at $O(\tau)$ we find

$$\frac{x_1^{(1)2} + 2x_2^{(1)}}{4}\operatorname{erfc}\Big(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big) - \frac{r-D}{2}\operatorname{erf}\Big(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big) + \frac{\sigma x_1^{(1)}}{2\sqrt{2\pi}}\Big(2 - \exp\Big(-\frac{x_1^{(1)2}}{2\sigma^2}\Big)\Big)$$
$$= \int_0^1 \Big[\frac{(x_2^{(1)} + r_1)\sqrt{1-\eta}}{4\pi\sqrt{\eta}} \exp\Big(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)$$
$$+ \Big(\frac{x_1^{(1)}\sigma}{4\sqrt{2\pi\eta}} - \frac{r-D}{4}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta, \tag{4.4}$$

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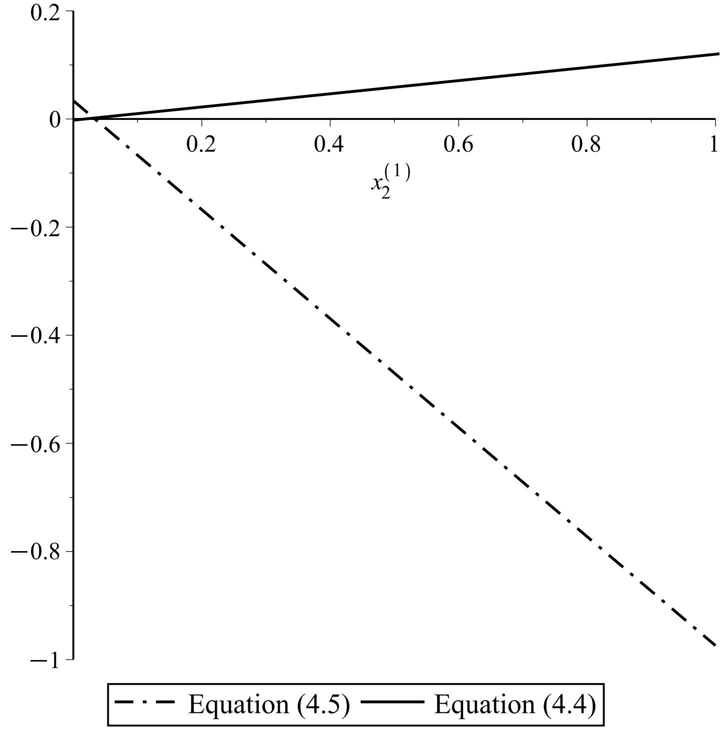

FIGURE 2. Plot of equations (4.4) (solid line) and (4.5) (dash-dotted line) showing their numerical root.

while from (3.2) at $O(\tau^{1/2})$ we find

$$\frac{\sigma}{\sqrt{2\pi}} - \frac{x_2^{(1)} + r_1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x_1^{(1)2}}{2\sigma^2}\right) = \int_0^1 \left[-\left(\frac{r-D}{2\sigma\sqrt{2\pi(1-\eta)}} + \frac{(x_2^{(1)} + r_1)x_1^{(1)}(1-\sqrt{\eta})}{4\sigma^2\pi\sqrt{\eta(1-\eta)}}\right)\right.$$
$$\left.\times \exp\left(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\right) + \frac{\sigma}{4\sqrt{2\pi\eta}} \operatorname{erfc}\left(-\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\right)\right] d\eta, \qquad (4.5)$$

which have a numerical root $x_2^{(1)} = 0.5516261057(r-D) + 0.04898978883\sigma^2$ (see Figure 2, which plots the LHS–RHS of (4.4) and the LHS–RHS of (4.5)). Parameters used for the plots were $r = 0.05, \sigma = 0.2$ and $D = 0.02$. Similar to the above procedure, the next term is found as $x_3^{(1)} = 0.413244516[(r-D)^2/\sigma] + 0.218773888\sigma(r-D) + 0.00303954446\sigma^3$.

5. Two-shout options

Although a detailed study of multiple-shout options is beyond the scope of this study, we will touch on the free boundary for a two-shout option. As we noted earlier, for a two-shout option the payoff at the free boundary $S_f^{(2)}(\tau)$ is the present value of

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the difference between the stock price and the strike price together with an at-the-money one-shout option. Using this payoff (2.12) and (2.13) in the formulae (2.4) and (2.5), respectively, it is possible to arrive at a set of integral equations somewhat similar to those presented above for the one-shout option. However, they are rather more complicated with these equations involving $S_f^{(1)}(\tau)$, which, in principle, is known as the solution to (3.1) and (3.2) for the call and (3.3) and (3.4) for the put as well as $S_f^{(2)}(\tau)$. As with the one-shout option, we solve these equations close to expiry to find expressions for the location of the free boundary $x_f^{(2)}(\tau) = \ln(S_f^{(2)}(\tau)/E^{(2)})$ in the limit $\tau \to 0$. In doing so, we will use the series we found for $x_f^{(1)}(\tau)$ earlier and this again is an example of how the pricing of shout options is a recursive problem: to find the free boundary $S_f^{(n)}(\tau)$ for an n-shout option, we first need to know $S_f^{(1)}(\tau), S_f^{(2)}(\tau), \ldots, S_f^{(n-1)}(\tau)$.

Returning to our analysis, we assume that $x_f^{(1)}(\tau)$ has the form (4.1) with the coefficients found earlier and that $x_f^{(2)}(\tau)$ has the form

$$x_f^{(2)}(\tau) \sim \sum_{n=1}^{\infty} x_n^{(2)} \tau^{n/2}. \tag{5.1}$$

In the following, we consider the two-shout call option and provide the results for the two-shout put option in Appendix B.

5.1. Two-shout call option At the leading order, we find the following pair of equations:

$$\begin{aligned}
&\frac{x_1^{(2)}}{2}\,\mathrm{erfc}\Big(\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big) + \frac{\sigma}{\sqrt{2\pi}}\Big(1 - \exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big)\Big) + \int_0^1 \frac{\sigma}{4\sqrt{2\pi\eta}}\,\mathrm{erfc}\Big(\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta \\
&\quad = \frac{\sigma}{4\sqrt{2\pi}} \int_0^1 \frac{1}{\sqrt{\eta}}\,\mathrm{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta \\
&\qquad + \frac{\sigma}{16\sqrt{2\pi}} \int_0^1\int_0^1 \frac{1}{\sqrt{\eta\gamma}}\,\mathrm{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\,\mathrm{erfc}\Big(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\,d\gamma\,d\eta
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
\frac{1}{2}\,\mathrm{erfc}\Big(\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big) &= \frac{1}{4\pi}\int_0^1 \frac{1}{\sqrt{\eta(1-\eta)}}\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\,d\eta \\
&\quad + \frac{1}{16\pi}\int_0^1\int_0^1 \frac{1}{\sqrt{\eta(1-\eta)\gamma}}\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big) \\
&\quad \times \mathrm{erfc}\Big(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\,d\gamma\,d\eta.
\end{aligned} \tag{5.3}$$

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The two equations (5.2) and (5.3) resemble the corresponding equations (4.2) and (4.3), respectively, for a one-shout call option but with several additional terms and have a numerical root $x_1^{(2)} = 0.478602511\sigma$. This root, as expected, is smaller than the root $x_1^{(1)} = 0.728600109\sigma$ for the one-shout call, because the early exercise payoff for a two-shout call is sweeter than that for a one-shout call and therefore a two-shout call is even more likely to be exercised early.

Continuing with our expansion at the next order, we find the pair of equations

$$\begin{aligned}
&\frac{x_1^{(2)2} + 2x_2^{(2)}}{4} \operatorname{erfc}\Big(\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big) - \frac{r-D}{2} \operatorname{erf}\Big(\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big) + \frac{\sigma x_1^{(2)}}{2\sqrt{2\pi}}\Big\{2 - \exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big)\Big\} \\
&\quad + \int_0^1 \Big[\frac{1}{4\pi}\Big(\frac{r_1\sqrt{1-\eta}}{\sqrt{\eta}} - \frac{x_2^{(1)}\sqrt{\eta}}{\sqrt{1-\eta}}\Big)\exp\Big(-\frac{\eta x_1^{(1)2}}{2\sigma^2(1-\eta)}\Big) \\
&\quad + \frac{1}{4}\Big(\frac{\sigma x_1^{(2)}}{\sqrt{2\pi\eta}} - r + D\Big)\operatorname{erfc}\Big(\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta \\
&= \int_0^1 \Big[\frac{(x_2^{(2)} + r_1)\sqrt{1-\eta}}{4\pi\sqrt{\eta}} \exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big) \\
&\quad + \Big(\frac{x_1^{(2)}\sigma}{4\sqrt{2\pi\eta}} - \frac{r-D}{4}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta \\
&\quad + \int_0^1\int_0^1 \Big[\frac{1}{8}\Big(\frac{\sigma x_1^{(2)}}{2\sqrt{2\pi\gamma\eta}} - r + D\Big)\operatorname{erfc}\Big(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big) \\
&\quad + \frac{1}{8\pi}\Big(\frac{r_1\sqrt{1-\gamma}}{\sqrt{\gamma}} - \frac{x_2^{(1)}\sqrt{\gamma}}{\sqrt{1-\gamma}}\Big)\exp\Big(-\frac{\gamma x_1^{(1)2}}{2\sigma^2(1-\gamma)}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big) \\
&\quad + \frac{(x_2^{(2)} + r_1)\sqrt{1-\eta}}{16\pi\sqrt{\gamma\eta}}\operatorname{erfc}\Big(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\Big]\,d\gamma\,d\eta
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\sigma}{\sqrt{2\pi}} - \frac{x_2^{(2)} + r_1}{\sigma\sqrt{2\pi}}\exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big) + \int_0^1 \frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta \\
&\quad = \int_0^1 \Big[-\Big(\frac{r-D}{2\sigma\sqrt{2\pi(1-\eta)}} + \frac{(x_2^{(2)} + r_1)x_1^{(2)}(1-\sqrt{\eta})}{\sigma^2\pi\sqrt{\eta(1-\eta)}}\Big)\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big) \\
&\qquad + \frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta \\
&\qquad + \int_0^1\int_0^1 \Big[\frac{\sigma}{16\sqrt{2\pi\gamma\eta}}\operatorname{erf}\Big(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)
\end{aligned}$$

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$$+\frac{r_1(1-\gamma)-x_2^{(1)}\gamma}{4\sigma\pi\sqrt{2\pi\gamma(1-\gamma)(1-\eta)}}\exp\left(-\frac{\gamma x_1^{(1)2}}{2\sigma^2(1-\gamma)}\right)\exp\left(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\right)$$
$$-\left(\frac{x_1^{(2)}(x_2^{(2)}+r_1)(1-\sqrt{\eta})}{16\pi\sigma^2\sqrt{\gamma\eta(1-\eta)}}+\frac{r-D}{4\sigma\sqrt{2\pi(1-\eta)}}\right)\mathrm{erfc}\left(\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\right)$$
$$\times\exp\left(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\right)\Bigg]\,d\gamma\,d\eta,$$

which have a numerical root $x_2^{(2)} = 0.3691038999(r-D) + 0.04142004125\sigma^2$.

6. Discussion

In this paper, we have used an integral equation approach due to Jacka [19], Kim [22], Kolodner [24] and McKean [29] to study shout options, which are exotic options with an early exercise feature allowing the holder to lock in the profit to date, while retaining the right to benefit from any further upside.

The principal results of this paper are concerned with the value of one-shout options and the location of the associated free boundary. In (2.10) and (2.11), we presented expressions for the values of a one-shout call and put option, respectively, where we followed Jacka [19] and Kim [22] and wrote the value as a sum of the corresponding European option, together with a term representing the value of an early shout. In addition to the solution for the one-shout calls, it is possible to use the formulae (2.4) and (2.5) recursively to price n-shout options for which the early exercise payoff is the difference between the current stock price and the strike price (paid at expiry), together with a new at-the-money $(n-1)$-shout option.

We then used the expression for the value of a one-shout option to write a pair of integral equations for the location of the optimal exercise boundary $S_f^{(1)}(\tau) = E^{(1)}\exp(x_f^{(1)}(\tau))$. As discussed by Kolodner [24], these equations are Volterra equations of the second kind. We were able to solve these equations close to expiry, finding that as $\tau \to 0$, the free boundary has the leading order $\tau^{1/2}$ behaviour which Tao [34] found, was the most common form for physical Stefan problems with $x_f^{(1)}(\tau) \sim x_1^{(1)}\tau^{1/2} + x_2^{(1)}\tau + \cdots$ and $x_1^{(1)} = 0.728600109\sigma$ for the call option and $x_1^{(1)} = -0.728600109\sigma$ for the put option.

It is interesting to compare this behaviour to that of the American options. For the shout options, the free boundary always starts from the strike price $E^{(n)}$ at expiry, while for the American options this only happens for the call option with $D \geq r$ and the put option with $D \leq r$. In addition, close to expiry, the free boundary for the shout options always behaves like $\tau^{1/2}$, while for the American options this form of behaviour prevails for the call option with $D < r$ and the put option with $D > r$ [12]. For the American call option with $D > r$ and the put option with $D < r$, the behaviour close to expiry involves logarithms [5]. In all cases, the free boundary close to expiry for shout options seems to be less steep than that for vanilla American options. It would seem likely that this is because an early shout is more probable for a shout option than an early exercise for a vanilla American on the same underlying stock with the same strike, as the rewards for early exercise are generally greater for a shout option than an American option. For an American option, early exercise involves a trade-off between receiving the payoff earlier and receiving benefits from any further upside, while, with a shout option, early exercise results in locking in a profit early when still benefitting from further upsides. As such, it appears paradoxical that although shout options are more complex contracts than vanilla American options, the analysis of one-shout options in the short time to expiry case is actually a little simpler than that of American options, primarily because logarithm terms in the free boundary are never present for the shouts.

Further, the payoff from a two-shout option is more appealing than that from a one-shout option and, because of this, we would expect that for shout call options, $E^{(2)} < S_f^{(2)}(\tau) < S_f^{(1)}(\tau)$, while, for shout put options, $E^{(2)} > S_f^{(2)}(\tau) > S_f^{(1)}(\tau)$, so that the free boundary for a two-shout option $S_f^{(2)}(\tau)$ is sandwiched between that for a one-shout option $S_f^{(1)}(\tau)$ and the strike price. Indeed, our analysis indicates that this is exactly what happens close to expiry with $x_f^{(2)}(\tau) \sim x_1^{(2)}\tau^{1/2} + x_2^{(2)}\tau + \cdots$ and $x_1^{(2)} = 0.478602511\sigma$ for the call option and the negative of that for the put option. This leads us to expect that for an n-shout call, $E_n < S_f^{(n)}(\tau) < S_f^{(n-1)}(\tau) < \cdots < S_f^{(2)}(\tau) < S_f^{(1)}(\tau)$, while, for an n-shout put, $E_n > S_f^{(n)}(\tau) > S_f^{(n-1)}(\tau) > \cdots > S_f^{(2)}(\tau) > S_f^{(1)}(\tau)$. Since this implies that close to expiry $S_f^{(n)}(\tau)$ must be less steep than $S_f^{(n-1)}(\tau)$ and so on, the form of $S_f^{(1)}(\tau)$ and $S_f^{(2)}(\tau)$ dictates that the latter free boundaries must also have the $\tau^{1/2}$ behaviour, with $0 < x_1^{(n)} < x_1^{(n-1)} < \cdots < x_1^{(2)} < x_1^{(1)}$ for the call option and $0 > x_1^{(n)} > x_1^{(n-1)} > \cdots > x_1^{(2)} > x_1^{(1)}$ for the put option.

In conclusion, we would like to re-iterate the motivation for reformulating a free boundary problem like the present one as an integral or integro-differential equation, regardless of the method by which this reformulation is achieved. There seem to be two principal advantages of this approach over applying Tao's method [34] and using an asymptotic expansion directly to the underlying PDE as was done by Dewynne et al. [12]. The first advantage is that once an equation for the location of the free boundary is derived, the problem of finding the location of the free boundary is decoupled from the problem of evaluating the option. The second advantage appears to be that the integral equation approach can handle multiple free boundaries and our own experience bears this out. Alobaidi and Mallier [1, 2] used such an approach to consider the American straddle, where exercise can take place on either of two free boundaries, with the region where it is optimal to hold the option lying between those two boundaries. In the present study, we touched on the issue of two-shout options and, indeed, n-shout options, where several "phase changes" occur sequentially. In both of these cases, the integral equation approach is able to handle the multiple free boundaries, but it is far less clear how Tao's method might be applied to such a situation.

Appendix A. Solution of integral equations for the put option close to expiry

Not surprisingly, the analysis for the put is very similar to that for the call. Once again we use the series (4.1) for $x_f^{(1)}(\tau)$, this time in (3.3) and (3.4). From (3.3), at $O(\tau^{1/2})$,

$$-\frac{x_1^{(1)}}{2}\operatorname{erfc}\Big(-\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)+\frac{\sigma}{\sqrt{2\pi}}\Big(1-\exp\Big(-\frac{x_1^{(1)2}}{2\sigma^2}\Big)\Big)$$
$$=\frac{\sigma}{4\sqrt{2\pi}}\int_0^1\frac{1}{\sqrt{\eta}}\operatorname{erfc}\Big(\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta, \tag{A.1}$$

while, from (3.4), at $O(\tau^0)$,

$$-\frac{1}{2}\operatorname{erfc}\Big(-\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)=-\frac{1}{4\pi}\int_0^1\frac{1}{\sqrt{\eta(1-\eta)}}\exp\Big(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\,d\eta. \tag{A.2}$$

These two equations (A.1) and (A.2), which differ from their counterparts for the call (4.2) and (4.3), respectively, in the sign of $x_1^{(1)}$, have a numerical root $x_1^{(1)}=-0.728600109\sigma$, which is negative of the value for the call.

Continuing with our expansion, at the next order, from (3.3), at $O(\tau)$,

$$-\frac{x_1^{(1)2}+2x_2^{(1)}}{4}\operatorname{erfc}\Big(-\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)-\frac{r-D}{2}\operatorname{erf}\Big(\frac{x_1^{(1)}}{\sqrt{2}\sigma}\Big)+\frac{\sigma x_1^{(1)}}{2\sqrt{2\pi}}\Big(2-\exp\Big(-\frac{x_1^{(1)2}}{2\sigma^2}\Big)\Big)$$
$$=\int_0^1\Big[-\frac{(x_2^{(1)}+r_1)\sqrt{1-\eta}}{4\pi\sqrt{\eta}}\exp\Big(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)$$
$$+\Big(\frac{x_1^{(1)}\sigma}{4\sqrt{2\pi\eta}}+\frac{r-D}{4}\Big)\operatorname{erfc}\Big(\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta$$
$$=0,$$

while, from (3.4), at $O(\tau^{1/2})$,

$$\frac{\sigma}{\sqrt{2\pi}}-\frac{x_2^{(1)}+r_1}{\sigma\sqrt{2\pi}}\exp\Big(-\frac{x_1^{(1)2}}{2\sigma^2}\Big)$$
$$=\int_0^1\Big[-\Big(\frac{r-D}{2\sigma\sqrt{2\pi(1-\eta)}}-\frac{(x_2^{(1)}+r_1)x_1^{(1)}(1-\sqrt{\eta})}{4\sigma^2\pi\sqrt{\eta(1-\eta)}}\Big)\exp\Big(-\frac{x_1^{(1)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)$$
$$+\frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(\frac{x_1^{(1)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta$$
$$=0,$$

which have a numerical root of $x_2^{(1)}=0.5516261057(r-D)+0.04898978883\sigma^2$, identical to that for the call. Similarly, the next value is found to be $x_3^{(1)}=-0.413244516[(r-D)^2/\sigma]-0.218773888\sigma(r-D)-0.00303954446\sigma^3$.

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Appendix B. Two-shout put options

Using $x_f^{(2)}(\tau)$ as in (5.1) and the payoff (2.13) at leading order, we find the pair of equations

$$
\begin{aligned}
&-\frac{x_1^{(2)}}{2}\operatorname{erfc}\Big(-\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big)+\frac{\sigma}{\sqrt{2\pi}}\Big(1-\exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big)\Big)\\
&\qquad+\int_0^1\frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta\\
&\quad=\frac{\sigma}{4\sqrt{2\pi}}\int_0^1\frac{1}{\sqrt{\eta}}\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta\\
&\qquad+\frac{\sigma}{16\sqrt{2\pi}}\int_0^1\int_0^1\frac{1}{\sqrt{\eta\gamma}}\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\,d\gamma\,d\eta
\end{aligned}
\tag{B.1}
$$

and

$$
\begin{aligned}
-\frac{1}{2}\operatorname{erfc}\Big(-\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big)&=-\frac{1}{4\pi}\int_0^1\frac{1}{\sqrt{\eta(1-\eta)}}\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\,d\eta\\
&\quad-\frac{1}{16\pi}\int_0^1\int_0^1\frac{1}{\sqrt{\eta(1-\eta)\gamma}}\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\\
&\quad\times\operatorname{erfc}\Big(\frac{-x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\,d\gamma\,d\eta.
\end{aligned}
\tag{B.2}
$$

If we compare these two equations to their counterparts (5.2) and (5.3), respectively, for the call, we see that the signs of $x_1^{(1)}$ and $x_1^{(2)}$ are reversed. The two equations (B.1) and (B.2) for the put have a numerical root $x_1^{(2)} = -0.478602511\sigma$, which again is the negative of the value for the call.

Continuing with our expansion, at the next order, we find the pair of equations

$$
\begin{aligned}
&-\frac{x_1^{(2)2}+2x_2^{(2)}}{4}\operatorname{erfc}\Big(-\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big)-\frac{r-D}{2}\operatorname{erf}\Big(\frac{x_1^{(2)}}{\sqrt{2}\sigma}\Big)+\frac{\sigma x_1^{(2)}}{2\sqrt{2\pi}}\Big(2-\exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big)\Big)\\
&\qquad+\int_0^1\Big[\frac{1}{4\pi}\Big(\frac{x_2^{(1)}\sqrt{\eta}}{\sqrt{1-\eta}}-\frac{r_1\sqrt{1-\eta}}{\sqrt{\eta}}\Big)\exp\Big(-\frac{\eta x_1^{(1)2}}{2\sigma^2(1-\eta)}\Big)\\
&\qquad+\frac{1}{4}\Big(\frac{\sigma x_1^{(2)}}{\sqrt{2\pi\eta}}+r-D\Big)\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta\\
&\quad=\int_0^1\Big[-\frac{(x_2^{(2)}+r_1)\sqrt{1-\eta}}{4\pi\sqrt{\eta}}\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)
\end{aligned}
$$

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$$
\begin{aligned}
&+ \Big(\frac{x_1^{(2)}\sigma}{4\sqrt{2\pi\eta}} + \frac{r-D}{4}\Big)\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta \\
&+ \int_0^1\int_0^1 \Big[\frac{1}{8}\Big(\frac{\sigma x_1^{(2)}}{2\sqrt{2\pi\gamma\eta}} + r - D\Big)\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big) \\
&+ \frac{1}{8\pi}\Big(\frac{-r_1\sqrt{1-\gamma}}{\sqrt{\gamma}} + \frac{x_2^{(1)}\sqrt{\gamma}}{\sqrt{1-\gamma}}\Big)\exp\Big(-\frac{\gamma x_1^{(1)2}}{2\sigma^2(1-\gamma)}\Big)\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big) \\
&- \frac{(x_2^{(2)}+r_1)\sqrt{1-\eta}}{16\pi\sqrt{\gamma\eta}}\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\Big]\,d\gamma\,d\eta
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\sigma}{\sqrt{2\pi}} - \frac{x_2^{(2)}+r_1}{\sigma\sqrt{2\pi}}\exp\Big(-\frac{x_1^{(2)2}}{2\sigma^2}\Big) + \int_0^1 \frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\eta}}{\sigma\sqrt{2(1-\eta)}}\Big)\,d\eta \\
&= \int_0^1 \Big[\Big(-\frac{r-D}{2\sigma\sqrt{2\pi(1-\eta)}} + \frac{(x_2^{(2)}+r_1)x_1^{(2)}(1-\sqrt{\eta})}{\sigma^2\pi\sqrt{\eta(1-\eta)}}\Big)\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big) \\
&+ \frac{\sigma}{4\sqrt{2\pi\eta}}\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big)\Big]\,d\eta \\
&+ \int_0^1\int_0^1 \Big[\frac{\sigma}{16\sqrt{2\pi\gamma\eta}}\operatorname{erf}\Big(-\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big)\operatorname{erfc}\Big(\frac{x_1^{(2)}(1-\sqrt{\eta})}{\sigma\sqrt{2(1-\eta)}}\Big) \\
&+ \frac{r_1(1-\gamma) - x_2^{(1)}\gamma}{4\sigma\pi\sqrt{2\pi\gamma(1-\gamma)(1-\eta)}}\exp\Big(-\frac{\gamma x_1^{(1)2}}{2\sigma^2(1-\gamma)}\Big)\exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big) \\
&+ \Big(\frac{x_1^{(2)}(x_2^{(2)}+r_1)(1-\sqrt{\eta})}{16\pi\sigma^2\sqrt{\gamma\eta(1-\eta)}} - \frac{r-D}{4\sigma\sqrt{2\pi(1-\eta}}\Big)\operatorname{erfc}\Big(-\frac{x_1^{(1)}\sqrt{\gamma}}{\sigma\sqrt{2(1-\gamma)}}\Big) \\
&\times \exp\Big(-\frac{x_1^{(2)2}(1-\sqrt{\eta})^2}{2\sigma^2(1-\eta)}\Big)\Big]\,d\gamma\,d\eta,
\end{aligned}
$$

which have a numerical root $x_2^{(2)} = 0.3691038999(r-D) + 0.04142004125\sigma^2$, identical to that for the call.

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