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Estimates of Hausdorff Dimension for the Non-Wandering Set of an Open Planar Billiard

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Abstract. The billiard flow in the plane has a simple geometric definition; the movement along straight lines of points except where elastic reflections are made with the boundary of the billiard domain. We consider a class of open billiards, where the billiard domain is unbounded, and the boundary is that of a finite number of strictly convex obstacles. We estimate the Hausdorff dimension of the nonwandering set M_0 of the discrete time billiard ball map, which is known to be a Cantor set and the largest invariant set. Under certain conditions on the obstacles, we use a well-known coding of M_0 [Mor91] and estimates using convex fronts related to the derivative of the billiard ball map [Sto03] to estimate the Hausdorff dimension of local unstable sets. Consideration of the local product structure then yields the desired estimates, which provide asymptotic bounds on the Hausdorff dimension's convergence to zero as the obstacles are separated.

1 Introduction

Billiards are the dynamical systems associated with the constant (unit) velocity movement of a point particle in a given domain, with reflections according to the law of geometric optics: 'the angle of incidence is equal to the angle of reflection' when it strikes the boundary. When the billiard domain is the exterior of pairwise disjoint strictly convex compact bodies $K_1, \dots, K_u \subseteq \mathbb{R}^2$ ($u \geq 3$) with C^3 boundaries and which do not eclipse each other (condition **(H)** in Section 2), the maximal invariant set M_0 of the billiard ball map B (from one reflection to the next) has a hyperbolic structure (*e.g.* [Sin70]). It is in fact known that M_0 can be coded by bi-infinite sequences of numbers indexing the bodies at the reflection points, and every point $x \in M_0$ is non-wandering in the sense that every neighbourhood eventually visits itself after enough reflections. The coding shows M_0 is a topological Cantor set, so is of Lebesgue measure zero, and other means must be used to compare such sets. There is a formula for one such yardstick, the Hausdorff dimension of M_0 ($\dim_{\mathrm{H}} M_0$), appearing in [MM83], though this is not well-suited to computation. Lopes and Markarian [LM96] have constructed measures with support contained in M_0 and good ergodic properties; in particular these imply a formula relating metric entropy, Lyapunov exponent and Hausdorff dimension of a certain measure, but (see [LM96, p. 670], also [MM83] for the usual two-dimensional case) Hausdorff dimensions of such measures are not nicely related to $\dim_{\mathrm{H}} M_0$. We use estimates involving convex curves (local unstable manifolds) to estimate $\dim_{\mathrm{H}} M_0$ from above and below; letting $K = \bigcup_i K_i$, our main result is the following theorem.

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Theorem 1.1

$$\frac{-2\ln(u-1)}{\ln\lambda} \leq \dim_H M_0 \leq \frac{-2\ln(u-1)}{\ln\mu} \tag{1.1}$$

where $\lambda^{-1} = 1 + d_{\max}(\frac{2\kappa_{\max}}{\cos\phi_0} + \frac{1}{d_{\min}})$ *and* $\mu^{-1} = 1 + 2d_{\min}\kappa_{\min}$.

Here $d_{\min} = \min_{i\neq j} d(K_i, K_j)$ and $d_{\max} \leq \operatorname{diam} K$, while $\kappa_{\min} > 0$ and $\kappa_{\max}$ are respectively the minimum and maximum curvatures of the boundary ∂K, and $\phi_0 \in [0,\pi)$ is an angle bounding above the angle between the reflected ray and the outwards normal at reflections on certain trajectories. The proof of Theorem 1.1 is completed in Section 4; more precise bounds are considered in 4.1.

In particular, it follows from the above result that $\dim_{\mathrm{H}} M_0$ is non-zero (as follows from [PT93, Chapter 4] and possibly can be derived in our case from [MM83]), and may be made arbitrarily small by moving the convex bodies apart—more precisely via the following result.

Theorem 1.2 *Fix* $A_i \in K_i$ $(i = 1,\dots,u)$ *and replace every* K_i *by* $K_i(r) := K_i + (r-1)A_i$ *and* M_0 *by the corresponding* $M_0(r)$ $(r > 0)$. *Then*

$$\dim_H M_0(r) = \frac{2\ln(u-1)}{\ln r} + \mathcal{O}\Big(\frac{1}{(\ln r)^2}\Big) \quad \text{as } r \to \infty.$$

A precise statement of error bounds for this decrease is also considered in Section 4. Section 2 concerns the symbolic spaces used for the calculation of the Hausdorff dimension, while Section 3 shows how bi-Lipschitz homeomorphisms with the local unstable manifold image X_0 are obtained from estimates of the evolution of the convex curves. Details of these estimates are given in an appendix; they use repeated applications of affine approximations to the billiard ball map (as does *e.g.* [BSC90]).

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We denote by S_t $(t \in \mathbb{R})$ the usual *billiard flow* (see [Bun89]) in the exterior $Q = \overline{\mathbb{R}^2 \setminus K}$ of the obstacle $K = \bigcup_i K_i$; $(p, w) = S_t(q, v)$ gives the position and velocity of a point mass at time t given initial position q and velocity v, and by convention $S_\tau(q, v)$ is defined as $\lim_{t\downarrow\tau} S_t(q, v)$ at points of discontinuity (reflections). Also denote by S^1 the unit circle in $\mathbb{R}^2$, by n the 'outwards' unit normal field of ∂K, by $\hat{Q} = \{(q, v) \in Q\times S^1 \mid q \in \operatorname{int} Q \text{ or } \langle n_K(q),\, v\rangle \geq 0\}$ the phase space of S_t, by π the canonical projection of $\hat{Q}$ onto Q, and by $M = \{(q, v) \in \partial K \times S^1 \mid \langle n_K(q),\, v\rangle \geq 0\}$ the boundary of $\hat{Q}$, a 2-dimensional compact manifold with u connected components. Let $t_j(x) \in [-\infty, \infty]$ $(j \in \mathbb{Z})$ denote the time of the j-th reflection of $x \in \hat{Q}$, let $\hat{Q}' = t_1^{-1}(0, \infty)$, let $M' = M \cap \hat{Q}'$ (again compact), and denote the *billiard ball map* by $B\colon M' \to M$, $x \to S_{t_1(x)}(x)$. Then B is invertible and smooth (which shall mean C^3) except where the image intersects the tangent bundle of K, and its restriction to $M_0 = \{x \in M \mid |t_j(x)| < \infty \text{ for all } j \in \mathbb{Z}\}$ is a bijection.

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Finally let $d_{\max} \leq \operatorname{diam} K$ be the maximum value of $t_1|_{M'}$ (note $d_{\min} = \min_{i \neq j} d(K_i, K_j)$ is the minimum). The unstable set of $x \in M_0$ in M under B is

$$Z = \left\{ y \in M \;\middle|\; |t_j(y)| < \infty \text{ for all } j \leq 0 \text{ and } d\big(B^j(y), B^j(x)\big) \to 0 \text{ as } j \to -\infty \right\},$$

however we write the *unstable set of* x (in M_0) to mean $W^{(\mathrm{u})}(x) = Z \cap M_0$, while keeping the ϵ-local unstable manifold (see below; as shown in Section 2, M_0 is a Cantor set, so cannot contain any manifolds) as $W^{(\mathrm{u})}_\epsilon(x) = \left\{ y \in Z \;\middle|\; d\big(B^j(y), B^j(x)\big) < \epsilon \text{ for all } j \leq 0 \right\}$.

Let $X\colon q(s)$, $s \in (0,1)$ be a smooth strictly convex curve in $\operatorname{int} Q$, with "outer" unit normal field n_X parametrized by $n(s) = n_X\big(q(s)\big)$, and let $\hat{X}$ be the corresponding curve in $\hat{Q}$ parametrized by $\big(q(s), n(s)\big)$. We assume no point of $\hat{X}$ has a forward trajectory anywhere tangent to K. Also let $k_0(s)$ be the curvature of X at $q(s)$, let $\hat{X}_t = S_t(\hat{X})$, let $X_t = \pi \hat{X}_t$ ($t \in \mathbb{R}$), let $t_j(s) = t_j\big(q(s), n(s)\big)$, and, where they are defined, let $q_j(s) = \pi B^j\big(q(s), n(s)\big)$ be the j-th reflection point of $\big(q(s), n(s)\big)$, $\phi_j(s)$ be the (acute) angle which the reflected ray makes with the outer normal to K at the j-th reflection, and let $d_j(s) = t_j(s) - t_{j-1}(s)$ be the distance between the $(j-1)$-th and j-th reflection points ($j \in \mathbb{Z}$). Finally let $\hat{X}_0 = \left\{ \big(q(s), n(s)\big) \;\middle|\; t_j(s) < \infty \text{ for all } j \geq 0 \right\}$ and $X_0 = \pi \hat{X}_0$, analagous to M_0. It is shown in [Sin70] that if $0 < t < t_1(s)$ for all s then the parametrization $\hat{X}_t\colon \big(p(s), w(s)\big)$ induced by $\hat{X}\colon \big(q(s), n(s)\big)$ is smooth, and X_t is strictly convex, with outer unit normal field parametrized by $w(s)$ and curvature parametrized by $\varkappa_t(s) = \frac{k_0(s)}{1 + t k_0(s)}$. If $j \in \mathbb{Z}$ and $t_j(s)$ is finite, then we can define $k_j(s) = \lim_{t \downarrow t_j(s)} \varkappa_t(s)$ so that for $t_j(s) < T < t_{j+1}(s)$ (between reflections), the following identity applies, and also (due to [Sin70], see also [Sin79]) a recurrence relation over multiple reflections.

$$\varkappa_T(s) = \frac{k_j(s)}{1 + \big(T - t_j(s)\big) k_j(s)} \tag{1.2}$$

$$k_{j+1}(s) = \frac{k_j(s)}{1 + d_j(s) k_j(s)} + 2 \frac{\kappa_K\big(q_{j+1}(s)\big)}{\cos \phi_{j+1}(s)}. \tag{1.3}$$

This curvature is achieved on the piecewise smooth curve $X_{t_j(s)}$; one of two components (X, K convex) is the strictly convex curve

$$Y = \{ p(w) \in X_{t_j(s)} \mid w \in (0,1), t_j(w) \leq t_j(s) \}$$

which has a (one-sided limiting) curvature of $k_j(s)$ at the endpoint $p(s) = q_j(s)$.

Since the situation locally near $x \in M_0$ is the same as for Sinai billiards, the existence of local stable and unstable manifolds follows from [Sin70] (see also [LM96]). Namely, a local unstable set $W^{(\mathrm{u})}_\epsilon(x)$ of a point $x \in M_0$ is a 'strictly convex' one-dimensional submanifold of M' in the sense that for any sufficiently small $\eta > 0$, $\hat{X} = S_\eta\big(W^{(\mathrm{u})}_\epsilon(x)\big)$ consists of a strictly convex curve (X) and its associated outer unit normal field, and if $\pi x \in K_i$ ($1 \leq i \leq u$), then $\pi W^{(\mathrm{u})}_\epsilon(x) \subseteq \partial K_i$. Local stable manifolds relate to strictly concave curves in an analogous way.

2 Coding of M_0 and X_0

In this section we consider a symbolic coding of M_0 and calculate some quantities for the symbolic spaces which will later be transferred to estimate $\dim_{\mathrm{H}}(M_0)$. Fixing $K = \bigcup_{i=1}^{u} K_i$ with $u \geq 2$, for each $x \in M_0$ we have a bi-infinite sequence of indices $\alpha = (\alpha_i)_{i=-\infty}^{\infty} \subseteq \{1, \ldots, u\}$; each α_i corresponding to the boundary ∂K_{α_i} on which the i-th reflection point $\pi B^i(x)$ lies. By convexity arguments $\alpha_i \neq \alpha_{i+1}$ for all i, so define the symbol space Σ as follows.

$$\Sigma = \Big\{ (\alpha_i)_{i=-\infty}^{\infty} \in \prod_{j=-\infty}^{\infty} \{1, \ldots, u\} \ \Big|\ \alpha_i \neq \alpha_{i+1} \text{ for all } i \in \mathbb{Z} \Big\}$$

Also let $f\colon M_0 \to \Sigma, x \mapsto \alpha$ denote the representation map. The (two-sided) subshift $\sigma\colon \Sigma \to \Sigma, (\alpha_i)_{i\in\mathbb{Z}} \mapsto (\alpha_{i+1})_{i\in\mathbb{Z}}$ (of finite type) is continuous under each metric d_θ ($\theta \in (0,1)$) on Σ,

$$d_\theta\big((\alpha_i)_{i\in\mathbb{Z}}, (\beta_i)_{i\in\mathbb{Z}}\big) = \begin{cases} 0, & \text{if } \alpha_i = \beta_i \text{ for all } i \in \mathbb{Z} \\ \theta^n, & \text{if } n = \max\{ j \geq 0 \mid \alpha_i = \beta_i \text{ for all } |i| < j\}, \end{cases}$$

as each d_θ induces the topology of Σ as a subspace of $\prod_{j=-\infty}^{\infty}\{1, \ldots, u\}$ with the product topology.

For the remainder of the current paper, we assume the following condition of Ikawa [Ika88] on K, with which q_1, q_2, q_3 are non-collinear whenever $i_1 \neq i_2 \neq i_3$ and $q_j \in K_{i_j}$ for $j = 1, 2, 3$.

(H) For $1 \leq i, j, k \leq u$, $i \neq k \neq j$ implies the convex hull of $K_i \cup K_j$ is disjoint from K_k.

As is well known, (M_0, B) is then conjugate to the subshift (Σ, σ); see [Mor91] and also related results in [BSC90]. In particular this may be derived from the following two lemmas, the first of which is clearly not in general true (as regards trajectories) without **(H)**.

Lemma 2.1 *If K satisfies condition* **(H)**, *then for any finite sequence of indices $1 \leq i_1, \ldots, i_n \leq u$ ($n \geq 3$) such that $i_j \neq i_{j+1}$, $j = 1, \ldots, n-1$, the corresponding function*

$$F\colon K_{i_1} \times \cdots \times K_{i_n} \to \mathbb{R}, \quad (q_1, \ldots, q_n) \mapsto \sum_{j=1}^{n-1} \|q_j - q_{j+1}\|$$

achieves its minimum at some $(p_1, \ldots, p_n)$ such that $p_j \in \partial K_{i_j}$ for all j. Specifically, $p_1, \ldots, p_n$ are the successive reflection points of a periodic billiard trajectory in Q which is normal to ∂K_{i_1} at p_1 and normal to ∂K_{i_n} at p_n.

Lemma 2.2 *If K satisfies* **(H)**, *there exist $C > 0$ and $\delta \in (0,1)$ such that any $x, y \in B^{-n}(M)$ ($n \geq 1$) with reflection points $q_j = \pi B^j(x)$, $p_j = \pi B^j(y)$ lying in the same components ∂K_{β_j} ($\beta_j \in \{1, \ldots, u\}$, $j = 0, \ldots, n$) must have*

$$|q_j - p_j| \leq C(\delta^j + \delta^{n-j}) \quad \text{for each } j = 0, \ldots, n.$$

The proof of Lemma 2.1 is essentially described in [Sjö90, Appendix B], [Mor91, pp. 824–825], while we refer to [PS92, Chapter 10] for the above formulation of Lemma 2.2; see also [Ika88, Section 3] in relation to both. Here we summarize the results on the coding of M_0.

Theorem 2.3 *If $u \geq 2$ and $\theta \in (0,1)$, f is a homeomorphism of M_0 (topology induced by M) onto (Σ, d_θ), and the shift σ is topologically conjugate to B: $B = f^{-1} \circ \sigma \circ f$.*

Proof Let $(\beta_i)_{i=-\infty}^{\infty} \in \Sigma$, $N \geq 3$, and denote by $M_\beta^{(N)}$ the set of points of M whose trajectories make at least N reverse and N forward reflections, consecutively from the components $K_{\beta_{-N}}, K_{\beta_{-N+1}}, \dots, K_{\beta_N}$. $M_\beta^{(N)}$ is closed in M', and nonempty by Lemma 2.1, so $f^{-1}\{\beta\} = \bigcap_{N=3}^{\infty} M_\beta^{(N)}$ is nonempty by bicompactness.

Now if $x, y \in f^{-1}\{\beta\}$ then

$$|\pi x - \pi y| = |\pi B^n(B^{-n}(x)) - \pi B^n(B^{-n}(y))| \leq C(\delta^n + \delta^{m-n})$$

for any $m \geq n$ by Lemma 2.2. Letting $m \to \infty$ and then $n \to \infty$ gives $\pi x = \pi y$, and similarly $\pi B(x) = \pi B(y)$, so f is a bijection. f^{-1} is continuous since if $\alpha \in \Sigma$, $x = f^{-1}(\alpha)$ and $U \subseteq M$ is an open neighborhood of x, then

$$\pi^{-1} B_{|\cdot|}(\pi x; \epsilon) \cup (\pi B)^{-1} B_{|\cdot|}(\pi(B(x)); \epsilon) \subseteq U$$

for sufficiently small $\epsilon > 0$, $\delta^{N-1} < \frac{\epsilon}{2C}$ for sufficiently large N, and each $y \in M_\alpha^{(N)}$ has

$$|\pi x - \pi y| = |\pi B^N(B^{-N}(x)) - \pi B^N(B^{-N}(y))| \leq C(\delta^N + \delta^{2N-N}) = 2C\delta^N < \epsilon,$$

and similarly $|\pi B(x) \pi B(y)| < \epsilon$. Since Σ is compact, f is a homeomorphism, and $B = f^{-1} \circ \sigma \circ f$ follows. ∎

Now assuming $u \geq 3$ to avoid the trivial case of a single 2-periodic orbit, it follows that M_0 is a (compact) topological Cantor set, B is topologically transitive on M_0, and its periodic points are dense in M_0. Hyperbolicity of M_0 follows from [Sin70] so M_0 is basic, and hence *is* the nonwandering set of B over M_0.

X_0 can of course now be coded by forward sequences via a coding map $\Upsilon \colon X_0 \to \Sigma_+$ defined in the same manner as f (that Υ is injective will be shown in Section 3), where Σ_+ is the compact ultrametric space under natural metrics $d_\theta \colon \Sigma_+ \times \Sigma_+ \to \mathbb{R}$ ($\theta \in (0,1)$) defined as follows.

$$\Sigma_+ = \Big\{ (\alpha_i)_{i=1}^{\infty} \in \prod_{j=1}^{\infty} \{1, \dots, u\} \;\Big|\; \alpha_i \neq \alpha_{i+1} \text{ for all } i \geq 1 \Big\}$$

$$d_\theta\big((\alpha_i)_{i=1}^{\infty}, (\beta_i)_{i=1}^{\infty}\big) = \begin{cases} 0, & \text{if } \alpha_i = \beta_i \text{ for all } i \geq 1 \\ \theta^n, & \text{if } n = \max\{j \geq 0 \mid \alpha_i = \beta_i \text{ for all } 1 \leq i \leq j\}. \end{cases}$$

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With the equivalence relations $\sim_m$ ($m \geq 1$) given by $(\alpha_i)_1^\infty \sim_m (\beta_i)_1^\infty \iff \alpha_i = \beta_i$ for all $1 \leq i \leq m$, and their equivalence classes the cylinders $[\alpha]_m$ (also define $\sim_0$ such that $[\alpha]_0 = \Sigma_+$ for all $\alpha \in \Sigma_+$), we can make the following calculations in (Σ_+, d_θ), useful in later sections where convex front estimates and Σ_+ are considered more than Σ. Calculations of exactly the same quantities are most likely available elsewhere, and the proofs are included only for completeness. Certainly the first lemma uses essentially the same argument as [Edg90, (6.2.1)] (apparently originally due to A. N. Kolmogorov).

Lemma 2.4 *For any $\alpha \in \Sigma_+$ and $n \in \mathbb{N}$, $\dim_H([\alpha]_n) = \frac{-\ln(u-1)}{\ln\theta}$.*

Proof Let $z = (u-1)^{-1}$, let $\mathcal{A}$ be the cover $\{[\alpha]_n \mid \alpha \in \Sigma_+, n \in \mathbb{N}\}$ of Σ_+, define $C\colon \mathcal{A} \to \mathbb{R}$, $[\alpha]_n \mapsto z^n$, and let $\mathcal{M}$ be the outer measure on Σ_+ constructed from $\mathcal{A}$ and C by

$$\mathcal{M}\colon \mathbb{P}(\Sigma_+) \to \mathbb{R}, B \mapsto \inf_{\mathcal{U}} \sum_{A\in\mathcal{U}} C(A),$$

where $\mathbb{P}(\Sigma_+) = \{B \mid B \subseteq \Sigma_+\}$ denotes the power set of Σ_+, and the infimum is taken over all countable $\mathcal{U} \subseteq \mathcal{A}$ which cover the set B. Now $\operatorname{diam}([\alpha]_n) = \sup_{\beta,\gamma\in[\alpha]_n} d_\theta(\beta,\gamma) = \theta^n$ (since $u \geq 3$). We will show that in fact $\mathcal{M}([\alpha]_n) = C([\alpha]_n) = (\operatorname{diam}[\alpha]_n)^{\ln z/\ln\theta}$ for any $[\alpha]_n \in \mathcal{A}$.

For any countable cover $\mathcal{U} = \{[x_i]_{n_i} \mid i \in I\} \subseteq \mathcal{A}$ of $[\alpha]_n$, since every element is an open ball and $[\alpha]_n$ is compact, we can assume $\mathcal{U}$ is a finite cover, without increasing $\sum_{A\in\mathcal{U}} C(A)$. We also assume every $A \in \mathcal{U}$ has $A \subseteq [\alpha]_n$ and the sets of $\mathcal{U}$ are pairwise disjoint. Letting $m = \max_{i\in I} n_i$, each $[x_i]_{n_i}$ is the disjoint union of $(u-1)^{m-n_i}$ classes of $\sim_m$, say represented by $y_{i,1}, \dots, y_{i,(u-1)^{m-n_i}}$. Then

$$C([x_i]_{n_i}) = (u-1)^{-n_i} = (u-1)^{m-n_i}(u-1)^{-m} = \sum_j C([y_{i,j}]_m).$$

Since the $[x_i]_{n_i}$ are disjoint, we have $[y_{i_1,j_1}]_m \cap [y_{i_2,j_2}]_m = \varnothing$ whenever $i_1 \neq i_2$. Since the $[x_i]_{n_i}$ cover $[\alpha]_n$, the $[y_{i,j}]_m$ are all the distinct classes of $\sim_m$ inside $[\alpha]_n$. Hence

$$C([\alpha]_n) = (u-1)^{m-n}(u-1)^{-m} = \sum_{i,j} C([y_{i,j}]_m) = \sum_i C([x_i]_{n_i})$$

Since $\mathcal{U}$ was arbitrary, this shows C is countably subadditive. It follows that $\mathcal{M}|_{\mathcal{A}} = C$, since for any $\mathcal{U}$ covering $[\alpha]_n$, the restriction to $[\alpha]_n$ is also a cover, and

$$C([\alpha]_n) \leq \sum_{A\in\mathcal{U}} C(A \cap [\alpha]_n) \leq \sum_{A\in\mathcal{U}} C(A).$$

Here we used the convention $C(\varnothing) = 0$ (we could have included it in $\mathcal{A}$), and the last inequality follows since for $A = [x_i]_{n_i}$, either $A \cap [\alpha]_n = \varnothing$ (if $x_i \notin [\alpha]_n$), or $A \subseteq [\alpha]_n$ if $x_i \in [\alpha]_n$ and $n_i \geq n$, or $[\alpha]_n \subseteq A$ if $x_i \in [\alpha]_n$ and $n_i < n$.

We now use this expression for $\mathcal{M}|_{\mathcal{A}}$ to show that when $s = \ln z/\ln\theta$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ (see *e.g.* [Edg90]) and $\mathcal{M}$ coincide on the whole

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of $\mathbb{P}(\Sigma_+)$. This is enough to show that $\dim_H([\alpha]_n) = s$, since $\mathcal{M}([\alpha]_n)$ is clearly finite and non-zero for any $\alpha \in \Sigma_+$, $n \in \mathbb{N}$.

For any $A \subseteq \Sigma_+$ with cardinality $|A| > 1$, there is a maximal $n \in \mathbb{N}$ such that $A \subseteq [\alpha]_n$ for some $\alpha \in \Sigma_+$. Then for any $\beta = (\beta_i)_{i=1}^\infty \in A$ there is a $\gamma = (\gamma_i)_{i=1}^\infty \in A$ such that $\beta_{n+1} \neq \gamma_{n+1}$, and $d_\theta(\beta,\gamma) = \theta^n = \operatorname{diam}([\alpha]_n) \leq \operatorname{diam} A \leq \operatorname{diam}([\alpha]_n)$. Consequently (and clearly also true for $|A| = 0, 1$) we have $\mathcal{M}(A) \leq C([\alpha]_n) = (\operatorname{diam}[\alpha]_n)^s = (\operatorname{diam} A)^s$.

Now, recall that for any $B \subseteq \Sigma_+$, $\mathcal{H}^s(B) = \lim_{\epsilon\downarrow 0} \mathcal{H}^s_\epsilon(B)$, where each $\mathcal{H}^s_\epsilon$ was constructed to be the largest outer measure such that $\mathcal{H}^s_\epsilon(E) \leq (\operatorname{diam} E)^s$ for all $E \in \mathcal{U} = \{A \subseteq \Sigma_+ \mid \operatorname{diam} A < \epsilon\}$. By letting ϵ tend to 0, we have $\mathcal{M} \leq \mathcal{H}^s_\epsilon$, and hence $\mathcal{M} \leq \mathcal{H}^s$.

For the converse, consider arbitrary $\alpha \in \Sigma_+$, $n \in \mathbb{N}$ and $\epsilon > 0$. For any $m > n$,

$$[\alpha]_n = \bigcup_{\beta\in[\alpha]_n} [\beta]_m = \bigcup_{i=1}^{(u-1)^{m-n}} [\beta_i]_m$$

where $\beta_1, \ldots, \beta_{(u-1)^{m-n}}$ are representatives of the equivalence classes of $\sim_m$ contained in $[\alpha]_n$. If m is sufficiently large that $\theta^m < \epsilon$, then

$$\begin{aligned}\mathcal{H}^s_\epsilon([\alpha]_n) &\leq \sum_{i=1}^{(u-1)^{m-n}} (\operatorname{diam}[\beta_i]_m)^s = \sum_{i=1}^{(u-1)^{m-n}} (\theta^m)^{\frac{\ln z}{\ln\theta}} = (u-1)^{m-n} z^m \\ &= (u-1)^{-n} = C([\alpha]_n) \quad \text{for any } \alpha \in \Sigma_+ \text{ and } n \in \mathbb{N}.\end{aligned}$$

Hence $\mathcal{H}^s_\epsilon \leq \mathcal{M}$ by maximality of $\mathcal{M}$ as an outer measure with $\mathcal{M}|_{\mathcal{A}} \leq C$. The result that $\dim_H([\alpha]_n) = s = -\ln(u-1)/\ln\theta$ follows. ∎

Another dimension (sometimes known as (upper) packing dimension) is of interest for the cylinders of Σ_+; denoted $\overline{\dim}_p$, it is constructed similarly to Hausdorff dimension, in the following way.

Definition 2.5 For metric space X and $s, \epsilon > 0$, define $P^s_\epsilon \colon \mathbb{P}(X) \to [0,\infty]$ for any $B \subseteq X$ by $P^s_\epsilon(B) = \sup \sum_{i\in I} (\operatorname{diam} A_n)^s$, the supremum being over all countable families of pairwise disjoint closed balls with diameter less than ϵ and centres in B (these will be called ϵ*-packings* of B, and always include the zero-radius packings). Also define $P^s \colon \mathbb{P}(X) \to [0,\infty], B \mapsto \inf_{\epsilon>0} P^s_\epsilon(B)$, and let the *s-dimensional packing outer measure* $\mathcal{P}^s$ be the outer measure constructed by $\mathcal{P}^s(B) = \inf_{\mathcal{U}} \sum_{A\in\mathcal{U}} P^s(A)$, where the infimum is over all covers $\mathcal{U}$ of A. Given $B \subseteq X$, $s \mapsto \mathcal{P}^s(B)$ is non-increasing and has at most one finite nonzero value. The *packing dimension* is defined by the following equation.

$$\overline{\dim}_p B = \begin{cases} 0, & \text{if } \mathcal{P}^s(B) = 0 \text{ for all } s > 0 \\ \infty, & \text{if } \mathcal{P}^s(B) = \infty \text{ for all } s > 0 \\ \inf\{s > 0 \mid \mathcal{P}^s(B) = 0\}, & \text{otherwise.} \end{cases}$$

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This dimension is monotonic, non-increasing under Lipschitz maps, and has the following property for cylinders of Σ_+.

Lemma 2.6 *For any $\alpha \in \Sigma_+$ and $n \in \mathbb{N}$, $\overline{\dim_p}([\alpha]_n) = \dim_H([\alpha]_n)$.*

Proof Let $A \subseteq \Sigma_+$, $\epsilon > 0$, and $z = (u-1)^{-1}$, $s = \ln z / \ln \theta$ as in the proof of Lemma 2.4. Note that the closed ball of radius $r > 0$ centred at $x \in A$ is $\overline{B}(x;r) = [x]_n = \overline{B}(x;\theta^n)$ where n is the minimum non-negative integer such that $\theta^n \le r$, so the equivalence classes $[x]_n$ $(n \ge 0)$ are exactly the closed balls of positive radius in Σ_+. For any $n \in \mathbb{N}$, $0 < \epsilon < \theta^n$, $\alpha \in \Sigma_+$ and for a ϵ-packing of $[\alpha]_n$ by $\overline{B}(x_i;r_i)$, $x_i \in [\alpha]_n$, $i \in I$ (I countable), where in addition $r_i < \epsilon$ for all i, suppose temporarily that $r_i \ne 0$ for all i. Then we have $\overline{B}(x_i;r_i) = [x_i]_{n_i} \subseteq [\alpha]_n$, where each n_i is minimal such that $\theta^{n_i} \le r_i < \epsilon < \theta^n$. As a result, we have the following inequality.

$$\begin{aligned}\sum_{i\in I}\big(\operatorname{diam}\overline{B}(x_i;r_i)\big)^s &= \sum_{i\in I}\theta^{n_i s} = \sum_{i\in I}(u-1)^{-n_i}\\ &= \sum_{i\in I}\mathcal{M}([x_i]_{n_i}) = \mathcal{M}\Big(\bigcup_{i\in I}[x_i]_{n_i}\Big) \le \mathcal{M}([\alpha]_n).\end{aligned}$$

This still holds if $r_i = 0$ for some $i \in I$, as the zero-radius balls do not contribute to the left hand side. Note also that the upper bound here *is* attained for some suitable packing. For, if $m \in \mathbb{N}$ is sufficiently large, $(u-1)^{-m} \le \epsilon$ and $[\alpha]_n$ is the disjoint union of the distinct equivalence classes of $\sim_m$ it contains, say having representatives $x_1, \ldots, x_{(u-1)^{m-n}}$. Now

$$\sum_{i=1}^{(u-1)^{m-n}} (\operatorname{diam}[x_i]_m)^s = \sum_{i=1}^{(u-1)^{m-n}} \theta^{ms} = (u-1)^{m-n} z^m = (u-1)^{-n}.$$

So $P^s_\epsilon([\alpha]_n) = z^n$, and since $\epsilon \in (0,\theta^n)$ was arbitrary, this also shows $P^s([\alpha]_n) = z^n$ and hence $\mathcal{P}^s([\alpha]_n) \le z^n = C([\alpha]_n)$ in the notation of the proof of Lemma 2.4. Since $[\alpha]_n$ was arbitrary, we have $\mathcal{P}^s \le \mathcal{M} = \mathcal{H}^s$.

It remains to show the converse, $\mathcal{H}^s \le \mathcal{P}^s$. Considering $A \subseteq \Sigma_+$, if $|A| = 0, 1$ then the result is obvious, as $\mathcal{M}(A) = 0$. If $|A| > 1$, then as in the previous proof there is some $[\alpha]_n \supseteq A$ such that $\operatorname{diam} A = \operatorname{diam}[\alpha]_n = \theta^n$. Let $\epsilon > 0$ and $m > n$ be sufficiently large that $\theta^m < \epsilon$, and let $\big\{ [x_i]_m \bigm| i \in \{1, \ldots, N\} \big\}$ be a finite ϵ-packing of A by pairwise disjoint closed balls with $x_i \in A$. Suppose there is some x in $A \setminus \bigcup_{i=1}^N [x_i]_m$; letting $x_{N+1} = x$ and $I = \{1, \ldots, N+1\}$ then gives a packing of A with $N+1$ balls. But there are only $(u-1)^{m-n}$ distinct equivalence classes of $\sim_m$ within $[\alpha]_n$, so the above induction must fail for some N. Then $\big\{ [x_i]_m \bigm| i \in \{1, \ldots, N\} \big\}$ is an ϵ-cover for A, so $\mathcal{M}(A) \le \sum_{i=1}^N C([x_i]_m) = \sum_{i=1}^N (\operatorname{diam}[x_i]_m)^s \le P^s_\epsilon(A)$. Since $\epsilon > 0$ was arbitrary in the above, we have $\mathcal{M}(A) \le P^s(A)$ for any $A \subseteq \Sigma_+$. Since $\mathcal{P}^s$ is the greatest outer measure such that $P^s \le \mathcal{P}^s$, this shows $\mathcal{H}^s = \mathcal{M} \le \mathcal{P}^s$. ■

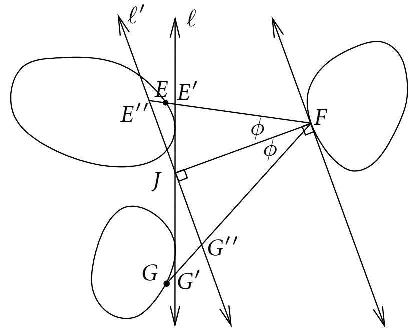

Figure 3.1: Triangles in the estimate of ϕ_0.

3 ϕ_0 and Hausdorff Dimension of X_0

Now we give the convex front estimates which will show the coding of forward trajectories strong enough to relate the previous calculations to $\dim_{\mathrm{H}} X_0$. For a convex front X with n transversal (forward) reflections (say from bodies $K_{\beta_1}, \dots, K_{\beta_n}$), we have $d_j(s) \in [d_{\min}, d_{\max}]$ and $\kappa_K\big(q_j(s)\big) \in [\kappa_{\min}, \kappa_{\max}]$ for any $1 \le j \le n$, and if every $\phi_j(s)$ is similarly bounded above by $\phi_0 < \frac{\pi}{2}$, then we may use (1.3) and $k_j(s) \ge 0$ $(0 \le j \le n-1)$ to obtain bounds $2\kappa_{\min} \le k_j(s) \le \frac{1}{d_{\min}} + \frac{2\kappa_{\max}}{\cos\phi_0}$ for all $1 \le j \le n$. It follows from **(H)** that a bound $\phi_0 \le \arccos(b/d_{\max})$ *does* exist, where b denotes the minimum distance between K_k and the convex hull of K_i and K_j $(i \ne k \ne j)$, if every point of $\hat{X}$ makes a reverse-time reflection.

For example, if $x \in \pi^{-1}(K_i)$, $B(x) \in \pi^{-1}(K_j)$, $B^2(x) \in \pi^{-1}(K_k)$, $E = \pi x$, $F = \pi B(x)$, $G = \pi B^2(x)$, and J is the first intersection of the normal ray from F with $\mathrm{Cvx}(K_i, K_k)$, then let ℓ be the line tangent to $\partial\,\mathrm{Cvx}(K_i, K_k)$ at J. The points E, G must lie in one closed halfspace of ℓ, while (by definition of J) F lies in the other. We can get an estimate for $\phi = \angle EFJ = \angle GFJ$ as follows. If $F \in \ell$, we get $E, G \in \ell$ and $\phi = 0$. If $F \notin \ell$, the line segments $\overline{EF}$ and $\overline{FG}$ must intersect ℓ, say at E' and G' respectively, and also the line ℓ' through J perpendicular to $\overline{FJ}$ must intersect the rays $\overrightarrow{FE}$, $\overrightarrow{FG}$, say at E'', G''. Then either $|E''F| \le |E'F|$ or $|G''F| \le |G'F|$; we without loss of generality assume the former, so that $\cos\angle EFJ = \frac{|FJ|}{|E''F|} \ge \frac{b}{d_{\max}}$. Essentially the same argument (with an inductive step for $F \in \ell$) can be used for $\dim M \ge 3$ without assuming smoothness of ∂K or convexity of $K_1, \dots, K_u$. By similar methods, there also exists an angle $\phi_1 \in (0, \pi)$ bounding above all angles between points of K_i, K_j, K_k where $i \ne j \ne k$.

As a result of the above bounds on $k_j(s)$, if we set

$$\lambda = \left(1 + d_{\max}\left(\frac{2\kappa_{\max}}{\cos\phi_0} + \frac{1}{d_{\min}}\right)\right)^{-1} \qquad \mu = (1 + 2d_{\min}\kappa_{\min})^{-1}$$

then the bounds $\delta_j(s) = \frac{1}{1+d_j(s)k_j(s)} \in [\lambda, \mu]$ hold for all s and $j = 1, \dots, n$.

Let $q(s)$ parametrize X and $p(s)$ parametrize $\Gamma = \{\pi B^n(x) \mid x \in \hat{X}\} \subseteq \partial K_{\beta_n}$ by arc length, then the distance between endpoints x_1, x_2 of X satisfies

$$\|x_1 - x_2\| = \left\| \int_X q'(s)\, \mathrm{d}s \right\| \leq \int_X \|q'(s)\|\, \mathrm{d}s = \int_\Gamma \|p'(s)\| \Big(\prod_{j=0}^{n-1} \delta_j(s) \Big)\, \mathrm{d}s$$

$$\leq \mu^{n-1} \int_X \delta_0(s)\, \mathrm{d}s \leq \mu^{n-1} |\Gamma| \leq \mu^{n-1} \max_i |\partial K_i|$$

where we used $\delta_0 < 1$ and $\|q'(s)\| = \|p'(s)\| \prod_{j=0}^{n-1} \delta_j(s)$ (from (A.1)) obtained by repeatedly using estimates of evolutes of X from one reflection to the next (see Section A for details). Assuming both endpoints $x_1, x_2 \in X$ have $(n+1)$-st forward reflections, but $y_1 = \pi B^{n+1}(x_1) \in \partial K_i$ and $\pi B^{n+1}(x_2) \in \partial K_j$, $i \neq j$, we can obtain a similar bound from below for $\|x_1 - x_2\|$. Specifically, let $[s_1, s_2]$ be an interval in which $s = s_1, s_2$ are the only values for which $\big(q(s), n(s)\big)$ has an $(n+1)$-st reflection, with $q(s_1) = x_1$ and $y_2 := q_{n+1}(s_2) \in \partial K_k$, say. By the assumption of first reflection points in the convex set K_{β_1}, X must be a simple arc, and the Lipschitz property for the inverse of its arc-length parametrization shows $\|x_1 - x_2\| \geq \mathrm{Const}_4 \int_X \|q'(s)\|\, \mathrm{d}s \geq \mathrm{Const}_4 \int_{s_1}^{s_2} \|q'(s)\|\, \mathrm{d}s$. If we suppose $t_{n+1}\big(q(s_1), n(s_1)\big) < \tau := t_{n+1}\big(q(s_2), n(s_2)\big)$ and that $q(s)$ corresponds to the parametrization $p(s)$ by arc length of $\pi S_\tau(\hat{X})$ (in particular its subcurve Y_{n+1} between $z = p(s_1)$ and y_2), then using (A.2) gives

$$\|x_1 - x_2\| \geq \mathrm{Const}\, \mathrm{Const}_4\, \lambda^n \int_{s_1}^{s_2} \|p'(s)\|\, \mathrm{d}s \geq \mathrm{Const}\, \mathrm{Const}_4\, \lambda^n \|y_2 - z\|.$$

To see that $\|y_2 - z\|$ is bounded from below, note that if the angle $\angle z y_1 y_2$ is obtuse then $\|y_2 - z\| \geq \|y_1 - y_2\| \geq d_{\min}$. If $\angle z y_1 y_2$ is acute, then the (unique) best approximation w to y_2 on the ray from $B^n\big(q(s_1), n(s_1)\big)$ must lie on the same side of y_1 as does z. Thus $\angle w y_1 y_2 = \angle z y_1 y_2$ and $\|y_2 - z\| \geq \|y_2 - w\| = \|y_1 - y_2\| \sin \angle w y_1 y_2$. Since $\pi - \angle z y_1 y_2 = \angle q_n(s_1) y_1 y_2$ is bounded above by $\phi_1 \in (0, \pi)$, we have $0 < \pi - \phi_1 < \angle z y_1 y_2 < \frac{\pi}{2}$, so $\|y_2 - z\| > d_{\min} \sin \phi_1$. In either case, we have $\|x_1 - x_2\| \geq \mathrm{Const}_5\, \lambda^n$ for a suitable positive constant.

Bounds such as those just derived may be recast as Lipschitz properties for the coding map $\Upsilon \colon X_0 \to \Upsilon(X_0) \subseteq \Sigma_+$ and a suitable inverse, albeit with respect to different metrics.

Proposition 3.1 *Suppose there are constants $c, C > 0$ such that $c\lambda^n \leq \|x - y\| \leq C\mu^n$ whenever $x, y \in X_0$ with $\Upsilon_j(x) = \Upsilon_j(y)$ for all $1 \leq j \leq n$ (some n), but $\Upsilon_{n+1}(x) \neq \Upsilon_{n+1}(y)$. Then $\Upsilon \colon X_0 \to \Sigma_+$ is injective and a Lipschitz homeomorphism from X_0 to $\big(\Upsilon(X_0), d_\lambda\big)$, and Υ^{-1} a Lipschitz homeomorphism from $\big(\Upsilon(X_0), d_\mu\big)$ onto X_0.*

Proof Certainly Υ is injective, since for any $x \in X_0$ and sufficiently large $n \geq 1$, there is some $z \in X_0$ such that $\Upsilon(z) \sim_n \Upsilon(x)$ but $\Upsilon_{n+1}(z) \neq \Upsilon_{n+1}(x)$, and so if $y \in X_0$ has $\Upsilon(x) = \Upsilon(y)$ then $\|x - y\| \leq \|x - z\| + \|y - z\| \leq 2C\mu^n \to 0$ as $n \to \infty$. Hence Υ^{-1} is well-defined (and injective).

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For distinct $x, y \in X_0$ and $n \geq 0$ maximal such that $\Upsilon_i(x) = \Upsilon_i(y)$ for all $i \leq n$, we have $d_\lambda\big(\Upsilon(x), \Upsilon(y)\big) = \lambda^n \leq \frac{1}{c}\|x - y\|$. Similarly, for distinct $\alpha, \beta \in \Upsilon(X_0)$, $x = \Upsilon^{-1}(\alpha)$, $y = \Upsilon^{-1}(\beta)$, and n as before, we have $\|\Upsilon^{-1}(\alpha) - \Upsilon^{-1}(\beta)\| \leq C\mu^n = Cd_\mu(\alpha, \beta)$. Finally, that the inverses

$$\Upsilon\colon X_0 \to \big(\Upsilon(X_0), d_\mu\big) \quad \text{and} \quad \Upsilon^{-1}\colon \big(\Upsilon(X_0), d_\lambda\big) \to X_0$$

are also continuous follows from continuity of the identity

$$I\colon \big(\Upsilon(X_0), d_\lambda\big) \to \big(\Upsilon(X_0), d_\mu\big)$$

(for $\epsilon > 0$ take $\mu^n < \epsilon$ and then $\lambda^m < \lambda^n \implies \mu^m < \mu^n$). ■

Our use of this correspondence is to estimate the Hausdorff dimension of X_0. Since for some $\alpha \in \Sigma_+$ and sufficiently large $n \geq 1$ the cylinder $[\alpha]_n$ is entirely contained in $\Upsilon(X_0)$, we have $\dim_{\mathrm{H}}([\alpha]_n, d_\lambda) \leq \dim_{\mathrm{H}} X_0 \leq \dim_{\mathrm{H}}(\Sigma_+, d_\mu)$, where the bounds can be calculated by Lemma 2.4.

4 Hausdorff Dimension of M_0

It remains to relate $\dim_{\mathrm{H}} X_0$ to the Hausdorff dimension of M_0. If $\hat{X} = S_\tau\big(W_\theta^{(\mathrm{u})}(x)\big)$ ($\tau > 0$) is the image of a local unstable manifold $W_\theta^{(\mathrm{u})}(x)$ ($\theta > 0$) of some $x \in M_0$ after a small evolution under the flow S_t, then

$$\dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{u})}(x) \cap M_0\big) = \dim_{\mathrm{H}} X_0 \in \Big[\frac{-\ln(u-1)}{\ln\lambda}, \frac{-\ln(u-1)}{\ln\mu}\Big]$$

(bi-Lipschitz image). We can also use these estimates for $\dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{s})}(x) \cap M_0\big)$; these dimensions are independent of x (by [MM83] or [PV88]) but in any case the bounds given are independent of x and θ, and $W^{(\mathrm{u})}(x) = \mathrm{Refl}\, W^{(\mathrm{s})}\big(\mathrm{Refl}(x)\big)$, where $\mathrm{Refl}\colon \hat{Q} \to \hat{Q}$ is the smooth (certainly bi-Lipschitz) involution given by

$$\mathrm{Refl}(q, v) = \begin{cases} (q, -v), & \text{for } q \in \operatorname{int} Q \\ \big(q, 2\langle n_K(q), v\rangle n_K(q) - v\big), & \text{for } q \in \partial K. \end{cases}$$

For Borel $A, B \subseteq \mathbb{R}^n$, the following inequalities (proofs appear in [Mat95]; the first is well-known ([Mar54]) and the second is due to [Tri82]) are known.

$$\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B \leq \dim_{\mathrm{H}}(A \times B) \leq \dim_{\mathrm{H}} A + \overline{\dim_{\mathrm{p}}} B.$$

Since $\overline{\dim_{\mathrm{p}}}(\Sigma_+, d_\theta) = \dim_{\mathrm{H}}(\Sigma_+, d_\theta)$ (see Lemma 2.6), for neighbourhoods $U \subseteq V$ of x and M_0 in M respectively, θ small enough that $W_\theta^{(\mathrm{u})}(x), W_\theta^{(\mathrm{s})}(x) \subseteq U$, and $\Upsilon\colon W_\theta^{(\mathrm{u})}(x) \times W_\theta^{(\mathrm{s})}(x) \to R$ the usual local product map to an open rectangular neighbourhood R of x, it is enough to note that Υ is C^1 to get

$$\begin{aligned} \frac{-2\ln(u-1)}{\ln\lambda} &\leq \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{u})}(x) \cap M_0\big) + \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{s})}(x) \cap M_0\big) \\ &\leq \dim_{\mathrm{H}} R \leq \frac{-2\ln(u-1)}{\ln\mu}. \end{aligned}$$

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That Υ *is* C^1 follows from [Rob75] (see also [dM73], and [PT93, Appendix I] for some comments). Now the separability of M_0 and the 'countable sup' property of Hausdorff dimension of Borel sets show $\dim_{\mathrm{H}} M_0 = \dim_{\mathrm{H}}(R \cap M_0)$ for some x and R, so Theorem 1.1 is proved. Alternatively, that $\dim_{\mathrm{H}} M_0 = \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{u})}(x) \cap M_0\big) + \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{s})}(x) \cap M_0\big)$ independent of $x \in M_0$ (or θ if it is small enough) follows directly from [MM83] or [PV88].

In particular, $\dim_{\mathrm{H}} M_0 > 0$ for any such system (already known from [PT93, Chapter 4]), but may be made arbitrarily small by choosing $d_{\min}$ large with respect to $\operatorname{diam}(K_i)$. Fixing arbitrarily $A_i \in K_i$ $(i = 1, \dots, u)$ and considering the obstacles $K_i(r) = K_i + (r-1)A_i$ $(r > 0)$ defined in Section 1, we show below that the corresponding $b(r) = \inf_{i \neq k \neq j} d\big(L_k, \operatorname{Cvx}(L_i, L_j)\big)$ is positive for large r, and hence $K(r) = \bigcup_i K_i(r)$ still satisfies **(H)**. In fact, we obtain an asymptotic as $r \to \infty$ for the dimension $\dim_{\mathrm{H}} M_0(r)$.

Theorem **(Full Statement of Theorem 1.2)** *Under the above conditions,*

$$\dim_H M_0(r) = \frac{2\ln(u-1)}{\ln r} + \mathcal{O}\Big(\frac{1}{(\ln r)^2}\Big) \quad \text{as } r \to \infty,$$

and for $\theta_1 \in (0, \pi)$ the minimum angle between distinct K_i, K_k, K_j and r sufficiently large,

$$\frac{\ln \frac{d_{\min}^2 \sin\theta_1}{2\kappa_{\max} \operatorname{diam}(K)^3}}{\ln r} \leq \frac{\ln r}{2\ln(u-1)} \dim_H M_0(r) - 1 \leq \frac{-\ln 2\kappa_{\min} d_{\min}}{\ln r}.$$

Proof If $r > 0$ is large enough, the minimum distance between any $L_i = K_i + (r-1)A_i$ and L_j is $d_{\min}(r) \geq (r-1)\|A_i - A_j\| - \max_{x \in K_i, y \in K_j} \|x - y\| \geq (r-1)d_{\min} - d_{\max}$, and similarly $d_{\max}(r) \leq (r-1)\max_{i,j}\|A_i - A_j\| + d_{\max} \leq (r-1)\operatorname{diam}(K) + d_{\max} \leq r \operatorname{diam}(K)$. The distance $b(r) = \inf_{i \neq k \neq j} d\big(L_k, \operatorname{Cvx}(L_i, L_j)\big)$ also changes approximately linearly in r for large r; for $i \neq j$, fixed $F_1 \in K_k$, $F = F_1 + (r-1)A_k$, ℓ a common tangent of L_i and L_j for which both components lie in the same closed halfspace (for definiteness, choose ℓ to be closest to F), and E, G the points of intersection of ℓ with L_i and L_j respectively, the height of triangle E, F, G is

$$\frac{|EF|\,|FG|}{|EG|} \sin\angle EFG \sim r \frac{\|A_i - A_k\|\,\|A_j - A_k\|}{\|A_i - A_j\|} \sin\angle A_i A_k A_j$$

as $r \to \infty$, say let $\alpha = \frac{\sin\theta_1}{\operatorname{diam}(K)} d_{\min}^2 < d_{\min}$. Then $b(r) \geq \alpha r$ for r sufficiently large (case $i = j$ is clear), and $\cos\phi_0(r) \geq \frac{\alpha}{\operatorname{diam}(K)}$. Now bounds on $\lambda(r)^{-1}$ and $\mu(r)^{-1}$, respectively from above and below, are given by the following expressions.

$$T_1(r) = 1 + \frac{2\kappa_{\max} \operatorname{diam}(K)^2}{\alpha} r + \frac{\operatorname{diam}(K) r}{(r-1)d_{\min} - d_{\max}} \qquad T_2(r) = 1 + 2\kappa_{\min} d_{\min} r.$$

It can be checked

$$1 = \lim_{r\to\infty} \frac{\ln r}{\ln T_1(r)} \leq \lim_{r\to\infty} \frac{\ln r}{-\ln\lambda(r)} \leq \lim_{r\to\infty} \frac{\ln r}{-\ln\mu(r)} \leq \lim_{r\to\infty} \frac{\ln r}{\ln T_2(r)} = 1,$$

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so

$$\begin{aligned}\liminf_{r\to\infty}(\ln r)^2\Big(\frac{\dim_{\mathrm{H}} M_0(r)}{2\ln(u-1)}-\frac{1}{\ln r}\Big) &= \liminf_{r\to\infty}\Big(\frac{\ln r}{2\ln(u-1)}\dim_{\mathrm{H}} M_0(r)-1\Big)\ln r\\ &\geq \ln\big(\liminf_{r\to\infty} r\lambda(r)\big)\lim_{r\to\infty}\frac{\ln r}{-\ln\lambda(r)}\\ &= \ln\Big(\frac{1}{2\kappa_{\max}}\liminf_{r\to\infty}\frac{r\cos\phi_0(r)}{d_{\max}(r)}\Big).\end{aligned}$$

Similarly

$$\begin{aligned}&\limsup_{r\to\infty}(\ln r)^2\Big(\dim_{\mathrm{H}} M_0(r)-\frac{2\ln(u-1)}{\ln r}\Big)\\ &\qquad\leq 2\ln(u-1)\ln\Big(\frac{1}{2\kappa_{\min}}\limsup_{r\to\infty}\frac{r}{d_{\min}(r)}\Big).\end{aligned}$$

Using some strict inequalities applying the bounds above, the inequalities for large r follow. ■

Unfortunately, a precise second asymptotic term for $\dim_{\mathrm{H}} M_0(r)$ is not forthcoming even in simple cases; the example considered in [LM96], of three unit radius circles with centres A_i equidistant from the origin and spaced at an angle of $\frac{2\pi}{3}$, has $\cos\phi_0(r)=\frac{1}{2}+\mathcal{O}(\frac{1}{r})$ and the above bounds (in terms of $d_{\min}(r)$, *etc.*) separated by a factor of 2.

4.1 Better Numerical Estimates

The continued fraction (from (1.3)) for $k_j(s)$ can in fact be used to obtain bounds for curvature significantly better than those used previously. Recall that if $\big(q(s),n(s)\big)\in\hat{X}$ made $j+2$ transversal reflections, we had

$$k_{j+1}(s)\in\Big[\frac{k_j(s)}{1+d_{\max}k_j(s)}+2\kappa_{\min},\frac{k_j(s)}{1+d_{\min}k_j(s)}+\frac{2\kappa_{\max}}{\cos\phi_0}\Big].$$

Notice for $\gamma,\theta>0$ that the map $f_{\gamma,\theta}\colon(0,\infty)\to\mathbb{R}, x\mapsto 2\gamma+\frac{x}{1+\theta x}$ has one fixed point x^*, since $(x^*-2\gamma)(1+\theta x^*)=x^*$ if and only if $x^*=\gamma\pm\sqrt{\gamma^2+2\gamma/\theta}$, of which only the greater solution is positive. Denoting the two solutions temporarily by $x^*_\pm$, we have

$$f_{\gamma,\theta}(x)-x=\frac{2\gamma+2\gamma\theta x-\theta x^2}{1+\theta x}=-\theta\frac{(x-x^*_+)(x-x^*_-)}{1+\theta x}$$

and hence $f_{\gamma,\theta}(x)>x$ when $0<x<x^*_+$, and $f_{\gamma,\theta}(x)<x$ when $x>x^*_+$. Since $f_{\gamma,\theta}$ is strictly increasing (see Figure 4.1 on page 128), x^*_+ attracts every $x>0$ monotonically under the iterated map $f_{\gamma,\theta}$. Since $\lim_{x\to\infty}f_{\gamma,\theta}(x)=2\gamma+1/\theta$ we also have $f_{\gamma,\theta}(x)\in(2\gamma,2\gamma+1/\theta)$ for all $x>0$.

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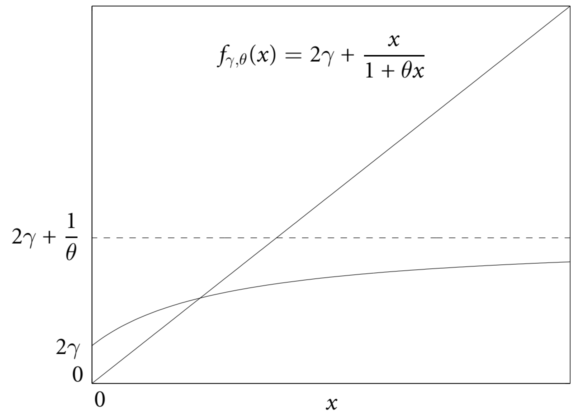

Figure 4.1: Plot of function $f_{\gamma,\theta}$.

Now consider the function given by what was previously known as x_+^*:

$$g\colon (0,\infty)\times(0,\infty)\to(0,\infty), (\gamma,\theta)\mapsto \gamma+\sqrt{\gamma^2+2\gamma/\theta}.$$

Clearly, $f_{\gamma_1,\theta_1}(x) \le f_{\gamma_2,\theta_2}(x)$ for all $x > 0$ whenever $0 < \gamma_1 \le \gamma_2$ and $\theta_1 \ge \theta_2 > 0$, and it follows that $g(\gamma_1,\theta_1) \le g(\gamma_2,\theta_2)$ under the same conditions. In fact on the (natural in our case) domain $[\kappa_{\min}, \frac{\kappa_{\max}}{\cos\phi_0}] \times [d_{\min}, d_{\max}]$, minimum and maximum values of g are respectively $g_{\min} = g(\kappa_{\min}, d_{\max})$ and $g_{\max} = g(\frac{\kappa_{\max}}{\cos\phi_0}, d_{\min})$. With these definitions, we have the following theorem.

Theorem 4.2

$$\frac{2\ln(u-1)}{\ln(1+d_{\max}g_{\max})} \le \dim_H M_0 \le \frac{2\ln(u-1)}{\ln(1+d_{\min}g_{\min})}.$$

Proof Firstly, either $k_0(s) \in [g_{\min}, g_{\max}]$, in which case $k_j(s) \in [g_{\min}, g_{\max}]$ for any $j \ge 0$ where it is defined, or $k_0(s)$ lies outside this interval. However, we can always use the former bounds for $\dim_H M_0$ estimates, as shown below.

Consider a particular $s \in (0,1)$ such that $q(s) \in X_0$, and any sequences $(\gamma_j)_1^\infty \subseteq [\kappa_{\min}, \frac{\kappa_{\max}}{\cos\phi_0}]$ and $(\theta_j)_1^\infty \subseteq [d_{\min}, d_{\max}]$, and inductively define $k_{j+1}(s) = f_{\gamma_j,\theta_j}\big(k_j(s)\big)$ for $0 \le j \le n-1$. For any open interval $U = (a,b) \supseteq [g_{\min}, g_{\max}]$ it can be shown there is a (minimal) $j_0(s) > 0$ such that $j \ge j_0(s) \implies k_j(s) \in U$, as follows. If $k_N(s) \le g_{\max}$ for some $N \ge 0$, then inductively

$$k_{j+1}(s) = f_{\gamma_j,\theta_j}\big(k_j(s)\big) \le f_{\frac{\kappa_{\max}}{\cos\phi_0},d_{\min}}\big(k_j(s)\big) \le f_{\frac{\kappa_{\max}}{\cos\phi_0},d_{\min}}(g_{\max}) = g_{\max}$$

for all $j \ge N$, and similarly for the lower bound, so if $k_N(s) \in [g_{\min}, g_{\max}]$ for some $N \ge 0$, the result follows. Suppose then that $k_j(s) \le a$ for all $j \ge 0$, and define $p_0(s) = k_0(s)$, $p_{j+1}(s) = f_{\kappa_{\min},d_{\max}}\big(p_j(s)\big)$, so that $\lim_{j\to\infty} p_j(s) = g_{\min}$. Since

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$p_j(s) \leq k_j(s)$ for all $j \geq 0$, this gives $k_j(s) > a$ for some j, a contradiction. The case $k_0(s) \geq b$ can be dealt with similarly under the definitions $m_0(s) = k_0(s)$, $m_{k+1}(s) = f_{\frac{\kappa_{\max}}{\cos\phi_0}, d_{\min}}\big(m_k(s)\big)$, and we get a suitable $j_0(s)$ for each point of X_0. Since X_0 is compact, $\inf_{q(s)\in X_0} k_0(s)$ is attained, say at $s = s_0$. Analogous to $j_0(s_0)$, there will also be some minimal j such that $p_j(s_0) \in U$ and $m_j(s_0) \in U$, say $j = l_0(s_0) \geq j_0(s_0)$.

Now consider the subset $\hat{Z}_n$ of $\hat{X}$ of points with at least n forward reflections, the connected component W_n of Z_n containing $q(s_0)$, and the curve Y_n of n-th reflections, where $n > l_0(s_0)$. Since $k_0'(s)$ is bounded (X is C^3) and the length of each connected component of Z_n tends to 0 as $n \to \infty$, for any $\epsilon > 0$ we can choose $n > l_0(s_0)$ sufficiently large that for $s \in q^{-1}(W_n)$, $|k_0(s_0) - k_0(s)| < \epsilon$. If $k_{0,\min}$ is the minimum curvature of X, then $f_{\kappa_{\min}, d_{\max}}$ and $f_{\frac{\kappa_{\max}}{\cos\phi_0}, d_{\min}}$ are contractions of $[k_{0,\min}, \infty)$ with respective fixed points $g_{\min}$ and $g_{\max}$, and contraction ratio at most $\alpha = (1 + d_{\min}\min\{k_{0,\min}, a\})^{-2} < 1$. By first making the choice of $\epsilon \leq \alpha^{-l_0(s_0)}\min\{p_{l_0(s_0)}(s_0) - a, b - p_{l_0(s_0)}(s_0), m_{l_0(s_0)}(s_0) - a, b - m_{l_0(s_0)}(s_0)\}$ and then $n = n_0$ as above, we can ensure $l_0(s) \leq l_0(s_0) < n_0$ is well-defined for all $s \in q^{-1}(W_{n_0})$, and hence so is $j_0(s) \leq l_0(s)$.

Now $k_j(s) \in (a, b)$ for all relevant s and each $j > n_0$ if we restrict to W_{n_0} rather than X. The condition of Proposition 3.1 can be modified to *there exist* $C, c > 0$ *and* $0 < \lambda < \mu < 1$ *such that* $c\lambda^{n-n_0} \leq \|\pi x - \pi y\| \leq C\mu^{n-n_0}$ *whenever the images under* Υ *of* $x, y \in \hat{X}_0$ *agree to exactly* $n \geq n_0$ *places*, where $\lambda = (1 + d_{\max}b)^{-1}$ and $\mu = (1 + d_{\min}a)^{-1}$, and proceeding as usual we get $\frac{-\ln(u-1)}{\ln\lambda} \leq \dim_{\mathrm{H}} X_0 \leq \frac{-\ln(u-1)}{\ln\mu}$. In the limit as $a \uparrow g_{\min}$ and $b \downarrow g_{\max}$, we have the result. ■

These estimates are always better than the previous values and hence give the same asymptotic for $\dim_{\mathrm{H}} M_0(r)$, but as before do not give a precise asymptotic for the error term. In fact, bounds on the error are the same: for $T_1(r) = 1 + d_{\max}(r)g_{\max}(r)$ and $T_2(r) = 1 + d_{\min}(r)g_{\min}(r)$, and provided $d_{\min}(r)$, $d_{\max}(r)$ and $\phi_0(r)$ are C^1 (certainly true in our case), calculations show $\liminf_{r\to\infty} \frac{r}{T_1(r)} = \frac{1}{2\kappa_{\max}} \liminf_{r\to\infty} \frac{r\cos\phi_0(r)}{d_{\max}(r)}$ and $\limsup_{r\to\infty} \frac{r}{T_2(r)} = \frac{1}{2\kappa_{\min}} \limsup_{r\to\infty} \frac{r}{d_{\min}(r)}$.

4.3 Concluding Remarks

The most desirable extension of the above results would be to weaken the condition **(H)**, which may be possible if only certain reflections of non-tangent trajectories are considered for the bound ϕ_0. Points of $X \setminus X_0$ however clearly are not subject to such a bound.

For K an obstacle in $\mathbb{R}^n$ ($n \geq 3$), M is no longer two-dimensional, and most of the above is no longer applicable. M_0 should remain zero-dimensional due to the coding, and then possibly the foliations of [Pix83] (if smooth enough) may be useful to relate the Hausdorff dimensions of X_0 and M_0, but the convex curve estimates are unlikely to translate so well, in part because in place of the curvature formulae (1.3) we have only an inequality relating k_{j+1} and k_j.

There are more general estimates of Hausdorff dimension for Cantor sets such as M_0 following from [PT93, Chapter 4] in terms of thickness $\tau(r)$ and denseness $\theta(r)$ of X_0, to which ours seem related. However, these are hard to use in our case without some modification to avoid problems with the uniformity of 'gap' sizes. Also possibly useful in determining more closely the behaviour of $\underline{\dim}_{\mathrm{H}} M_0$ are [MM83] and [PV88], from which we have that $\dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{u})}(x)\cap M_0\big) = \overline{\dim}_{\mathrm{p}}\big(W_\theta^{(\mathrm{u})}(x)\cap M_0\big)$ independent of $x \in M_0$, similarly for stable local manifolds, and $\dim_{\mathrm{H}} M_0 = \overline{\dim}_{\mathrm{p}} M_0 = \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{u})}(x) \cap M_0\big) + \dim_{\mathrm{H}}\big(W_\theta^{(\mathrm{s})}(x) \cap M_0\big)$, which as indicated above eliminate the need for the calculation of $\overline{\dim}_{\mathrm{p}}$ in Lemma 2.6. We also have continuity of the Hausdorff dimension with C^1 perturbations of B, which implies continuity under certain perturbations of K (if **(H)** is still satisfied), and may be useful in reducing the smoothness assumptions used.

Finally, there are questions relating to the behaviour of $\dim_{\mathrm{H}} M_0$ as components are added or removed (possibly infinitely many) from the obstacle K (*i.e.*, u is changed), while retaining or losing conditions like **(H)**. For instance, it is not immediately clear whether there exists a (possibly unbounded) K with a countably infinite number of components that still satisfies (or nearly satisfies) **(H)**, or how M_0 would behave for such a system.

A Details of Main Estimate

In this section, included for completeness only as it essentially follows [Sto03], we describe one way of deriving the following estimates used in Section 3.

$$\|p'(s)\| \prod_{j=0}^{n-1} \delta_j(s) = \|q'(s)\|, \tag{A.1}$$

$$\frac{\|p'(s)\|}{1 + \big(\tau - t_n(s)\big)\, k_n(s)} \prod_{j=0}^{n-1} \delta_j(s) = \|q'(s)\|. \tag{A.2}$$

In both (A.1) and (A.2), q parametrises a convex curve X with n reflections, and p parametrises a certain curve near the n-th or $(n+1)$-st reflection, as described below.

First, choose arbitrary distinct $x_1, x_2 \in \hat{X}$, let $\tau_j = \max\{t_j(x_1), t_j(x_2)\}$, and let $l_j \in \{1, 2\}$ be the (minimal for $j = 0$) index of the x_i for which $t_j(x_{l_j}) = \tau_j$. Denote by $\hat{Y}_j$ the oriented (piecewise smooth) subcurve of $S_{\tau_j}(\hat{X})$ with endpoints the respective images of x_1 and x_2. This is the earliest evolute of the corresponding subcurve of X that lies after the j-th reflection; notice that one of the endpoints lies on ∂K_{β_j}. Also denote by $p_j(x) = \pi S_{\tau_j}(x)$ the image on Y_j of any $x \in \hat{X}$ with πx lying on the curve between πx_1 and πx_2.

Secondly, let $\epsilon_j = \|p_j(x_1) - p_j(x_2)\| = \epsilon_j^{(l_j)}$ and $\delta_j = \delta_j^{(l_j)}$ where, for $i = 1, 2$, $\epsilon_j^{(i)}$ and $\delta_j^{(i)}$ are defined by

$$\epsilon_j^{(i)} = \|\pi S_{t_j(x_i)}(x_1) - \pi S_{t_j(x_i)}(x_2)\| \quad \text{and} \quad (\delta_j^{(i)})^{-1} = 1 + \big(t_{j+1}(x_i) - t_j(x_i)\big)\, k_j(x_i).$$

For suitable strictly convex smooth curves Y and Z (such that every outwards normal trajectory from Y reflects transversally from Z with reflection angle bounded

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above by some $\phi \in (0, \frac{\pi}{2})$), and for any $y_1, y_2 \in Y$, we have the following inequalities regarding the billiard in the exterior of $\mathrm{Cvx}(Z)$ (these follow from *e.g.* [Ika88, Lemma 3.7]).

$$\text{(A.3)} \qquad \big(1 + t_1(y_1)\kappa\big) D - cD^2 \le \|\pi S_{t_1(y_i)}(y_1) - \pi S_{t_1(y_i)}(y_2)\| \le \big(1 + t_1(y_1)\kappa\big) D + CD^2$$

$$\text{(A.4)} \qquad \big(1 + t_1(y_1)\kappa\big) D - cD^2 \le \|\pi S_{t_1(y_1)}(y_1) - \pi S_{t_1(y_2)}(y_2)\| \le \big(1 + t_1(y_1)\kappa\big) D + CD^2.$$

Here $D = \|q_1 - q_2\|$ is dependent on $q_i = \pi y_i$ $(i = 1, 2)$, $c, C > 0$ are some constants dependent only on ϕ_0 and the minima and maxima of the curvatures of Y and Z, and κ is the curvature of Y at q_1.

Returning to the billiard in the exterior of K, assuming $\tau_j < \min_{i=1,2} t_{j+1}(x_i)$ and $\tau_{j+1} < \min_{i=1,2} t_{j+2}(x_i)$ we may apply (A.3) to $\hat{Y} = \hat{Y}_j$, $Z = \partial K_{\beta_{j+1}}$, $y_1 = S_{\tau_j}(x_1)$ and $y_2 = S_{\tau_j}(x_2)$ (so $\pi y_i = p_j(x_i)$ for $i = 1, 2$), and further assuming $l_j = l_{j+1} = 1$ we have the following inequality $(0 \le j \le n - 1)$.

$$\text{(A.5)} \qquad \frac{\epsilon_j}{\delta_j} - \mathrm{Const}_1(\epsilon_j)^2 \le \epsilon_{j+1} \le \frac{\epsilon_j}{\delta_j} + \mathrm{Const}_2(\epsilon_j)^2.$$

This can also be shown to hold for any l_j, l_{j+1} by using symmetry in x_1 and x_2, switching y_1 and y_2 if necessary, and finding constants such as $\mathrm{Const}_3 = d_{\max}k_{\max} - d_{\min}k_{\min} + \mathrm{Const}_1$ so that $\epsilon_j^{(1)}(\delta_j^{(1)})^{-1} - \mathrm{Const}_3(\epsilon_j^{(1)})^2 \le \epsilon_j^{(1)}(\delta_j^{(2)})^{-1} - \mathrm{Const}_1(\epsilon_j^{(1)})^2$. For $j = n - 1$ we use (A.4) to get the following analogous inequality, where we can clearly assume constants $C, c > 0$ also valid in (A.5) for $j = 0, \dots, n-1$; independent of the particular j.

$$\text{(A.6)} \qquad \frac{\epsilon_{n-1}}{\delta_{n-1}} - c\epsilon_{n-1}^2 \le \|q_n(x_1) - q_n(x_2)\| \le \frac{\epsilon_{n-1}}{\delta_{n-1}} + C\epsilon_{n-1}^2.$$

Now no longer assuming the above conditions on τ_j (except for $j = 0$), let $\Gamma = \pi B^n(\hat{X}) \subseteq \partial K_{\beta_n}$ and suppose both x_1 and x_2 have $(n + 1)$-st forward reflections, but $y_1 = \pi B^{n+1}(x_1) \in \partial K_m$ and $y_2 = \pi B^{n+1}(x_2) \in \partial K_{m'}$ where $m \ne m'$, with $t_{n+1}(s) = +\infty$ for all $q(s)$ on the subcurve between them (so reflections at y_1 and y_2 are tangencies). The curve just after 'reflection' ($\hat{Y}_{n+1} = S_{\tau_{n+1}}(\hat{X})$ if we extend the definitions for $j \le n$) then has the endpoint $p_{n+1}(x_{l_{n+1}})$ on ∂K; assume this is y_2. Consider corresponding parametrizations $X : q(s)$ and $Y_{n+1} : p(s) = \pi S_\tau\big(q(s), n_X\big(q(s)\big)\big)$, each smooth between s_1, s_2 such that $x_i = q(s_i)$ $(i = 1, 2)$. For arbitrary $q(s)$ and $q(s_0)$ on the subcurve between x_1 and x_2, we have corresponding $\check{\tau}_j = \max\{t_j(s), t_j(s_0)\}$, curves $\check{Y}_j$ *etc.*, to which we can apply (A.5) (for $0 \le j \le n - 1$) if s is close enough to s_0 that $\check{\tau}_j < \min\{t_{j+1}(s), t_{j+2}(s_0)\}$ and $\check{\tau}_{j+1} < \min\{t_{j+2}(s), t_{j+2}(s_0)\}$ hold for each j. Clearly $\lim_{s\to s_0} \check{\delta}_j = \delta_j(s_0)$ for $j \le n - 1$, it is easy to check $\lim_{s\to s_0} \check{\epsilon}_j = 0$ $(0 \le j \le n)$, and since $\check{Y}_n$ evolves to a subcurve of Y_{n+1} in time $\tau_{n+1} - \check{\tau}_n < d_{\max}$ and the curvature of $\check{Y}_n$ is bounded below (the limiting curvature of $S_{-t}(\hat{\check{Y}}_n)$ at any $s_3 \in [s, s_0]$ as $t \uparrow \check{\tau}_n - t_n(s_3)$) is bounded below by $2\kappa_{\min}$, and (1.2) gives a lower bound

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on $\kappa_{\check{Y}_n}(s_3)$), by [Ika88] or [Sto03, Lemma 1] there are $\epsilon, C, c > 0$ such that if $\check{\epsilon}_n < \epsilon$ then we have the following.

$$\text{(A.7)} \quad \big(1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)\big)\,\check{\epsilon}_n - c\check{\epsilon}_n^2 \le \|p(s)-p(s_0)\| \le \big(1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)\big)\,\check{\epsilon}_n + C\check{\epsilon}_n^2.$$

By choosing s sufficiently close to s_0 we can assume $(\check{\delta}_j)^{-1} \ge c\check{\epsilon}_j$ for $0 \le j \le n-1$ and $1+(\tau-\check{\tau}_n)k_n(s) \ge c\check{\epsilon}_n$, so rearranging (A.5) and (A.7) gives (for $0 \le j \le n-1$) the following.

$$\text{(A.8)} \qquad \frac{\check{\epsilon}_{j+1}}{\frac{1}{\check{\delta}_j} + C\check{\epsilon}_j} = \frac{\check{\delta}_j}{1+C\check{\delta}_j\check{\epsilon}_j}\check{\epsilon}_{j+1} \le \check{\epsilon}_j \le \frac{\check{\delta}_j}{1-c\check{\delta}_j\check{\epsilon}_j}\check{\epsilon}_{j+1},$$

$$\text{(A.9)} \qquad \frac{\|p(s)-p(s_0)\|}{1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)+C\check{\epsilon}_n} \le \check{\epsilon}_n \le \frac{\|p(s)-p(s_0)\|}{1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)-c\check{\epsilon}_n}.$$

Starting with $j = 0$, applying (A.8) n times and then (A.9) gives the following inequalities.

$$\frac{\|p(s)-p(s_0)\|}{1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)+C\check{\epsilon}_n}\prod_{j=0}^{n-1}\frac{\check{\delta}_j}{1+C\check{\delta}_j\check{\epsilon}_j}$$
$$\le \check{\epsilon}_0 \le \frac{\|p(s)-p(s_0)\|}{1+(\tau-\check{\tau}_n)\kappa_{\check{Y}_n}(s)-c\check{\epsilon}_n}\prod_{j=0}^{n-1}\frac{\check{\delta}_j}{1-c\check{\delta}_j\check{\epsilon}_j}.$$

Dividing by $s - s_0$ and taking limits (note $\check{\epsilon}_0 = \|q(s)-q(s_0)\|$) then gives equation (A.2). The other result is obtained similarly at $s = s_0$; consider the parametrization by arc length $\Gamma\colon p(s)$ and use the estimate (A.8) n times, followed by the estimate

$$\frac{\check{\delta}_{n-1}}{1+C\check{\delta}_{n-1}\check{\epsilon}_{n-1}}\|q_n(s_1)-q_n(s_2)\| \le \check{\epsilon}_{n-1} \le \frac{\check{\delta}_{n-1}}{1-c\check{\delta}_{n-1}\check{\epsilon}_{n-1}}\|q_n(s_1)-q_n(s_2)\|$$

obtained by rearrangement from (A.6).

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