THE STEINITZ-GROSS THEOREM ON SUMS OF VECTORS

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1. Introduction. $\alpha_1, \alpha_2, \ldots, \alpha_p$ are n-dimensional vectors,

$$\sum_{\pi=1}^{p} \alpha_\pi = 0, \qquad |\alpha_\pi| \leqslant 1 \quad (1 \leqslant \pi \leqslant p);$$

they are arranged to form a closed polygon

$$OA_1A_2 \ldots A_{p-1}O \ (\overrightarrow{OA_1} = \alpha_1, \ldots, \overrightarrow{A_{\pi-1}A_\pi} = \alpha_\pi, \ldots, \overrightarrow{A_{p-1}O} = \alpha_p).$$

Denote by $R(\alpha_1, \alpha_2, \ldots, \alpha_p)$ the radius of the smallest circumscribed hypersphere with centre at O; by $\bar{R}(\alpha_1, \alpha_2, \ldots, \alpha_p)$ the minimum of

$$R(\alpha_1, \alpha_{\pi_2}, \ldots, \alpha_{\pi_{p-1}}, \alpha_p)$$

for all possible reorderings

$$\alpha_{\pi_2}, \ldots, \alpha_{\pi_{p-1}}$$

of $\alpha_2, \ldots, \alpha_{p-1}$; and by c_n the least possible constant such that

$$\bar{R}(\alpha_1, \alpha_2, \ldots, \alpha_p) \leqslant c_n$$

for all possible choices of p and $\alpha_1, \alpha_2, \ldots, \alpha_p$.

Steinitz **(1)** proved that $c_n \leqslant 2(n+1)$; using induction with respect to n, Gross **(2)** obtained the weaker estimate $c_n \leqslant 2^n - 1$; by the same method Bergström **(3)** obtained the result $c_n^2 \leqslant 4c_{n-1}^2 + 1$. Trivially, $c_1 = 1$. $c_2 = \sqrt{2}$ was proved independently by Gross **(2)**, Bergström **(4)**, and Damsteeg and Halperin **(5)**. For $n \geqslant 3$ the exact values of c_n are not known; from Bergström's estimate it follows that $c_3 \leqslant 3$, $c_4 \leqslant \sqrt{37}$; for $n \geqslant 5$, Steinitz's estimate gives the best result.

By a refinement of Steinitz's original method it will be shown in this paper that, for $n \geqslant 3$, $c_n < n$ (Theorem 1), and particularly, $c_3 \leqslant (5 + 2\sqrt{3})^{\frac{1}{2}} = 2.90\ldots$ (Theorem 2).

The lower estimate $c_n \geqslant \frac{1}{2}(n+6)^{\frac{1}{2}}$ given by Damsteeg and Halperin **(5)**, and other examples make it likely that the true order of c_n is $n^{\frac{1}{2}}$.

2. Notation. Greek letters except $\kappa, \lambda, \mu, \nu, \pi$ denote n-dimensional vectors ($n \geqslant 3$); $a, b, c, d, e, f, g, x, y, z$ real numbers; $i, j, k, l, m, n, p, q, r, s, t, \kappa, \lambda, \mu, \nu, \pi$ natural numbers.

$|\alpha|$ denotes the length of α; $\alpha\beta$ the scalar product of α, β.

The vectors $\theta_1, \theta_2, \ldots, \theta_m$ will be called *positively dependent* (p.d.) if they are linearly dependent with non-negative coefficients; *positively independent* (p.i.) means not p.d.

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3. Lemmas.

(I) *From any* $m(>n+1)$ p.d. *vectors* $\theta_1, \theta_2, \ldots, \theta_m$, $n+1$ p.d. *vectors* $\theta_{\mu_1}, \theta_{\mu_2}, \ldots, \theta_{\mu_{n+1}}$ *can be selected.*

Proof. It is sufficient to show that $m-1$ of the given vectors are p.d. If, in the given relation

$$\sum_{\mu=1}^{m} d_\mu \theta_\mu = 0 \qquad d_\mu \geqslant 0,$$

(at least one d_μ being positive), one d_μ is zero, this is trivial. If all $d_\mu > 0$, choose any linear relation between $\theta_1, \ldots, \theta_{n+1}$,

$$\sum_{\nu=1}^{n+1} a_\nu \theta_\nu = 0 \qquad (\text{not all } a_\nu = 0),$$

and consider the relation

$$\sum_{\nu=1}^{n+1} (d_\nu - x a_\nu)\, \theta_\nu + \sum_{\nu=n+2}^{m} d_\nu \theta_\nu = 0.$$

For $x = 0$ all coefficients are positive; hence x can be chosen such that one coefficient vanishes, the others remaining non-negative (and d_m positive); the ensuing relation expresses the p.d. of $m-1$ of the vectors.

(I.1)

$$\theta_{\mu_1}, \ldots, \theta_{\mu_{n+1}}$$

in (I) *may be prescribed to include* θ_1.

Proof. Suppose θ_1 is not already included. Let

$$\sum_{i=1}^{n+1} b_i \theta_{\mu_i} = 0$$

be the relation expressing the p.d. of

$$\theta_{\mu_1}, \ldots, \theta_{\mu_{n+1}}.$$

If one $b_i = 0$, the term $0 \cdot \theta_1$ may be substituted for $b_i \theta_{\mu_i}$. If all $b_i > 0$, consider any linear relation between $\theta_{\mu_1}, \ldots, \theta_{\mu_n}, \theta_1$:

$$\sum_{i=1}^{n} e_i \theta_{\mu_i} + e_{n+1}\theta_1 = 0 \qquad (\text{not all } e_i = 0).$$

It may be assumed that $e_{n+1} \geqslant 0$. If all $e_i \geqslant 0$,

$$\theta_{\mu_1}, \ldots, \theta_{\mu_n}, \theta_1$$

are p.d. If one $e_i < 0$, consider the relation

$$\sum_{i=1}^{n} (b_i + x e_i)\, \theta_{\mu_i} + b_{n+1}\theta_{\mu_{n+1}} + x e_{n+1}\theta_1 = 0.$$

For $x = 0$ all coefficients in the first sum are positive; hence $x > 0$ can be determined so that one coefficient vanishes, the others remaining non-negative (and b_{n+1} positive). The following corollary is obvious:

(I.2) *In* (I) *and* (I.1) θ_m *may be excluded from*

$$\theta_{\mu_1}, \ldots, \theta_{\mu_{n+1}}$$

unless $\theta_1, \ldots, \theta_{m-1}$ *are* p.i.

(II) *If* $m > n$,

$$\theta = \sum_{\mu=1}^{m} d_\mu \theta_\mu \neq 0, \qquad 0 \leqslant d_\mu \leqslant 1,$$

then θ can be expressed in the form

$$\theta = \sum_{\mu=l}^{m} d_\mu' \theta_\mu', \qquad \begin{cases} 0 < d_\mu' \leqslant 1, & \mu < l + n, \\ d_\mu' = 1, & \mu \geqslant l + n, \end{cases}$$

where $1 \leqslant l \leqslant m$, *and the* θ_μ' *are a rearrangement of the* θ_μ.

Proof. Let r be the number of d_μ's with $0 < d_\mu < 1$. If $r \leqslant n$, then the required relation is obtained from the given one by omitting the terms with coefficient 0. It is therefore sufficient to show that, for $r \geqslant n + 1$, the value of r can be diminished. Suppose that $0 < d_\mu < 1$ for $1 \leqslant \mu \leqslant n + 1$; using a linear relation

$$\sum_{\mu=1}^{n+1} a_\mu \theta_\mu = 0 \qquad (\text{not all } a_\mu = 0),$$

form

$$\theta = \sum_{\mu=1}^{n+1} (d_\mu - x a_\mu)\, \theta_\mu + \sum_{\mu=n+2}^{m} d_\mu \theta_\mu.$$

For $x = 0$ the first $n + 1$ coefficients lie between 0 and 1; hence x can be chosen such that one coefficient becomes equal to 0 or 1, the others remaining $\geqslant 0$, $\leqslant 1$. As $\theta \neq 0$, the final representation of θ contains at least one term, i.e., $l \leqslant m$.

(II.1) *The representation*

$$\theta = \sum_{\mu=l}^{m} d_\mu' \theta_\mu'$$

in (II) *may be so chosen that either* $\theta_1 = \theta_l'$ *or* θ_1 *does not occur at all.*

Proof. Suppose θ_1 occurs in the relation obtained. (II.1) is obvious if the coefficient of θ_1 is less than 1 or if fewer than n coefficients are less than 1 (only a trivial reordering of the θ_μ' being required). If the coefficient of θ_1 is 1, and exactly n coefficients are less than 1, i.e.,

$$0 < d_\mu' < 1 \text{ for } \mu = l, \ldots, l + n - 1,$$

$$\theta_1 = \theta_s', \quad s \geqslant l + n, \quad d_s = 1,$$

use a linear relation

$$\sum_{\mu=l}^{l+n-1} b_\mu \theta_\mu + b_s \theta_s' = 0 \qquad (b_\mu, b_s \text{ not all } 0),$$

to form

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$$\theta = \sum_{\mu=l}^{l+n-1} (d_\mu' - xb_\mu)\,\theta_\mu' + (1 - xb_s)\,\theta_1 + \sum_{\substack{\mu=l+n\\ \mu\neq s}}^{m} d_\mu'\theta_\mu'.$$

It may be assumed that $b_s \geqslant 0$; letting x increase from 0, either θ_1 can be eliminated from the relation, or one of the first n coefficients can be made equal to 0 or 1 (the others remaining $\geqslant 0, \leqslant 1$); in this case θ_1 can be incorporated in the first n terms and be renamed θ_l'.

(III) *If* $k \geqslant 2$, $|\theta_\kappa| \leqslant 1$ $\quad (1 \leqslant \kappa \leqslant k)$,

$$\eta = d_1\theta_1 - \sum_{\kappa=2}^{k} d_\kappa\theta_\kappa, \qquad 0 \leqslant d_1 \leqslant 1, 0 \leqslant d_\kappa < 1\ (\kappa > 1),$$

then

$$|\theta_1 + \theta_\kappa| > 1, \qquad 1 < \kappa < k$$

implies

$$|\eta| < \sqrt{(k^2 - 3k + 3)} + 1.$$

Proof. For $k = 2$, $|\eta| \leqslant |d_1\theta_1| + |d_2\theta_2| < 2 = \sqrt{1} + 1$; for $k \geqslant 3$,

$$(\theta_1 + \theta_\kappa)^2 = \theta_1^2 + 2\theta_1\theta_\kappa + \theta_\kappa^2 > 1, \qquad 1 < \kappa < k,$$

implies

$$-2\theta_1\theta_\kappa < \theta_1^2 + \theta_\kappa^2 - 1 \leqslant 1,$$

whence

$$|\eta| \leqslant \left| d_1\,\theta_1 - \sum_{\kappa=2}^{k-1} d_\kappa\theta_\kappa \right| + |d_k\theta_k|$$

$$< \left\{ d_1^2\theta_1^2 - \sum_{\kappa=2}^{k-1} d_1\,d_\kappa 2\theta_1\theta_\kappa + \left(\sum_{\kappa=2}^{k-1} d_\kappa\theta_\kappa \right)^2 \right\}^{\frac{1}{2}} + 1$$

$$< \{1 + (k-2) + (k-2)^2\}^{\frac{1}{2}} + 1 = (k^2 - 3k + 3)^{\frac{1}{2}} + 1.$$

(III.1) *If the condition* $|\theta_1 + \theta_k| > 1$ *is added in* (III), *then*

$$|\eta| < (k^2 - k + 1)^{\frac{1}{2}}.$$

Proof. By obvious modification of the proof of (III).

(IV) *If* $k \geqslant 2$, $|\theta_\kappa| \leqslant 1$ $\quad (1 \leqslant \kappa \leqslant k)$,

$$\eta' = \sum_{\kappa=2}^{k} d_\kappa\,\theta_\kappa, \quad \zeta' = \sum_{\kappa=2}^{k} (1 - d_\kappa)\theta_\kappa, \quad 0 \leqslant d_\kappa \leqslant 1,$$

$$\eta = \theta_1 + \eta', \qquad \zeta = -\theta_1 + \zeta',$$

then $|\zeta| > 1$, *implies*

$$|\eta| < k - (2 - \sqrt{2}).$$

Proof. For $k = 2$,

$$\zeta^2 = (-\theta_1 + (1 - d_2)\theta_2)^2 = \theta_1^2 - 2(1 - d_2)\theta_1\theta_2 + (1 - d_2)^2\theta_2^2 > 1$$

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implies $1 - d_2 > 0$ and

$$2\theta_1\theta_2 < \frac{\theta_1^2 - 1}{1 - d_2} + (1 - d_2)\,\theta_2^2 \leqslant (1 - d_2)\,\theta_2^2,$$

whence

$$\eta^2 = \theta_1^2 + 2d_2\theta_1\theta_2 + d_2^2\theta_2^2 < \theta_1^2 + d_2\theta_2^2 \leqslant 2,$$

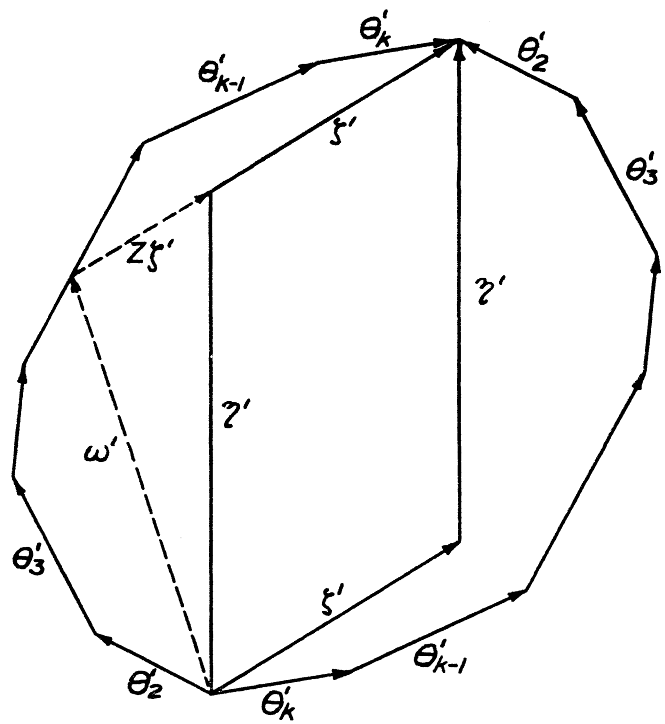
i.e.,

$$|\eta| < \sqrt{2} = 2 - (2 - \sqrt{2}).$$

Let $k \geqslant 3$. As $\eta' = 0$ would imply $|\eta| = |\theta_1| \leqslant 1$, it may be assumed that $\eta' \neq 0$. Similarly, $|\zeta| > 1$, implies $\zeta' \neq 0$. Let $\theta_2', \ldots, \theta_k'$ be the projections of $\theta_2, \ldots, \theta_k$ into a plane containing η' and ζ'. Then

$$\eta' = \sum_{\kappa=2}^{k} d_\kappa\,\theta_\kappa', \quad \zeta' = \sum_{\kappa=2}^{k} (1 - d_\kappa)\,\theta_\kappa', \quad \eta' + \zeta' = \sum_{\kappa=2}^{k} \theta_\kappa', \quad |\theta_\kappa'| \leqslant 1.$$

It may be assumed that the component of every θ_κ' $(2 \leqslant \kappa \leqslant k)$, and hence the component of ζ', in the η'-direction is positive, as otherwise $|\eta'| \leqslant k - 2$ and $|\eta| = |\theta_1 + \eta'| \leqslant k - 1 < k - (2 - \sqrt{2})$. The θ_κ' may then be so renumbered



that they form a convex polygon which encloses the parallelogram formed by η', ζ'. Defining ω' as shown in the Figure,

(1) $$\eta' = \omega' + z\zeta', \qquad z \geqslant 0,$$

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where

(2) $$|\omega'| \leqslant \sum_{\kappa=2}^{k-1} |\theta_\kappa'| \leqslant k-2,$$

(3) $$|\omega'| + (z+1)|\zeta'| \leqslant \sum_{\kappa=2}^{k} |\theta_\kappa'| \leqslant k-1.$$

By assumption,

$$\zeta^2 = (-\theta_1 + \zeta')^2 = \theta_1^2 - 2\theta_1\zeta' + \zeta'^2 > 1,$$

whence

$$2\theta_1\zeta' < \theta_1^2 + \zeta'^2 - 1 \leqslant \zeta'^2,$$

and

(4) $$(z\zeta' + \theta_1)^2 = z^2\zeta'^2 + 2z\zeta'\theta_1 + \theta_1^2 < (z^2+z)\zeta'^2 + 1$$
$$< (z+1)^2\zeta'^2 + 1 \leqslant (k-1-|\omega'|)^2 + 1,$$

by (3). By (1), (4),

$$|\eta| = |\eta' + \theta_1| = |\omega' + z\zeta' + \theta_1|$$
$$\leqslant |\omega'| + |z\zeta' + \theta_1| < |\omega'| + ((k-1-|\omega'|)^2 + 1)^{\frac{1}{2}}.$$

The last expression increases with $|\omega'|$ and takes its greatest value, by (2), for $|\omega'| = k-2$, i.e.,

$$|\eta| < k - 2 + \sqrt{2}.$$

(V) *If*

$$\eta = \xi + \sum_{\mu=1}^{m} \theta_\mu,\ |\xi| < a, \quad |\eta| \leqslant b,\ b > 0,\ |\theta_\mu| \leqslant 1\ (1 \leqslant \mu \leqslant m),$$
$$1 \leqslant m \leqslant 2a(a-b)$$

(which implies $a > b$), then $\theta_\nu = \theta_1'$ can be selected such that $|\xi + \theta_1'| < a$.

Proof. Select $\theta_1' = \theta_\nu$ such that $(\xi+\theta_\nu)^2 \leqslant (\xi+\theta_\mu)^2$ for $1 \leqslant \mu \leqslant m$; then

$$(\xi + \theta_1')^2 \leqslant \frac{1}{m}\sum_{\mu=1}^{m}(\xi+\theta_\mu)^2 = \frac{1}{m}\left(m\xi^2 + 2\xi(\eta-\xi) + \sum_{\mu=1}^{m}\theta_\mu^2\right)$$
$$\leqslant \left(1-\frac{2}{m}\right)\xi^2 + \frac{2}{m}\xi\eta + 1 < \left(1-\frac{2}{m}\right)a^2 + \frac{2}{m}ab + 1$$
$$= a^2 - \frac{2a(a-b)-m}{m} \leqslant a^2,$$

provided that $m \geqslant 2$. For $m = 1$, $\theta_1' = \theta_1$, $|\xi + \theta_1'| = |\eta| \leqslant b < a$.

(V.1) *Under the conditions of* (V) *a rearrangement $\theta_1', \ldots, \theta_m'$ of $\theta_1, \ldots, \theta_m$ exists such that*

$$\left|\xi + \sum_{\mu=1}^{q}\theta_\mu'\right| < a, \qquad 1 \leqslant q \leqslant m.$$

Proof. Successive application of (V).

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It can easily be verified that the conditions of (V) and (V.1) are satisfied in the following two cases:

(V.2) $a = (n^2 - 3n + 3)^{\frac{1}{2}} + 1, \quad b = (k^2 - 3k + 3)^{\frac{1}{2}} + 1,$
$$2 \leqslant k \leqslant n - 1, \quad 1 \leqslant m \leqslant 2n - k.$$

(V.3) $a = (n^2 - 3n + 3)^{\frac{1}{2}} + 1, \quad b = 1, \quad 1 \leqslant m \leqslant 2n^2 - 4n + 3.$

(VI) *If* $m \geqslant 1, |\theta_\mu| \leqslant 1 \ (1 \leqslant \mu \leqslant m), a > 0, b \geqslant 0,$
$$\eta = \xi + \sum_{\mu=1}^{m} \theta_\mu, \quad \eta^2 < a^2, \quad \xi^2 < a^2 + b^2,$$
then $\theta_\nu = \theta_1'$ *can be selected such that*
$$(\xi + \theta_1')^2 < a^2 + b_1^2, \quad b_1^2 = \frac{m-1}{m} b^2 + 1.$$

Proof. For $m = 1$, $(\xi + \theta_1')^2 = (\xi + \theta_1)^2 = \eta^2 < a^2 < a^2 + 1 = a^2 + b_1^2$. If $m \geqslant 2$, select $\theta_1' = \theta_\nu$ as in (V); then
$$\begin{aligned}(\xi + \theta_1')^2 &\leqslant \left(1 - \frac{2}{m}\right)\xi^2 + \frac{2}{m}\xi\eta + 1 \\ &< \left(1 - \frac{2}{m}\right)(a^2 + b^2) + \frac{2}{m} a(a^2 + b^2)^{\frac{1}{2}} + 1 \\ &\leqslant \left(1 - \frac{2}{m}\right)(a^2 + b^2) + \frac{2}{m}(a^2 + \tfrac{1}{2}b^2) + 1 \\ &= a^2 + \frac{m-1}{m} b^2 + 1 = a^2 + b_1^2.\end{aligned}$$

(VII) *If* $m \geqslant 1, |\theta_\mu| \leqslant 1 \quad (1 \leqslant \mu \leqslant m),$
$$\eta = \xi + \sum_{\mu=1}^{m} \theta_\mu, \quad |\eta| < a, \quad |\xi| < a,$$
then a rearrangement $\theta_1', \ldots, \theta_m'$ *of* $\theta_1, \ldots, \theta_m$ *exists such that*
$$f(m)^2 = \max_{1 \leqslant q \leqslant m} \left(\xi + \sum_{\mu=1}^{q} \theta_\mu'\right)^2 < a^2 + \tfrac{3}{2} + e^{-1}(m - \tfrac{1}{2}),$$
for $m \geqslant 1$, *and in particular,*
$$f(1)^2 < a^2, \qquad f(2)^2 < a^2 + 1,$$
$$f(3)^2 < a^2 + \tfrac{3}{2}, \quad f(4)^2 < a^2 + \tfrac{11}{6}.$$

Proof. Applying (VI), with $b = 0$, θ_1' can be selected such that
$$\xi_1^2 = (\xi + \theta_1')^2 < a^2 + b_1^2, \qquad b_1^2 = 1;$$
applying (VI) again, θ_2' can be selected such that
$$\xi_2^2 = (\xi_1 + \theta_2')^2 = \left(\xi + \sum_{\mu=1}^{2} \theta_\mu'\right)^2 < a^2 + b_2^2, \qquad b_2^2 = \frac{m-2}{m-1} b_1^2 + 1;$$
and continued application of (VI) will lead to

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$$\xi_q^2 = (\xi_{q-1} + \theta_q')^2 = \left(\xi + \sum_{\mu=1}^{q} \theta_\mu'\right)^2 < a^2 + b_q^2, \quad b_q^2 = \frac{m-q}{m-q+1} b_{q-1}^2 + 1,$$

for $q \leqslant m$. Hence,

$$f(m)^2 < a^2 + b_r^2, \qquad b_r^2 = \max_{1 \leqslant q \leqslant m} b_q^2.$$

Now

$$(5) \qquad b_q^2 = (m-q) \sum_{\kappa=1}^{q} \frac{1}{m-\kappa},$$

and

$$b_{q+1}^2 - b_q^2 = 1 - \frac{b_q^2}{m-q} = 1 - \sum_{\kappa=1}^{q} \frac{1}{m-\kappa},$$

i.e., b_q^2 first increases, then decreases, and reaches its maximum b_r^2 when

$$1 - \sum_{\kappa=1}^{r} \frac{1}{m-\kappa} \leqslant 0 \leqslant 1 - \sum_{\kappa=1}^{r-1} \frac{1}{m-\kappa},$$

i.e.

$$(6) \qquad \sum_{\kappa=1}^{r-1} \frac{1}{m-\kappa} \leqslant 1 \leqslant \sum_{\kappa=1}^{r} \frac{1}{m-\kappa}.$$

Now

$$\sum_{\lambda=s}^{t} \frac{1}{\lambda} < \int_{s-\frac{1}{2}}^{t+\frac{1}{2}} \frac{dx}{x} = \log \frac{t+\frac{1}{2}}{s-\frac{1}{2}},$$

hence, by (6),

$$1 < \log \frac{m-\frac{1}{2}}{m-r-\frac{1}{2}},$$

whence

$$(7) \qquad m - r < e^{-1}(m - \tfrac{1}{2}) + \tfrac{1}{2},$$

and

$$f(m)^2 < a^2 + b_r^2 = a^2 + 1 + (m-r) \sum_{\kappa=1}^{r-1} \frac{1}{m-\kappa}$$
$$< a^2 + 1 + (e^{-1}(m - \tfrac{1}{2}) + \tfrac{1}{2})\, 1 = a^2 + \tfrac{3}{2} + e^{-1}(m - \tfrac{1}{2}),$$

by (5), (6), (7). The relation $f(1)^2 < a^2$ is trivial. For

$m = 2$, $b_1^2 = b_2^2 = 1$, whence $f(2)^2 < a^2 + 1$;

$m = 3$, $b_1^2 = 1$, $b_2^2 = \frac{3}{2}$, $b_3^2 = 1$, whence $f(3)^2 < a^2 + \frac{3}{2}$;

$m = 4$, $b_1^2 = 1$, $b_2^2 = \frac{5}{3}$, $b_3^2 = \frac{11}{6}$, $b_4^2 = 1$, whence $f(4)^2 < a^2 + \frac{11}{6}$.

(VII.1) *If, in* (VII),

$$a = (n^2 - 3n + 3)^{\frac{1}{2}} + 1, \qquad n \geqslant 3, \quad m \leqslant n,$$

then

$$\left| \xi + \sum_{\mu=1}^{q} \theta_\mu' \right| < g(n) < n, \qquad 1 \leqslant q \leqslant m,$$

where $g(n)$ is defined in the following proof.

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Proof. For $n = 3$,

$$\left(\xi + \sum_{\mu=1}^{q} \theta_\mu'\right)^2 < (\sqrt{3} + 1)^2 + \tfrac{3}{2} = \tfrac{11}{2} + 2\sqrt{3} = g(3)^2, \quad g(3) < 2.995 < 3.$$

For $n = 4$,

$$\left(\xi + \sum_{\mu=1}^{q} \theta_\mu'\right)^2 < (\sqrt{7} + 1)^2 + \tfrac{11}{6} = \tfrac{59}{6} + 2\sqrt{7} = g(4)^2, g(4) < 3.89 < 4.$$

For $n \geqslant 5$,

$$\left(\xi + \sum_{\mu=1}^{q} \theta_\mu'\right)^2 < \{(n^2 - 3n + 3)^{\frac{1}{2}} + 1\}^2 + e^{-1}(n - \tfrac{1}{2}) + \tfrac{3}{2} = g(n)^2,$$

where

$$g(n)^2 < (n - \tfrac{1}{3})^2 + e^{-1}(n - \tfrac{1}{2}) + \tfrac{3}{2} = n^2 - (\tfrac{2}{3} - e^{-1})n + \tfrac{29}{18} - \tfrac{1}{2e}$$

$$\leqslant n^2 - \tfrac{31}{18} + \tfrac{9}{2}e^{-1} < n^2,$$

i.e., $g(n) < n$.

4. THEOREM 1. *For $n \geqslant 3$, $c_n < n$.*

The proof is in several steps.

4.1. Let

$$\sum_{\pi=1}^{p} \alpha_\pi = 0, \qquad |\alpha_\pi| \leqslant 1.$$

A rearrangement

$$\delta_1 = \alpha_1, \quad \delta_2 = \alpha_{\pi_2}, \quad \ldots, \quad \delta_{p-1} = \alpha_{\pi_{p-1}}, \quad \delta_p = \alpha_p$$

is to be constructed such that

$$\left|\sum_{\pi=1}^{q} \delta_\pi\right| < g(n) < n$$

for $1 \leqslant q \leqslant p$. We use induction with respect to p. For $p = 1$, in fact for $p \leqslant 2n - 1$, the result is trivial as no reordering is necessary:

$$\left|\sum_{\pi=1}^{q} \alpha_\pi\right| = \left|\sum_{\pi=q+1}^{p} \alpha_\pi\right| \leqslant \min(q, p - q) \leqslant \min(q, 2n - 1 - q) \leqslant n - 1.$$

In the following it will be assumed that the result is true for $p' < p$.

If a partial sum

$$\zeta = \alpha_1 + \sum_{i=2}^{q} \alpha_{\pi_i}, \qquad 2 \leqslant q \leqslant p - 2,$$

has a modulus $\leqslant 1$, then the result may be applied to

$$\alpha_1 + \sum_{i=2}^{q} \alpha_{\pi_i} + (-\zeta) = 0 \qquad (p' = 1 + q < p),$$

and to

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$$\zeta + \sum_{i=q+1}^{p-1} \alpha_{\pi_i} + \alpha_p = 0 \qquad (p' = p - q + 1 < p),$$

prescribing α_1 and $-\zeta$ in the first case, ζ and α_p in the second case, as first and last vectors of the rearrangement; combining the two arrangements and omitting the vectors $-\zeta$ and ζ, the desired rearrangement of the α_π is obtained. In the following we may therefore make the assumptions:

(VIII) *If ζ is a partial sum of the α_π containing exactly one of α_1, α_p and at least* 1, *at most $p - 3$ other vectors, then* $|\zeta| > 1$.

In particular,

(VIII.1) $$|\alpha_1 + \alpha_\pi| > 1, \qquad 2 \leqslant \pi \leqslant p - 1.$$

Also,

(VIII.2) *No partial sum is* 0, *except possibly $\alpha_1 + \alpha_p$ and*

$$\sum_{\pi=2}^{p-1} \alpha_\pi.$$

For let ζ be a partial sum other than the above, and $\zeta = 0$. The following cases may arise: (a) ζ contains neither α_1 nor α_p; in this case $|\zeta + \alpha_1| \leqslant 1$, contradicting (VIII); (b) ζ contains one of α_1, α_p; this directly contradicts (VIII) unless $\zeta = \alpha_1$ or $\zeta = \alpha_p$ or $\zeta = \alpha_1 + \alpha_2 + \ldots + \alpha_{p-1}$ or $\zeta = \alpha_2 + \alpha_3 + \ldots + \alpha_p$, which implies $\alpha_1 = 0$ or $\alpha_p = 0$ and reduces the number of vectors to $p' = p - 1$; (c) ζ contains both α_1 and α_p and at least another α_π; removal of α_p gives $|\zeta - \alpha_p| \leqslant 1$, again contradicting (VIII).

4.2. The desired rearrangment of the α_π will be obtained in three stages:

(1) a rearrangement $\beta_1, \beta_2, \ldots, \beta_p$;

(2) a trivial alteration $\gamma_1, \gamma_2, \ldots, \gamma_p$ of (1) obtained by placing α_1 first; here certain *special* partial sums

$$\sum_{\kappa=1}^{q} \gamma_\kappa, \quad \sum_{\kappa=1}^{q'} \gamma_\kappa, \ldots$$

with not too distantly spaced values of $q, q', \ldots$ have a modulus less than n (more precisely, less than a bound somewhat smaller than n);

(3) the final rearrangement $\delta_1, \delta_2, \ldots, \delta_p$ obtained from (2) by reordering the vectors within each group $\gamma_{q+1}, \ldots, \gamma_{q'}$ leading from one special partial sum to the next.

The β_π, γ_π, δ_π will be defined inductively as follows. Suppose an index i, $1 \leqslant i \leqslant p$, has been found such that

(i) β_ν have been selected from the α_π for $\nu < i$;

(ii) the non-selected vectors, $\epsilon_i, \ldots, \epsilon_p$, say, satisfy a relation

$$\sum_{\nu=i}^{p} e_\nu \epsilon_\nu = 0,$$

where

(iii) $0 < e_\nu \leqslant 1$ for $\nu < i + n$, $e_\nu = 1$ for $\nu \geqslant i + n$;

(iv) α_p is one of the ϵ_ν; and if the ϵ_ν other than α_p are p.d. then $\alpha_p = \epsilon_p$;

(v) if α_1 is one of the ϵ_ν, then $\alpha_1 = \epsilon_1$;

(vi a) if α_1 is one of the ϵ_ν, then $\gamma_1, \ldots, \gamma_i$ are the vectors $\alpha_1, \beta_1, \ldots, \beta_{i-1}$; and

$$\xi = \sum_{\nu=1}^{i} \gamma_\nu = \alpha_1 + \sum_{\nu=1}^{i-1} \beta_\nu$$

is the special partial sum belonging to the index i;

(vi b) if α_1 is one of the β_ν, $\alpha_1 = \beta_r$ say, then $\gamma_1, \ldots, \gamma_{i-1}$ are the vectors $\alpha_1, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_{i-1}$; and

$$\xi = \sum_{\nu=1}^{i-1} \gamma_\nu = \sum_{\nu=1}^{i-1} \beta_\nu$$

is the special partial sum belonging to i;

(vii) $|\xi| < (n^2 - 3n + 3)^{\frac{1}{2}} + 1$;

(viii) $\delta_1, \ldots, \delta_{i-1}, (\delta_i)$ are a rearrangement of $\gamma_1, \ldots, \gamma_{i-1}, (\gamma_i)$ with $\delta_1 = \gamma_1 = \alpha_1$;

(ix) $$\left|\sum_{\nu=1}^{q} \delta_\nu\right| < g(n) < n \qquad q = 1, \ldots, i-1, (i).$$

Such an index i will be called a *special* index.

The index $i = 1$ is special: (i) is void as no β's have to be selected; the given relation

$$\sum_{\pi=1}^{p} \alpha_\pi = 0$$

plays the role of (ii) $(\alpha_\pi = \epsilon_\pi)$; (iii), (iv), (v) are satisfied; defining $\delta_1 = \gamma_1 = \alpha_1 = \xi$, (vi) and (viii) are satisfied; (vii) and (ix) are trivial.

To every special index i, with $i < p - 2n$, a new special index $j > i$ will now be constructed (the construction will preserve the vectors $\beta_\nu, \gamma_\nu, \delta_\nu$ already selected for the index i).

4.3. Relation (ii) contains $p - (i-1) > 2n + i - (i-1) = 2n + 1$ terms. Applying (I) to $\epsilon_i, \ldots, \epsilon_p$, we select $n + 1$ p.d. vectors

$$\epsilon_{\mu_1}, \ldots, \epsilon_{\mu_{n+1}},$$

where we *include*

$$\epsilon_i = \epsilon_{\mu_{n+1}},$$

by (I.1), and *exclude* α_p, by (I.2), if possible (i.e., certainly when $\alpha_p = \epsilon_p$ and $\epsilon_i, \ldots, \epsilon_{p-1}$ are p.d.). If the relation of expressing p.d. is

(8) $$a_0 \epsilon_i + \sum_{j=1}^{n} a_j \epsilon_{\mu_j} = 0, \qquad a_j \geqslant 0, \text{ not all } a_j = 0,$$

then, for all x,

(9) $$\sum_{\nu=i}^{p} e_\nu \epsilon_\nu - x(a_0 \epsilon_i + \sum_{j=1}^{n} a_j \epsilon_{\mu_j}) = 0.$$

For $x = 0$ all coefficients are positive and $\leqslant 1$; hence a positive value of x can be determined for which (at least) one coefficient becomes 0, the others remaining $\geqslant 0$, $\leqslant 1$. At most $2n$ coefficients can be less than 1 (those of $\epsilon_i, \ldots, \epsilon_{i+n-1}, \epsilon_{\mu_1}, \ldots, \epsilon_{\mu_n}$), so that at least two coefficients remain equal to 1. Renaming the ϵ_ν: $\epsilon_i', \epsilon_{i+1}', \ldots, \epsilon_p'$, taking first the vector or vectors with coefficient 0, then the remaining ϵ_ν from $\epsilon_i, \ldots, \epsilon_{i+n-1}, \epsilon_{\mu_1}, \ldots, \epsilon_{\mu_n}$, and then the remaining ones with coefficient 1, (9) will read

(10) $$\sum_{\nu=i+1}^{p} e_\nu' \epsilon_\nu' = 0, \quad 0 \leqslant e_\nu' \leqslant 1 \text{ for } \nu < i + 2n, \quad e_\nu' = 1 \text{ for } \nu \geqslant i + 2n.$$

4.4. Put

(11) $$\epsilon = \sum_{\nu=i+1}^{i+2n-1} e_\nu' \epsilon_\nu' \qquad (0 \leqslant e_\nu' \leqslant 1),$$

so that (10) may be written

(12) $$\epsilon + \sum_{\nu=i+2n}^{p} \epsilon_\nu' = 0.$$

α_1 cannot be contained in the partial sum

$$\sum_{\nu=i+2n}^{p} \epsilon_\nu';$$

for if α_1 occurs in (9), then $\alpha_1 = \epsilon_i$ by (v), i.e. α_1 is one of $\epsilon_i', \ldots, \epsilon'_{i+2n-1}$; by (VIII.2) the partial sum cannot vanish, whence $\epsilon \neq 0$. By (II) ϵ can be written in the form

(13) $$\epsilon = \sum_{\nu=i+l}^{i+2n-1} f_\nu \phi_\nu, \; 0 < f_\nu \leqslant 1 \text{ for } \nu < i + l + n, \; f_\nu = 1 \text{ for } \nu \geqslant i + l + n,$$

where

(14) $$1 \leqslant l \leqslant 2n - 1,$$

and $\phi_{i+1}, \ldots, \phi_{i+2n-1}$ is a rearrangement of $\epsilon'_{i+1}, \ldots, \epsilon'_{i+2n-1}$. By (II.1) it may be assumed that

(15) if α_1 is still present in (13), then $\alpha_1 = \phi_{i+l}$.

Define

(16) $$j = i + l;$$

then, by (14),

(17) $$i + 1 \leqslant j \leqslant i + 2n - 1.$$

It will now be shown that j is a special index. The properties (i), . . ., (ix) relating to j will be denoted by (i′), . . ., (ix′).

4.5. (i′) By (i), β_ν is defined for $\nu < i$; defining

$$\beta_i = \epsilon_i', \quad \beta_{i+1} = \phi_{i+1}, \quad \ldots, \quad \beta_{j-1} = \phi_{j-1},$$

β_ν are selected for $\nu < j$.

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The non-selected vectors are $\phi_j, \ldots, \phi_{i+2n-1}$ and $\epsilon'_{i+2n}, \ldots, \epsilon_p'$ which will be renamed $\phi_{i+2n}, \ldots, \phi_p$. Substituting (13) into (12), we get

(ii′) $$\sum_{\nu=j}^{p} f_\nu \phi_\nu = 0,$$

where

(iii′) $$0 < f_\nu \leqslant 1 \text{ for } \nu < j + n, \quad f_\nu = 1 \text{ for } \nu \geqslant j + n;$$

note also that

(18) $$f_\nu = 1 \text{ for } \nu \geqslant i + 2n;$$

in particular,

(19) $$f_{p-1} = f_p = 1.$$

(iv′) α_p is one of the ϕ_ν $(\nu \geqslant j)$. For, *either* the ϵ_μ other than α_p are p.i.; then, *a fortiori*, the ϕ_ν other than α_p are p.i.; but (ii′) expresses the p.d. of the ϕ_ν other than α_p unless α_p is present in (ii′); *or* the ϵ_ν other than α_p are p.d.; then $\alpha_p = \epsilon_p$ by (iv), and α_p was excluded from (8), so that $\alpha_p = \epsilon_p = \epsilon_p' = \phi_p$. This latter case certainly arises if the ϕ_ν other than α_p are p.d., for this implies the p.d. of the ϵ_μ other than α_p.

(v′) If α_1 is one of the ϕ_ν $(\nu \geqslant j)$, then $\alpha_1 = \phi_j$, by (15), (16).

(vi′ a) If α_1 is one of the ϕ_ν, then $\gamma_1, \ldots, \gamma_j$ are the vectors $\alpha_1, \beta_1, \ldots, \beta_{j-1}$: and

$$\eta = \sum_{\nu=1}^{j} \gamma_\nu = \alpha_1 + \sum_{\nu=1}^{j-1} \beta_\nu$$

will be defined as the special partial sum belonging to the index j;

(vi′ b) if $\alpha_1 = \beta_r$, $r < j$, then $\gamma_1, \ldots, \gamma_{j-1}$ are the vectors $\alpha_1, \beta_1, \ldots, \beta_{r-1}, \beta_{r+1}, \ldots, \beta_{j-1}$; and

$$\eta = \sum_{\nu=1}^{j-1} \gamma_\nu = \sum_{\nu=1}^{j-1} \beta_\nu.$$

These definitions are consistent with the definitions (vi).

4.6. We now investigate the special partial sum η.

In case (vi′ a)

$$\eta = \alpha_1 + \sum_{\nu=1}^{j-1} \beta_\nu = \alpha_1 - \sum_{\nu=j}^{p} \phi_\nu = - \sum_{\nu=j+1}^{p} \phi_\nu \qquad \text{by (v′)}$$

$$= - \sum_{\nu=j+1}^{p} \phi_\nu + \sum_{\nu=j}^{p} f_\nu \phi_\nu \qquad \text{by (ii′)}$$

$$= f_j \alpha_1 - \sum_{\nu=j+1}^{p} (1 - f_\nu) \phi_\nu,$$

and as $f_\nu = 1$ for $\nu \geqslant \min(j + n, i + 2n)$ by (iii′) and (18),

(20) $$\eta = f_j \alpha_1 - \sum_{\nu=j+1}^{j+k-1} (1 - f_\nu) \phi_\nu,$$

where

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$$j + k - 1 = \min(j + n - 1, i + 2n - 1)$$

i.e., by (16),

(21) $$k = \min(n, 2n - (j - i)) = \min(n, 2n - l),$$

whence, by (14), $1 \leqslant k \leqslant n$. The case $k = 1$ can be excluded as it would imply $|\eta| = |f_j \alpha_1| \leqslant 1$ where η is a partial sum with $j = 2n + i - 1$ terms ($2n \leqslant j \leqslant p - 2$), including α_1, excluding α_p, which contradicts (VIII). Thus,

(22) $$2 \leqslant k \leqslant n, \quad 1 \leqslant l \leqslant 2n - 2, \quad i + 1 \leqslant j \leqslant i + 2n - 2.$$

As $|\alpha_1| \leqslant 1$, $|\phi_\nu| \leqslant 1$, $0 < f_j \leqslant 1$, $0 \leqslant 1 - f_\nu < 1$, and, by (VIII.1) $|\alpha_1 + \phi_\nu| > 1$, except, possibly, for $\phi_\nu = \alpha_p$, (20) satisfies the conditions of (III), and we have

(23) $$|\eta| < (k^2 - 3k + 3)^{\frac{1}{2}} + 1.$$

As $k \leqslant n$, this implies

(vii′) $|\eta| < (n^2 - 3n + 3)^{\frac{1}{2}} + 1.$

In case (vi′b),

$$\eta = \sum_{\nu=1}^{j-1} \beta_\nu = -\sum_{\nu=j}^{p} \phi_\nu = -\sum_{\nu=j}^{p} \phi_\nu + \sum_{\nu=j}^{p} f_\nu \phi_\nu \qquad \text{by (ii′)}$$

$$= \sum_{\nu=j}^{p} (1 - f_\nu)(-\phi_\nu) = \sum_{\nu=j}^{j+k-1} (1 - f_\nu)(-\phi_\nu),$$

by (iii′) and (18), where k is defined by (21); $k = 1$ would imply $|(1 - f_j)\phi_j| = |\eta| \leqslant 1$, hence can be excluded as above; thus, (22) will hold and η may be written

(24) $$\eta = \sum_{\nu=j}^{j+k-2} (1 - f_\nu)(-\phi_\nu) + (1 - f_{j+k-1})(-\phi_{j+k-1}).$$

We may assume that $\alpha_p = \phi_p$ or $\alpha_p = \phi_{j+k-1}$, so that the partial sum

$$\zeta = \sum_{\nu=j+k-1}^{p} \phi_\nu$$

contains α_p, but not α_1, and $f = p - (j + k - 1)$ further terms; $j \geqslant 2$, $k \geqslant 2$ imply $f \leqslant p - 3$; (21) and $i < p - 2n$ imply $f > 1$; hence, by (VIII),

$$|\zeta| > 1.$$

Now,

(25) $$\zeta = \sum_{\nu=j+k-1}^{p} \phi_\nu - \sum_{\nu=j}^{p} f_\nu \phi_\nu$$

$$= \sum_{\nu=j}^{j+k-2} f_\nu(-\phi_\nu) - (1 - f_{j+k-1})(-\phi_{j+k-1}).$$

(24), (25) satisfy the conditions of (IV); hence,

(26) $$|\eta| < k - (2 - \sqrt{2}),$$

which implies (23) and (vii′).

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4.7. It remains to establish (viii′) and (ix′). By (vi) and (vi′), the three possibilities are:

$$(27)\qquad \begin{cases} \eta = \xi + \sum_{\nu=i}^{j-1} \gamma_\nu, \\ \eta = \xi + \sum_{\nu=i+1}^{j} \gamma_\nu, \\ \eta = \xi + \sum_{\nu=i+1}^{j-1} \gamma_\nu. \end{cases}$$

The γ_ν contained in ξ have already been rearranged as δ_ν according to (viii) to satisfy (ix); it therefore remains to reorder the γ_ν under the summation sign in (27). There are m such γ_ν, where $m = j - i = l$ or $m = j - i - 1 = l - 1$, i.e., $1 \leqslant m \leqslant l$. (The case $m = 0$ is trivial, since then $\eta = \xi$, $\beta_i = \epsilon_i = \alpha_1$ and the vectors considered in (viii′), (ix′) are identical with those of (viii), (ix).) We distinguish two cases:

(1) $2 \leqslant k \leqslant n - 1$. By (21), $k = 2n - l$, $1 \leqslant m \leqslant 2n - k$. Together with (vii) and (23), these are the conditions of (V.2) for (27) which guarantee the required reordering (viii′) of the γ_ν satisfying (ix′), the bound obtained being $(n^2 - 3n + 3)^{\frac{1}{2}} + 1$.

(2) $k = n$. By (21), $n \leqslant 2n - l$, whence $m \leqslant l \leqslant n$, and by (vii) and (vii′), (27) satisfies the conditions of (VII.1) which guarantee the required reordering (viii′) of the γ_ν satisfying (ix′), the bound $g(n)$ being defined as in the proof of (VII.1).

As $g(n)$ is greater than $(n^2 - 3n + 3)^{\frac{1}{2}} + 1$, the bound $g(n)$ may also be used in case (1).

This completes the proof that j is a special index.

4.8. The procedure of selecting the β_ν, γ_ν, δ_ν can be continued until a special index i is reached for which $i \geqslant p - 2n$. In this case $\delta_1, \ldots, \delta_{i-1}$ or $\delta_1, \ldots, \delta_i$ have been correctly selected, and the corresponding special partial sum is

$$\xi = \sum_{\nu=1}^{i-1} \delta_\nu$$

or

$$\xi = \sum_{\nu=1}^{i} \delta_\nu.$$

If the remaining vectors are called $\gamma_i, \ldots, \gamma_p = \alpha_p$ or $\gamma_{i+1}, \ldots, \gamma_p = \alpha_p$ respectively, then

$$\eta = -\alpha_p = \xi + \sum_{\nu=i}^{p-1} \gamma_\nu$$

or

$$\eta = -\alpha_p = \xi + \sum_{\nu=i+1}^{p-1} \gamma_\nu$$

satisfies the conditions of (V.3), because the number of γ_ν is $m = p - i$ or $p - i - 1$, whence $m \leqslant 2n < 2n^2 - 4n + 3$ (for $n \geqslant 3$). Reordering the γ_ν according to (V.3), and choosing α_p as the last vector, the rearrangement of the given vectors is completed.

5. THEOREM 2. $c_3 \leqslant (5 + 2\sqrt{3})^{\frac{1}{2}} \simeq 2.91$.

Proof. For any special index i $(1 \leqslant i \leqslant p - 7)$, relation (ii) of §4.2 reads $(n = 3)$

$$\sum_{\nu=i}^{i+2} e_\nu \epsilon_\nu + \sum_{\nu=i+3}^{p} \epsilon_\nu = 0, \qquad 0 < e_\nu \leqslant 1.$$

We shall prove (cf. (vii)) that

$$|\xi| < 1 + \sqrt{2},$$

unless both α_1, α_p are present in

$$\sum_{\nu=i}^{i+2} e_\nu \epsilon_\nu$$

and the coefficient of the third vector is less than 1 (in this case (vii) gives $|\xi| < 1 + \sqrt{3}$). If α_1 is not present in

$$\sum_{\nu=i}^{i+2} e_\nu \epsilon_\nu,$$

the reasoning of 4.6, (vi′b) applies leading to (26), which for $k \leqslant 3$ gives the estimate $1 + \sqrt{2}$. If α_1 is present, α_p absent, then

$$\xi = e_i\alpha_1 - (1 - e_{i+1})\,\epsilon_{i+1} - (1 - e_{i+2})\,\epsilon_{i+2};$$

if

$$\zeta = \epsilon_{i+3} + \ldots + \epsilon_p = -\,e_i\alpha_1 - e_{i+1}\epsilon_{i+1} - e_{i+2}\epsilon_{i+2},$$

then $|\zeta| > 1$, by (VIII), and $|\xi| < 1 + \sqrt{2}$, by (IV). If, finally, α_1, α_p are both present, but the coefficient of the third vector is 1, then

$$|\xi| = |e_i\alpha_1 - (1 - e_{i+1})\,\alpha_p| \leqslant 2 < 1 + \sqrt{2}.$$

The relation between the special partial sums ξ, η belonging to two successive special indices i, j is given by (27), where $m \leqslant l \leqslant 2n - 2 = 4$. We distinguish two cases:

(1) One of the two partial sums, say η, has modulus less than $1 + \sqrt{2}$, i.e.,

$$|\xi| < 1 + \sqrt{3}, \quad |\eta| < 1 + \sqrt{2}.$$

If the γ_ν in (27) are called $\theta_1, \ldots, \theta_m$, then

$$\eta = \xi + \sum_{\mu=1}^{m} \theta_\mu.$$

Let $\theta_1', \ldots, \theta_m'$ be the rearrangement of $\theta_1, \ldots, \theta_m$ according to the principle used in (V). Then, for $m = 4$,

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$$(\xi + \theta_1')^2 \leqslant \tfrac{1}{2}\xi^2 + \tfrac{1}{2}|\xi| \cdot |\eta| + 1$$

$$< \tfrac{1}{2}(\sqrt{3} + 1)^2 + \tfrac{1}{2}(\sqrt{3} + 1)(\sqrt{2} + 1) + 1 < 8.04,$$

$$|\xi + \theta_1'| < 2.84,$$

$$(\xi + \theta_1' + \theta_2')^2 \leqslant \tfrac{1}{3} \times 8.04 + \tfrac{2}{3} \times 2.84 \times 2.42 + 1 < 8.27,$$

$$|\xi + \theta_1' + \theta_2'| < 2.88,$$

$$(\xi + \theta_1' + \theta_2' + \theta_3')^2 \leqslant 2.88 \times 2.42 + 1 < 7.97,$$

$$|\xi + \theta_1' + \theta_2' + \theta_3'| < 2.83;$$

the maximum estimate, 2.88, is less than $(5 + 2\sqrt{3})^{\frac{1}{2}}$. The cases $m < 4$ are treated in the same way.

(2) The estimate $1 + \sqrt{2}$ is not available for either of ξ, η. This means, by (vii), that both (ii) and (ii′) contain α_1 and α_p in their first three terms, the coefficient of the third term being less than 1. By (iv), the ϵ_ν other than α_p are p.i.; hence (8) contains $\alpha_1 = \epsilon_i$, $\alpha_p = \epsilon_{i+1}$, and two other vectors ϵ_{μ_1}, ϵ_{μ_2}. In the transition from (ii) via (8)–(13) to (ii′), α_1, α_p are retained together with at least one of ϵ_{i+2}, ϵ_{μ_1}, ϵ_{μ_2}, i.e., at most two vectors are eliminated. Hence, $m = l = j - i \leqslant 2$; $m = 1$ means $\eta = \xi + \theta_1$ which requires no reordering; $m = 2$ means $\eta = \xi + \theta_1 + \theta_2$, and θ_1' can be selected from θ_1, θ_2 such that

$$(\xi + \theta_1')^2 < (1 + \sqrt{3})^2 + 1 = 5 + 2\sqrt{3},$$

$$|\xi + \theta_1'| < (5 + 2\sqrt{3})^{\frac{1}{2}}.$$

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