APPLYING THE CAUCHY INTEGRAL

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Introduction. It could perhaps be reasonably maintained that for most students of calculus the definite integral $\int_a^b f(x)dx$ is in fact the Cauchy Integral. That is to say, $\int_a^b f(x)dx = F(b) - F(a)$, where F is a primitive of f.

If done heuristically, the discussion of the Riemann integral serves two purposes. Firstly, it indicates that every continuous function has a primitive. In the second place it provides the student with an intuitive basis which will probably make him accept, for example, the definition of the surface area of a surface of revolution as $\int_a^b 2\pi x f(x)\sqrt{1+[f'(x)]^2}\,dx$.

If the definite integral $\int_a^b f(x)dx$ is defined to be $F(b) - F(a)$, where $F'(x) = f(x)$ for all but a countable number of points of (a, b), then there is the problem of using this definition to justify, for example, the definition of surface area. In a recent calculus book by Lang this problem is solved for area, work and moments. The other applications of the definite integral are treated by means of Riemann sums.

I propose to show that the standard applications of the definite integral can be discussed by using this definition of the integral, that is by using the Cauchy Integral. An advantage of this procedure is that the problem of infinitesimals of higher order, which occurs in the method of shells and surface area, is completely eliminated. In other words, the use of the Principle of Duhamel is avoided.

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Examples

1. Let $0 \leq a \leq b$ and let $f(x)$ be a continuous non-negative function on $[a, b]$. Denote by V_c^d, $a \leq c \leq d \leq b$, the volume of the solid obtained by revolving $y = f(x)$, $c \leq x \leq d$, about the y-axis. Then it appears that

(I_1) $\pi m_{cd}(d^2 - c^2) \leq V_c^d \leq \pi M_{cd}(d^2 - c^2)$, and

(I_2) $V_c^d + V_d^e = V_c^e$, $c \leq d \leq e$, where M_{cd} (m_{cd}) is the maximum (minimum) of f on $[c, d]$.

From these properties, it follows easily that

$$m_h \frac{\mu(h)}{h} \leq \frac{V_a^{x+h} - V_a^x}{h} \leq M_h \frac{\mu(h)}{h}, \quad \text{where}$$

M_h (m_h) is the maximum (minimum) of f on the closed interval bounded by x and $x+h$, and $\mu(h) = \pm\int_x^{x+h} 2\pi t \, dt$ according as $h > 0$ or $h < 0$. Letting $h \to 0$, it follows from the squeezing process that $\frac{d}{dx} V_a^x = 2\pi x f(x)$. Hence, $V_a^b = \int_a^b 2\pi x f(x) dx$.

2. Let $x(t)$, $y(t)$ be two continuously differentiable functions with $a \leq t \leq b$. The length L_c^d of the arc defined by x and y for $c \leq t \leq d$ is defined to be $\sup \sum_{i=1}^{n} \sqrt{[x(t_i)-x(t_{i-1})]^2 + [y(t_i)-y(t_{i-1})]^2}$, where P ranges over the partitions $c = t_o < t_1 < \ldots < t_n = d$ of $[c, d]$.

Denote by M'_{cd} and N'_{cd} $(m'_{cd}$ and $n'_{cd})$ the maximum (minimum) of $(x')^2$ and $(y')^2$ on $[c, d]$. Then,

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(I_1) $\sqrt{m'_{cd} + n'_{cd}} \cdot (d-c) \le L_c^d \le \sqrt{M'_{cd} + N'_{cd}} \cdot (d-c)$, and

(I_2) $L_c^d + L_d^e = L_c^e$ if $c \le d \le e$.

The difference quotient satisfies

$$\sqrt{m'_h + n'_h} \le \frac{L_a^{t+h} - L_a^t}{h} \le \sqrt{M'_h + N'_h}\ ,$$ where M'_h and N'_h (m'_h and n'_h) denote the maximum (minimum) of $(x')^2$ and $(y')^2$ on the closed interval bounded by x and $x+h$. Hence,

$$\frac{d}{dt} L_a^t = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$ and so

$$L_a^b = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt.$$

3. Consider the surface obtained by rotating the curve in example 2 about the x-axis. Assume $y(t) \ge 0$, $a \le t \le b$.

Denote by S_c^d the area of the surface obtained by rotating the part of the curve corresponding to $c \le t \le d$ about the x-axis. Then, assuming that the area of the surface determined increases if the curve is replaced by a longer curve which is further away from the x-axis than the original one, it can be seen that

(I_1) $2\pi n_{cd} \sqrt{m'_{cd} + n'_{cd}}\ (d-c) \le S_c^d \le 2\pi N_{cd} \sqrt{M'_{cd} + N'_{cd}}\ (d-c)$, and

(I_2) $S_c^d + S_d^e = S_c^e$ if $c \le d \le c$, where $N_{cd}(n_{cd})$ is the maximum (minimum) of y on $[c, d]$.

From this it follows that

$$2\pi\, n_h \sqrt{m'_h + n'_h} \le \frac{S_a^{t+h} - S_a^t}{h} \le 2\pi N_h \sqrt{M'_h + N'_h}\ .$$

and so $\frac{d}{dt} S_a^t = 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2}$. Hence

$$S_a^b = \int_a^b 2\pi y(t) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.$$

The principle of Duhamel. These three examples, together with the examples of area and work discussed in Lang's Calculus book, should suffice to show how one can apply the Cauchy Integral directly.

However, it is perhaps of some interest to extract from these examples a general principle which covers them all. Since it accomplishes the same thing as Duhamel's Principle it could be given the same name.

Let $A \subseteq R^n$ be a closed set with the following property: if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are in A then the vectors u and v are in A where $u_i = \max\{x_i, y_i\}$ and $v_i = \min\{x_i, y_i\}$. Denote by φ a continuous real-valued function on A.

Let $[a, b]$ be a closed interval in R and let $\theta(t)$, $a \leq t \leq b$, be a continuous real-valued function. If $[c, d] \subseteq [a, b]$ denote $\int_c^d \theta(t)dt$ by $\mu([c, d])$.

Assume $f_1, f_2, \ldots, f_n$ are n real-valued functions on $[a, b]$ with $(f_1(t), f_2(t), \ldots, f_n(t)) \in A$ if $a \leq t \leq b$. For $a \leq c \leq d \leq b$, let M_{cd}^i (m_{cd}^i) denote the maximum (minimum) of f_i on $[c, d]$, and let M_{cd} (m_{cd}) denote the vector $(M_{cd}^1, M_{cd}^2, \ldots, M_{cd}^n)$ $((m_{cd}^1, m_{cd}^2, \ldots, m_{cd}^n))$. The conditions imposed on A ensure that M_{cd} and m_{cd} are in A.

THEOREM. Assume that with any pair of numbers $c \leq d$ in $[a, b]$ we can associate a number I_c^d with the following properties:

(I_1) $\varphi(m_{cd})\,\mu([c,d] \leq I_c^d \leq \varphi(M_{cd})\,\mu([c,d])$; and

(I_2) $I_c^d + I_d^e = I_c^e$ if $c \leq d \leq e$.

Then, $\frac{d}{dt} I_a^t = \varphi(f_1(t), f_2(t), \ldots, f_n(t))\ \theta(t)$ and hence

$$I_a^b = \int_a^b \varphi(f_1(t), f_2(t), \ldots, f_n(t))\ \theta(t)dt.$$

Conversely, if φ is increasing (i.e. $x_i \leq y_i$, $1 \leq i \leq n$, implies $\varphi(x) \leq \varphi(y)$) and θ is non-negative, then

$$I_c^d = \int_c^d \varphi(f_1(t), f_2(t), \ldots, f_n(t))\ \theta(t)dt$$

satisfies (I_1) and (I_2).

Proof. Let m_h denote $m_{t(t+h)}$ if $h > 0$ and $m_{(t+h)t}$ if $h < 0$. Similarly define M_h and $\mu(h)$.

Then, it follows from properties (I_1) and (I_2) that

$$\varphi(m_h)\ \frac{\mu(h)}{h} \leq \frac{I_a^{t+h} - I_a^t}{h} \leq \varphi(M_h)\ \frac{\mu(h)}{h} \quad \text{for } h \neq 0.$$

Hence, taking the limits as $h \to 0$, we obtain by the squeezing process that

$$\frac{d}{dt} I_a^t = \varphi(f_1(t), f_2(t), \ldots, f_n(t))\ \theta(t).$$

For the converse, the hypotheses on φ and θ imply that (I_1) is satisfied. It is clear that (I_2) is satisfied.

REMARK. To cover the situations discussed in the examples it is necessary that A be allowed to be a proper subset of R^n. Note that in the second example, $n = 2$,

$$A = \{(u,v) \mid u \geq, v \geq 0\}, \quad \text{and} \quad \varphi(u,v) = \sqrt{u+v}.$$

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