ASYMMETRIC PERIODIC ORBITS IN THE THREE-BODY PROBLEM AND THEIR STABILITY

A. Tsouroplis and C.G. Zagouras
University of Patras, Patras, Greece

ABSTRACT

An algorithm for the numerical determination of asymmetric periodic solutions of the planar general three body problem is described. The elements of the "variational" matrix which are used in this algorithm are computed by numerical integration of the corresponding "variational" equations. These elements are also used in the study of the linear iso-energetic stability. A number of asymmetric periodic orbits are presented and their stability parameters are given.

1. NUMERICAL DETERMINATION OF ASYMMETRIC PERIODIC SOLUTIONS

We use a rotating system of dimensionless coordinates with origin at the center of mass of the two more massive bodies P_1 and P_2. The position of the three-body system is fully determined in terms of the coordinates x,y of the third body P_3, the distance x_2 of P_2 from the origin and the angle θ between the rotating and a non-rotating system.

In the rotating coordinate system the Equations of motion of the planar general three body problem are

$$\begin{aligned} \ddot{x} &= Bx + x\dot{\theta}^2 + 2\dot{\theta}\dot{y} + \ddot{\theta}y + \mu A x_2 \\ \ddot{y} &= (B+\dot{\theta}^2)y - x\ddot{\theta} - 2\dot{x}\dot{\theta} , \\ \ddot{x}_2 &= (m_3B^* + \dot{\theta}^2)x_2 - (1-m_3)(1-\mu)^3/x_2^2 + m_3(1-\mu)Ax, \\ \ddot{\theta} &= -2\dot{\theta}\dot{x}_2/x_2 + m_3(1-\mu)Ay/x_2, \end{aligned} \qquad (1)$$

or in first-order form:

$$\frac{dx_1}{dt} = x_4 \triangleq f_1, \quad \frac{dx_2}{dt} = x_5 \triangleq f_2, \quad \frac{dx_3}{dt} = x_6 \triangleq f_3$$

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$$\frac{dX_4}{dt} = BX_1 + X_1X_8^2 + 2X_8X_5 + X_8X_2 + \mu AX_3 \triangleq f_4,$$

$$\frac{dX_5}{dt} = (B + X_8^2)X^2 - X_1\dot{X}_8 - 2X_4X_8 \triangleq f_5,$$

$$\frac{dX_6}{dt} = (m_3B^* + X_8^2)X_3 - (1-m_3)(1-\mu)^3/X_3^2 + m_3(1-\mu)AX_1 \triangleq f_6, \qquad (2)$$

$$\frac{dX_7}{dt} = X_8 \triangleq f_7,$$

$$\frac{dX_8}{dt} = -2X_8X_6/X_3 + m_3(1-\mu)AX_2/X_3 \triangleq f_8,$$

where

$$(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = (x, y, x_2, \dot{x}, \dot{y}, \dot{x}_2, \theta, \dot{\theta}).$$

A periodic solution $\underline{X}(\underline{X}_o;\, t)$ of the above Equations will satisfy

$$X_i(\underline{X}_o;\, t+T) = X_i(\underline{X}_o; t), \quad i \neq 7 \qquad (3)$$

where T is the period and $\underline{X}_o = (X_{o1}, \ldots, X_{o8})$ is the initial-conditions vector. Further, without loss of generality, we shall fix initial values of y, θ and $\dot{\theta}$ as follows: $y_o = 0$, $\theta_o = 0$, $\dot{\theta}_o = 1$.
The periodicity conditions are written in the form:

$$x(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = x_0\,, \qquad (a)$$

$$y(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = y_0\,, \qquad (b)$$

$$x_2(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = x_{20}\,, \qquad (c)$$

$$\dot{x}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = \dot{x}_0\,, \qquad (d) \quad (4)$$

$$\dot{y}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = \dot{y}_0\,, \qquad (e)$$

$$\dot{x}_2(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = x_{20}, \qquad (f)$$

$$\dot{\theta}(x_0, x_{20}, \dot{x}_0, \dot{y}_0, \dot{x}_{20};\, T) = \dot{\theta}_0 \qquad (g)$$

In practice condition (4b) is satisfied "by force" since we start and terminate the numerical integration when the orbit crosses the Ox axis. Further, due to the integrals of the problem only four of the remaining six periodicity conditions are trully independent. Essentially, therefore, the periodicity conditions are only four and in this work we

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have used the conditions (4a, c,d,f).

From these periodicity conditions corrector-predictor algorithms can be established for the numerical determination of entire series of asymmetric periodic solutions. In the corrector phase we assume an initial state vector $\underline{X}_0$ which approximately leads to a periodic orbit of (approximate) period T, and seek to adjust this state vector by differential corrections to improve iteratively the accuracy of periodicity.

If we integrate the Equations of motion and stop at the second crossing with the Ox-axis (after one full revolution), we have in general

$$\underline{X}(\underline{X}_0 \; ; \; T) \neq \underline{X}_0 \, .$$

We seek corrections $\delta\underline{X}_0 = (\delta x_0, 0, \delta x_{02}, \delta\dot{x}_0, \delta\dot{y}_0, \delta\dot{x}_{20}, 0, 0)$ such that

$$\underline{X}\,(\underline{X}_0 + \delta\underline{X}_0 \; ; \; T + \delta T) = \underline{X}_0 + \delta\underline{X}_0 \, . \tag{5}$$

Expanding in Taylor series and neglecting terms of order higher that the first, we shall have

$$X_i + \frac{\partial X_i}{\partial X_{01}}\delta X_{01} + \frac{\partial X_i}{\partial X_{03}}\delta X_{03} + \frac{\partial X_i}{\partial X_{04}}\delta X_{04} + \frac{\partial X_i}{\partial X_{05}}\delta X_{05} + \frac{\partial X_i}{\partial X_{06}} X_{06} + \frac{\partial X_i}{\partial T}\delta T = X_{0i} + \delta X_{0i} \, ,$$

$$(i = 1,2,3,4,6). \tag{6}$$

For i=2 we obtain in particular,

$$\frac{\partial X_2}{\partial X_{01}}\delta X_{01} + \frac{\partial X_2}{\partial X_{03}}\delta X_{03} + \frac{\partial X_2}{\partial X_{04}}\delta X_{04} + \frac{\partial X_2}{\partial X_{05}}\delta X_{05} + \frac{\partial X_2}{\partial X_{06}}\delta X_{06} + \frac{\partial X_2}{\partial T}\delta T = 0 \, , \tag{7}$$

since, for t=T, $x_2 = y = 0$ while $\delta X_{02} = \delta y_0 = 0$. Solving now Equations (7) for δT and substituting into relations (6) we get

$$X_i + u_{i1}\delta X_{01} + u_{i3}\delta X_{03} + u_{i4}\delta X_{04} + u_{i5}\delta X_{05} + u_{i6}\delta X_{06} = X_{0i} + \delta X_{0i}, \quad i = 1,3,4,6. \tag{8}$$

where

$$u_{ij} = \frac{\partial x_i}{\partial x_{0j}} - \frac{\partial x_2}{\partial x_{0j}} \frac{f_i}{f_2}, \qquad i=1,3,4,6, \tag{9}$$

("variations at the crossing"; Markellos, 1977).

Assuming X_{04} constant or equivalently $\delta X_{04} = 0$, Equations (8) become

$$\begin{aligned}
(u_{11}-1)\,\delta X_{01} + u_{13}\delta X_{03} + u_{15}\delta X_{05} + u_{16}\delta X_{06} &= X_{01}-X_1, \\
u_{31}\delta X_{01} + (u_{33}-1)\delta X_{03} + u_{35}\delta X_{05} + u_{36}\delta X_{06} &= X_{03}-X_3, \\
u_{41}\delta X_{01} + u_{43}\delta X_{03} + u_{45}\delta X_{05} + u_{46}\delta X_{06} &= X_{04}-X_4, \\
u_{61}\delta X_{01} + u_{63}\delta X_{03} + u_{65}\delta X_{05} + (u_{66}-1)\delta X_{06} &= X_{06}-X_6 \;.
\end{aligned} \tag{10}$$

This system is the corrector of the algorithm. It is solved for the corrections δX_{01}, δX_{03}, δX_{05}, δX_{06}, which are then added to the corresponding components of the initial state vector to obtain a better approximation to the periodic orbit with period $T + \delta T$.

After repeated applications of the corrector we find (assuming convergence) the periodic (to the desired accuracy) solution characterized by the value X_{04} which is kept constant during the correction process.

We then proceed to a single application of the predictor:

$$\begin{aligned}
(u_{11}-1)\Delta X_{01} + u_{13}\Delta X_{03} + u_{15}\Delta X_{05} + u_{16}\Delta X_{06} &= -u_{14}\Delta X_{04}, \\
u_{31}\Delta X_{01} + (u_{33}-1)\Delta X_{03} + u_{35}\Delta X_{05} + u_{36}\Delta X_{06} &= -u_{34}\Delta X_{04}, \\
u_{41}\Delta X_{01} + u_{43}\Delta X_{03} + u_{45}\Delta X_{05} + u_{46}\Delta X_{06} &= (1-u_{44})\Delta X_{04}, \\
u_{61}\Delta X_{01} + u_{63}\Delta X_{03} + u_{65}\Delta X_{05} + (u_{66}-1)\Delta X_{06} &= -u_{64}\Delta X_{04}.
\end{aligned} \tag{11}$$

This predictor is designed to obtain the approximate initial state vector $\underline{X}_0 + \Delta\underline{X}_0$ corresponding to another periodic orbit (along the family), characterized by the value $X^*_{04} = X_{04} + \Delta X_{04}$, where the "increment" $\Delta\underline{X}_{04}$ is arbitrary but small so that convergence of the subsequent application of the corrector is secured. The values of the "sensitivities" u_{ij} involved in Equations (10) and (11) are computed from relations (9), where the "variations" $\partial x_i/\partial x_{0j}$ are known through numerical integration of the linear variational Equations:

$$\frac{dV}{dt} = P\,V,$$

where

$$V = (v_{ij}) = (\partial X_i/\partial X_{0j})$$

and

$$P = \left(\frac{\partial f_i}{\partial x_j}\right), \qquad i,j = 1,\dots,8.$$

2. STABILITY

IF $\underline{X}_0$ is the vector, in phase space, corresponding to a periodic orbit and $\underline{X}_0 + \delta\underline{X}_0$ is the vector of a neighboring orbit corresponding to the same value of the energy and angular momentum integrals, then a transformation T is constructed which transforms the initial state $\underline{X}_0$ to the state $\underline{X}$ when the orbit crosses the surface of section $X_2=Y=0$ for the second time (simple orbits). This transformation is expressed as

$$\underline{X} = \underline{\sigma}(\underline{X}_0), \tag{13}$$

where

$$\underline{\sigma} = (\sigma_1, \sigma_3, \sigma_4, \sigma_6). \tag{14}$$

After linearization, the transformation (13) is written

$$\delta\underline{X} = A\,\delta\underline{X}_0 \tag{15}$$

where

$$\delta\underline{X} = (\delta X_1, \delta X_3, \delta X_4, \delta X_6)^T,$$

$$\delta\underline{X}_0 = (\delta X_{01}, \delta X_{03}, \delta X_{04}, \delta X_{06})^T, \tag{16}$$

and A is the 4x4 matrix with elements the first partial derivatives of the functions $(\sigma_1, \sigma_3, \sigma_4, \sigma_6)$ with respect to the initial conditions, i.e.

$$A = (\alpha_{ij}) = \left(\frac{\partial \sigma_i}{\partial x_{0j}}\right), \quad i,j = 1,3,4,6 \tag{17}$$

The conditions for stability are:

$$\Delta > 0, \quad |p| < 2, \quad |q| < 2, \tag{18}$$

where

$$\Delta = \alpha^2 - 4(\beta - 2), \quad p = \frac{1}{2}(\alpha + \sqrt{\Delta}), \quad q = \frac{1}{2}(\alpha - \sqrt{\Delta}) \tag{19}$$

TABLE I: The series A_{20} of asymmetric periodic orbits of the planar general three body problem for $\mu = 0.25$ and $x_{04} = -0.17292$.

	m_3	x_{01}	x_{03}	x_{05}	x_{06}	E	p	q
1	0.000103	-2.33048	0.749905	1.90139	-0.00037	-0.093803	-1.998	-39.19
2	0.001203	-2.32672	0.749145	1.89876	-0.002430	-0.094241	-2.037	-36.25
3	0.014009	-2.31258	0.743557	1.89264	-0.013446	-0.097700	-2.629	-36.98
4	0.036509	-2.31138	0.736489	1.90284	-0.024301	-0.102142	-3.983	-46.22
5	0.050629	-2.31501	0.732556	1.91394	-0.029892	-0.104733	-5.030	-54.27
6	0.078649	-2.32598	0.725141	1.93860	-0.038778	-0.109138	-7.234	-73.04
7	0.100269	-2.33609	0.719556	1.95911	-0.044647	-0.112165	-9.119	-90.75
8	0.119689	-2.34574	0.714556	1.97801	-0.049404	-0.114642	-10.93	-109.3
9	0.134089	-2.35307	0.710838	1.99217	-0.052676	-0.116338	-12.37	-124.7
10	0.150109	-2.36131	0.706676	2.00797	-0.056093	-0.118088	-14.07	-143.9
11	0.170189	-2.37167	0.701409	2.02777	-0.060084	-0.120083	-16.38	-171.2
12	0.190009	-2.38185	0.696143	2.04723	-0.063737	-0.121838	-18.87	-201.8
13	0.200000	-2.38694	0.693459	2.05699	-0.065480	-0.122642	-20.21	-218.9

The period of the orbits varies from T = 12.4773 (orbit 1) to T = 10.9992 (orbit 13).

and

$$\alpha = -(\alpha_{11} + \alpha_{33} + \alpha_{44} + \alpha_{66}) \quad (20)$$

$$\beta = \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{14} \\ \alpha_{41} & \alpha_{44} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{16} \\ \alpha_{61} & \alpha_{66} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{34} \\ \alpha_{43} & \alpha_{44} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{36} \\ \alpha_{63} & \alpha_{66} \end{vmatrix} + \begin{vmatrix} \alpha_{44} & \alpha_{46} \\ \alpha_{64} & \alpha_{66} \end{vmatrix}, \quad (21)$$

(Hadjidemetriou, 1975). The elements a_{ij} can be determined as functions of the elements v_{ij} of the "variational" matrix through the expressions:

$$a_{1i} = (v_{1i} - \frac{x_4}{x_5} v_{2i}) + (v_{15} - \frac{x_4}{x_5} v_{25})D_{i5} + (v_{18} - \frac{x_4}{x_5} v_{28})D_{i8},$$

$$a_{3i} = (v_{3i} - \frac{x_6}{x_5} v_{2i}) + (v_{35} - \frac{x_6}{x_5} v_{25})D_{i5} + (v_{38} - \frac{x_6}{x_5} v_{28})D_{i8},$$

$$a_{4i} = (v_{4i} - \frac{\dot{x}_4}{x_5} v_{2i}) + (v_{45} - \frac{\dot{x}_4}{x_5} v_{25})D_{i5} + (v_{48} - \frac{\dot{x}_4}{x_5} v_{28})D_{i8},$$

$$a_{6i} = (v_{6i} - \frac{\dot{x}_6}{x_5} v_{2i}) + (v_{65} - \frac{\dot{x}_6}{x_5} v_{25})D_{i5} + (v_{68} - \frac{\dot{x}_6}{x_5} v_{28})D_{i8},$$

$$(i = 1,3,4,6) \quad (22)$$

where

$$D_{i5} = -(F_{1i}F_{28} - F_{2i}F_{18})/D,$$

$$D_{i8} = -(F_{2i}F_{15} - F_{1i}F_{25})/D, \quad (23)$$

$$D = F_{15}F_{28} - F_{18}F_{25},$$

and

$$F_{1j} = \frac{\partial F_1}{\partial x_j} = \frac{\partial E}{\partial x_j}, \quad F_{2j} = \frac{\partial F_2}{\partial x_j} = \frac{\partial P}{\partial x_j}, \quad j = 1,3,4,6 \quad (24)$$

with $F_1 = E$ and $F_2 = P$ denoting respectively the energy and angular momentum integrals.

3. PRELIMINARY RESULTS

Applying the above technique, we started the computation of asymmetric periodic solutions of the general three body problem using initial conditions of such solutions of the restricted problem given by Markellos (1977) for values of the mass parameter μ in the interval (0,0.5). We chose as starting point an orbit belonging to the bifurcation series A_{20} for $\mu = 0.25$ with initial conditions $x_0 = -2.3310$. $\dot{x}_0 = -0.17292$, $\dot{y} = 1.9017$ and Jacobi constant C = 2.67054. The periodic solutions obtained are members of a continuous series formed by gradual increase of the mass of the third body m_3, in the interval (0,0.2), while the value of the mass parameter μ is kept constant: $\mu = 0.25$. Sample numerical results are given in Table I. As can be seen in the last column of the Table, all orbits are unstable, in the linear "isoenergetic" sence.

4. REFERENCES

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