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BEST APPROXIMATION BY POLYNOMIALS

SUNG GUEN KIM

In this paper we show that if E is a separable Banach space, F is a reflexive Banach space, and $n, k \in \mathbb{N}$, then every continuous polynomial of degree n from E into F has at least one element of best approximation in the Banach subspace of all continuous $k-$homogeneous polynomials from E into F.

1. INTRODUCTION AND NOTATION

We recall the basic definitions needed to discuss polynomials defined between Banach spaces E and F over the real or complex field K. We write B_E for the closed unit ball of E and the dual space of E is denoted by E^*. For $n \in \mathbb{N}$, we let $\mathcal{L}(^nE : F)$ denote the Banach space of all continuous n-linear maps from $E^n := E \times \cdots \times E$ into F endowed with the norm

$$\|L\| := \sup\Big\{\big\|L(x_1, \ldots, x_n)\big\| \ : \|x_j\| \leqslant 1, j = 1, \ldots, n\Big\}.$$

A map $L \in \mathcal{L}(^nE : F)$ is symmetric if $L(x_1, \ldots, x_n) = L(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $x_1, \ldots, x_n \in E$ and $\sigma \in \mathcal{S}_n$, where $\mathcal{S}_n$ denotes the set of permutations of the first n natural numbers. We let $\mathcal{L}_s(^nE : F)$ denote the Banach subspace of all continuous symmetric n-linear maps from E^n into F. A map $P : E \to F$ is a continuous *n-homogeneous polynomial* if there is a unique $L \in \mathcal{L}_s(^nE : F)$ such that $P(x) = L(x, \ldots, x)$ for all $x \in E$. In this case it is convenient to write $L = \check{P}$. More generally, a continuous polynomial of degree n from E into F is a map $P : E \to F$ of the form

$$P = P_0 + P_1 + \cdots + P_n$$

where P_0 is a constant function, P_j $(1 \leqslant j \leqslant n)$ is a continuous $j-$homogeneous polynomial, and P_n is not identically zero. This abstract definition of a polynomial of degree n between Banach spaces agrees with the classical definition when $E = K^n, F = K$:

$$P(x_1, \ldots, x_n) = \sum_{k=0}^{n} \sum_{k_1+\cdots k_n=k} a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

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where the indices $k_1, \dots, k_n$ are restricted to the non-negative integers and the coefficients $a_{k_1 \cdots k_n}$ are in K. We let $\mathcal{P}(E : F)$ denote the normed space of continuous polynomials of E into F endowed with the norm

$$\|P\| := \sup_{x \in B_E} \|P(x)\|$$

and the collection of all continuous k-homogeneous polynomials of E into F is a Banach subspace which we denote by $\mathcal{P}({}^kE : F)$. Note that $\mathcal{P}({}^0E : F) = F$ and $\mathcal{P}({}^1E : F) = \mathcal{L}({}^1E : F) = \mathcal{L}(E : F)$, which is the Banach space of bounded linear operators from E into F. For general background on polynomials, we refer to ([**2, 4**]).

We recall that if M is a nonempty set in a normed space F and $x \in F$, then any element $y_0 \in M$ with the property

$$\|x - y_0\| = \operatorname{dist}(x, M) := \inf_{y \in M} \|x - y\|$$

is called an *element of best approximation* of x in M. In the particular case when M is a $k-$dimensional linear subspace of F it is well known that every $x \in F$ has at least one element of best approximation in M. Holmes and Kripke [**7**] proved that every bounded linear operator between Hilbert spaces has a best approximation in the space of compact linear operators. For the best approximation theory in a normed space, we refer to [**8**].

In this paper we prove the following results. Let k be a natural number.

(1) Suppose that E, F are *complex* normed spaces. Let $n \neq k$ be a natural number and $P_0 \in \mathcal{P}({}^nE : F)$. Then $\|P_0\| = \inf_{Q \in \mathcal{P}({}^kE:F)} \|P_0 - Q\|$.

In *real* normed spaces it is not true.

(2) If E is a separable Banach space and F is a reflexive Banach space, then for every $P_0 \in \mathcal{P}(E : F)$ there exists some $Q_0 \in \mathcal{P}({}^kE : F)$ such that

$$\|P_0 - Q_0\| = \inf_{Q \in \mathcal{P}({}^kE:F)} \|P_0 - Q\|.$$

2. Results

Lemma 1. *Let F be a complex normed space. Let $a, b_1, \dots, b_m \in E$, $c > 0$, $m \in \mathbb{N}$, and $n_1, \dots, n_m$ nonzero distinct integers. Suppose*

$$\|a + \sum_{j=1}^{m} z^{n_j} b_j\| \leqslant c \text{ for all } z \in \mathbb{C} \text{ with } |z| = 1.$$

Then $\|a\| \leqslant c$.

Proof: Let $g(z) = a + \sum_{j=1}^{m} z^{n_j} b_j$ for $z \in \mathbb{C}\backslash\{0\}$. Then there is some $k \in \mathbb{N}$ such that $f(z) = z^k g(z)$ is a holomorphic function. Since $\|f(z)\| = \|g(z)\| \leqslant c$ for $z \in \mathbb{C}$ with $|z| = 1$, we have $k!\|a\| = \|f^{(k)}(0)\| \leqslant k!c$ by the Cauchy inequality. ∎

Theorem 2. *Suppose E, F are complex normed spaces. Let $k, k_1, \ldots, k_m$ be distinct natural numbers and $0 \neq P_j \in \mathcal{P}({}^{k_j}E : F)$ for each $j = 1, \ldots, m$. Then*

$$\max\{\|P_1\|, \ldots, \|P_m\|\} \leqslant \inf_{Q \in \mathcal{P}({}^kE:F)} \Big\| \sum_{j=1}^m P_j - Q \Big\| \leqslant \Big\| \sum_{j=1}^m P_j \Big\|.$$

Proof: The right inequality is obvious. For the left inequality, let $\varepsilon > 0$ and $j_0 \in \{1, \ldots, m\}$. Choose $x_0 \in S_E$ such that $\|P_{j_0}(x_0)\| > \|P_{j_0}\| - \varepsilon$. Let $Q \in \mathcal{P}({}^kE : F)$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. It follows that

$$\begin{aligned}\Big\| \sum_{j=1}^m P_j - Q \Big\| &\geqslant \Big\| \sum_{j=1}^m P_j(\lambda x_0) - Q(\lambda x_0) \Big\| \\ &= \Big\| \lambda^{k_{j_0}} P_{j_0}(x_0) + \sum_{j \neq j_0} \lambda^{k_j} P_j(x_0) - \lambda^k Q(x_0) \Big\| \\ &= \Big\| P_{j_0}(x_0) + \sum_{j \neq j_0} \lambda^{k_j - k_{j_0}} P_j(x_0) - \lambda^{k - k_{j_0}} Q(x_0) \Big\|.\end{aligned}$$

By Lemma 1 we have

$$\|P_{j_0}\| - \varepsilon < \|P_{j_0}(x_0)\| \leqslant \Big\| \sum_{j=1}^m P_j - Q \Big\|,$$

showing $\|P_{j_0}\| \leqslant \Big\| \sum_{j=1}^m P_j - Q \Big\|$ because $\varepsilon > 0$ is arbitrary. Since $Q \in \mathcal{P}({}^kE : F)$ is arbitrary, we have

$$\|P_{j_0}\| \leqslant \inf_{Q \in \mathcal{P}({}^kE:F)} \Big\| \sum_{j=1}^m P_j - Q \Big\|.$$

Since $j_0 \in \{1, \ldots, m\}$ is arbitrary, we complete the proof of theorem. ∎

Corollary 3. *Suppose E, F are complex normed spaces. Let $n \neq k$ be a natural number and $P_0 \in \mathcal{P}({}^nE : F)$. Then*

$$\|P_0\| = \operatorname{dist}\big(P_0, \mathcal{P}({}^kE : F)\big) := \inf_{Q \in \mathcal{P}({}^kE:F)} \|P_0 - Q\|.$$

Remark 4. In the *real* case Corollary 3 is not true. Indeed, let $E, F = \mathbb{R}, P_0(x) = x^2$. Then $1 = \|P_0\|$. We claim that

$$\inf_{Q \in \mathcal{P}({}^4\mathbb{R})} \|P_0 - Q\| = \Big\| x^2 - \frac{\sqrt{2}+1}{2} x^4 \Big\| = \frac{\sqrt{2}-1}{2} < 1 = \|P_0\|.$$

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PROOF OF CLAIM: It follows that

$$\begin{aligned}\inf_{Q\in\mathcal{P}({}^4\mathbb{R})}\|P_0-Q\| &= \inf_{a\in\mathbb{R}}\|x^2-ax^4\|\\ &= \min\{\inf_{a\leqslant 0}\|x^2-ax^4\|, \inf_{a\geqslant 0}\|x^2-ax^4\|\}\\ &= \min\{1, \inf_{a\geqslant 0}\|x^2-ax^4\|\}\\ &= \min\{\min_{0\leqslant a\leqslant 1}\|x^2-ax^4\|, \inf_{a>1}\|x^2-ax^4\|\}\\ &= \min\Big\{\min_{0\leqslant a\leqslant 1}\|x^2-ax^4\|, \min_{1\leqslant a\leqslant(\sqrt{2}+1)/2}\|x^2-ax^4\|, \inf_{(\sqrt{2}+1)/2<a}\|x^2-ax^4\|\Big\}\\ &= \min\Big\{\min_{0\leqslant a\leqslant 1}\frac{1}{4a}, \min_{1\leqslant a\leqslant(\sqrt{2}+1)/2}\frac{1}{4a}, \inf_{(\sqrt{2}+1)/2<a} a-1\Big\}\\ &= \min\Big\{\frac{1}{4}, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}-1}{2}\Big\} = \frac{\sqrt{2}-1}{2} = \Big\|x^2-\frac{\sqrt{2}+1}{2}x^4\Big\|,\end{aligned}$$

showing

$$\|P\| = (2\sqrt{2}+2)\inf_{Q\in\mathcal{P}({}^4\mathbb{R})}\|P-Q\| \text{ for each } P\in\mathcal{P}({}^2\mathbb{R}).$$

∎

THEOREM 5. *Suppose that E, F are real normed spaces. Let $k, n \in \mathbb{N}$ with $k+n$ is an odd integer and $P_0 \in \mathcal{P}({}^nE : F)$. Then*

$$\|P_0\| = \inf_{Q\in\mathcal{P}({}^kE:F)}\|P_0-Q\|.$$

PROOF: It suffices to show that $\max\{\|P_0\|, \|Q\|\} \leqslant \|P_0-Q\|$ for each $Q \in \mathcal{P}({}^kE : F)$. Let $Q \in \mathcal{P}({}^kE : F)$ and $x_0 \in B_E$. Then:

$$\big\|P_0(x_0)-Q(x_0)\big\| \leqslant \|P_0-Q\|$$

and

$$\big\|P_0(-x_0)-Q(-x_0)\big\| = \big\|(-1)^nP_0(x_0)-(-1)^kQ(x_0)\big\| = \big\|P_0(x_0)+Q(x_0)\big\| \leqslant \|P_0-Q\|.$$

By the triangle inequality, we have

$$\max\Big\{\big\|P_0(x_0)\big\|, \big\|Q(x_0)\big\|\Big\} \leqslant \|P_0-Q\|.$$

Since $x_0 \in B_E$ is arbitrary, we have

$$\max\{\|P_0\|, \|Q\|\} \leqslant \|P_0-Q\|.$$

∎

The following is an extension of the Banach-Steinhaus type theorem for continuous homogeneous polynomials due to Mazur and Orlicz (see [2]).

THEOREM 6. *Let E and F be Banach spaces. Suppose $\langle Q_n\rangle$ is a sequence in $\mathcal{P}({}^kE : F)$. If $\langle Q_n(x)\rangle$ converges weakly to $Q(x) \in F$ for each $x \in E$, then $Q \in \mathcal{P}({}^kE : F)$.*

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PROOF: Let $\check{Q}_n$ be the symmetric k-linear map associated to Q_n for each $n \in \mathbb{N}$. It is easy to show that by the polarisation formula weak $-\lim_{n\to\infty} \check{Q}_n(x_1, \dots, x_k)$ exists in F for $x_1, \dots, x_k \in E$. Let

$$A(x_1, \dots, x_k) = \text{weak} - \lim_{n\to\infty} \check{Q}_n(x_1, \dots, x_k) \quad \text{for } x_1, \dots, x_k \in E.$$

Then A is a k-linear map and $Q(x) = A(x, \dots, x)$ for each $x \in E$.

CLAIM. $\sup_{n\in\mathbb{N}} \|Q_n\| < \infty$.

Since the sequence $\langle \check{Q}_n(x_1, \dots, x_k)\rangle$ is weakly bounded in F for each $x_1, \dots, x_k \in E$, $\langle \check{Q}_n(x_1, \dots, x_k)\rangle$ is norm-bounded in F for each $x_1, \dots, x_k \in E$. Note that $\mathcal{L}(^kE : F)$ is isometric isomorphic to the space $\mathcal{L}\big(E : \mathcal{L}(^{k-1}E : F)\big)$. If we consider $\langle \check{Q}_n\rangle$ as a sequence in $\mathcal{L}\big(E : \mathcal{L}(^{k-1}E : F)\big)$, then by induction and the Uniform Boundedness Principle, we obtain $\sup_{n\in\mathbb{N}} \|Q_n\| \leqslant \sup_{n\in\mathbb{N}} \|\check{Q}_n\| < \infty$.

We claim that Q is continuous.

Let $x \in E$ with $\|x\| = 1$. By the Hahn-Banach theorem there is $x^* \in E^*$ with $\|x^*\| = 1$ such that $\big|x^*\big(Q(x)\big)\big| = \big\|Q(x)\big\|$.

It follows that

$$\big\|Q(x)\big\| = \Big|x^*\big(Q(x)\big)\Big| = \lim_{n\to\infty} \Big|x^*\big(Q_n(x)\big)\Big| \leqslant \|x^*\| \liminf_{n\to\infty} \big\|Q_n(x)\big\| \leqslant \sup_{n\in\mathbb{N}} \|Q_n\|.$$

Since $x \in E$ with $\|x\| = 1$ was arbitrary we have $\|Q\| \leqslant \sup_{n\in\mathbb{N}} \|Q_n\| < \infty$. Thus $Q \in \mathcal{P}(^kE : F)$. ∎

Here is the main result.

THEOREM 7. *Suppose E is a separable Banach space and F is a reflexive Banach space. Let $k \in \mathbb{N}$ and $P_0 \in \mathcal{P}(E : F)$. Then there exists $Q_0 \in \mathcal{P}(^kE : F)$ such that $\|P_0 - Q_0\| = \inf_{Q\in\mathcal{P}(^kE:F)} \|P_0 - Q\|$.*

PROOF: Let $d = \operatorname{dist}\big(P_0, \mathcal{P}(^kE : F)\big)$. By the definition of d, there exists a sequence $\langle Q_n\rangle$ in $\mathcal{P}(^kE : F)$ such that $\|P_0 - Q_n\| \to d$. Note that $\langle Q_n\rangle$ is bounded in $\mathcal{P}(^kE : F)$. Suppose that $\{e_i\}$ be a countable dense subset of B_E. Since $\langle Q_n\rangle$ is bounded in $\mathcal{P}(^kE : E)$, $\big\langle Q_n(e_i)\big\rangle$ is bounded in F for each $i \in \mathbb{N}$. Since F is reflexive, $\big\langle Q_n(e_i)\big\rangle$ is relatively weak-compact in F, so there is a subsequence $\big\langle Q_{n1}\big\rangle$ of $\langle Q_n\rangle$ such that $Q_{n1}(e_1)$ converges weakly to $y_1 \in F$. Similarly, there is a subsequence $\langle Q_{n2}\rangle$ of $\langle Q_{n1}\rangle$ such that $Q_{n2}(e_2)$ converges weakly to $y_2 \in F$ and $Q_{n2}(e_1)$ converges weakly to y_1. Continuing this process, we can construct subsequences $\langle Q_{ni}\rangle$ of $\langle Q_n\rangle$ for each i such that $\big\langle Q_{ni}(e_j)\big\rangle$ converges weakly to $y_j \in F$ for $1 \leqslant j \leqslant i$. By Cantor's diagonal process we have weak $-\lim_{n\to\infty} Q_{nn}(e_i)$ exists in F for each i. We claim that for each $x \in E$, weak $-\lim_{n\to\infty} Q_{nn}(x)$ exists in F. By the homogeneity of Q_{nn}, it suffices to show that for each $x \in B_E$, weak $-\lim_{n\to\infty} Q_{nn}(x)$ exists in F. Let $x \in B_E$.

We claim that $\langle Q_{nn}(x)\rangle$ is weakly convergent in F.

Since F is weakly complete it is enough to show that $\langle Q_{nn}(x)\rangle$ is weakly Cauchy. Let $0 < \varepsilon < 1, x^* \in F^*$. Let

$$I = \|x^*\|(2k^k)/(k!)\sup_n \|Q_n\| \sum_{0\leqslant j\leqslant k-1} {}_kC_j.$$

Then there is e_l such that $\|x - e_l\| < \min\{\varepsilon/(2I), 1\}$. Pick N_0 such that for $n, m > N_0$ we have

$$\left|x^*(Q_{nn}(e_l)) - x^*(Q_{mm}(e_l))\right| < \varepsilon/2.$$

It follows that for $n, m > N_0$,

$$\begin{aligned}
\left|x^*(Q_{nn}(x)) - x^*(Q_{mm}(x))\right| &\leqslant \left|x^*(Q_{nn}(x) - Q_{nn}(e_l))\right| \\
&\quad + \left|x^*(Q_{nn}(e_l)) - x^*(Q_{mm}(e_l))\right| + \left|x^*(Q_{mm}(e_l) - Q_{mm}(x))\right| \\
&\leqslant \|x^*\| \sum_{0\leqslant j\leqslant k-1} {}_kC_j \|\check{Q}_{nn}(e_l^j, (x-e_l)^{k-j})\| + \frac{\varepsilon}{2} \\
&\quad + \|x^*\| \sum_{0\leqslant j\leqslant k-1} {}_kC_j \|\check{Q}_{mm}(e_l^j, (x-e_l)^{k-j})\| \quad \text{(by the binomial theorem)} \\
&\leqslant \|x^*\|\,\|\check{Q}_{nn}\| \sum_{0\leqslant j\leqslant k-1} {}_kC_j \|e_l\|^j \|x-e_l\|^{k-j} + \frac{\varepsilon}{2} \\
&\quad + \|x^*\|\|\check{Q}_{mm}\| \sum_{0\leqslant j\leqslant k-1} {}_kC_j \|e_l\|^j \|x-e_l\|^{k-j} \\
&\leqslant 2\|x^*\|\|x-e_l\| \frac{k^k}{k!} \sup_n \|Q_{nn}\| \Big(\sum_{0\leqslant j\leqslant k-1} {}_kC_j \|x-e_l\|^{k-j-1}\Big) + \frac{\varepsilon}{2} \\
&\leqslant 2\,\|x^*\|\,\|x-e_l\|\,\frac{k^k}{k!} \sup_n \|Q_n\| \Big(\sum_{0\leqslant j\leqslant k-1} {}_kC_j\Big) + \frac{\varepsilon}{2} \\
&\leqslant \varepsilon,
\end{aligned}$$

where $\check{Q}_{mm}$ is the symmetric k-linear map associated to Q_{mm}. Define $Q_0(x) = \text{weak} - \lim_{n\to\infty} Q_{nn}(x)$ for $x \in E$. By Theorem 6 we have $Q_0 \in \mathcal{P}({}^kE : F)$.

We claim that $\|P_0 - Q_0\| = d$.

Let $x \in E$ with $\|x\| = 1$. By the Hahn-Banach theorem there is $x^* \in E^*$ with $\|x^*\| = 1$ such that

$$\left|x^*((P_0 - Q_0)(x))\right| = \|(P_0 - Q_0)(x)\|.$$

We have

$$\begin{aligned}
\|(P_0 - Q_0)(x)\| &= \left|x^*(P_0(x)) - x^*(Q_0(x))\right| = \left|x^*(P_0(x)) - \lim_{n\to\infty} x^*(Q_{nn}(x))\right| \\
&= \lim_{n\to\infty} \left|x^*((P_0 - Q_{nn})(x))\right| \leqslant \|x^*\| \liminf_{n\to\infty} \|(P_0 - Q_{nn})(x)\| \\
&\leqslant \lim_{n\to\infty} \|P_0 - Q_{nn}\| = d.
\end{aligned}$$

Since $x \in E$ with $\|x\| = 1$ was arbitrary we have $\|P_0 - Q_0\| \leqslant d$. By $Q_0 \in \mathcal{P}({}^kE : F)$ and the definition of d, we have $\|P_0 - Q_0\| \geqslant d$, showing $\|P_0 - Q_0\| = d$. ∎

It is clear that if $P_0 \in \mathcal{P}(E)$ and Q, R are different elements of best approximation of P_0 in $\mathcal{P}({}^kE)$, then every element of the line segment of Q and R is a best approximation of P_0 in $\mathcal{P}({}^kE)$. We do not know if elements of best approximation in Theorem 7 are unique. In [**1**] it was shown that $\mathcal{P}({}^kl_p)$ is reflexive if and only if $k < p < \infty$. We recall that a Banach space E is polynomially reflexive ([**3, 5**]) if for every $n \in \mathbb{N}$, $\mathcal{P}({}^nE)$ is a reflexive space. In ([**1, 6**]) it was shown that $E = T^*$, $l_\infty \check{\otimes} l_p$ $(2 < p < \infty)$, $l_\infty \check{\otimes} T^*$ are polynomially reflexive where T^* be the original Tsirelson space.

REMARK 8. Suppose E is a Banach space such that $\mathcal{P}({}^kE)$ is reflexive for some $k \in \mathbb{N}$. Let M be a nonempty, closed, convex subset of $\mathcal{P}({}^kE)$ and $P_0 \in \mathcal{P}(E)$. Then there exists $Q_0 \in M$ such that $\|P_0 - Q_0\| = \operatorname{dist}(P_0, M)$.

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Department of Mathematics
Kyungpook National University
Taegu
Korea (702-701)
e-mail: sgk317@knu.ac.kr

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