

Exponential polynomials with Fatou and non-escaping sets of finite Lebesgue measure

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Abstract. We give conditions ensuring that the Fatou set and the complement of the fast escaping set of an exponential polynomial f both have finite Lebesgue measure. Essentially, these conditions are designed such that $|f(z)| \geq \exp(|z|^\alpha)$ for some $\alpha > 0$ and all z outside a set of finite Lebesgue measure.

Key words: exponential polynomial, fast escaping set, Fatou set, Julia set, Lebesgue measure

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1. Introduction and results

Let f be a transcendental entire function, and let f^n denote the n th iterate of f . The *Fatou set* $F(f)$ of f is the set of all $z \in \mathbb{C}$ such that the iterates $(f^n)_{n \in \mathbb{N}}$ form a normal family in a neighbourhood of z , and the *Julia set* $J(f)$ is the complement of $F(f)$. These sets have an important role in complex dynamics. Clearly, $F(f)$ is open, and $J(f)$ is closed. Moreover, $J(f)$ is always non-empty, and either $J(f) = \mathbb{C}$ or $J(f)$ has empty interior. An introduction to the dynamics of transcendental entire functions can be found in [2]. The *escaping set* of f is defined by

$$I(f) := \{z : f^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Eremenko [7] showed that $I(f)$ is always non-empty, and that $J(f) = \partial I(f)$. For $r > 0$, let $M(r, f) := \max_{|z|=r} |f(z)|$ denote the *maximum modulus* of f , and let $M^n(r, f)$ be its n th iterate with respect to r . The *fast escaping set* $A(f)$ is a subset of the escaping set. It was introduced by Bergweiler and Hinkkanen [5] and is defined by

$$A(f) := \{z : \text{there exists } l \in \mathbb{N} \text{ such that } |f^n(z)| \geq M^{n-l}(R, f) \text{ for } n > l\},$$

where R is chosen such that $M(r, f) > r$ for $r \geq R$. For a thorough discussion of the fast escaping set, see [12].

We are interested in the Lebesgue measure of the sets defined above. McMullen [9] showed that the Julia set of $f(z) = \sin(az + b)$, $a \neq 0$, has positive Lebesgue measure. In fact, it can be seen from the proof that $J(f) \cap A(f)$ also has positive measure. Sixsmith [14] proved that if $f(z) = \sum_{j=1}^q a_j \exp(\omega_q^j z)$, where $q \geq 2$, $a_j \in \mathbb{C} \setminus \{0\}$, and $\omega_q = \exp(2\pi i/q)$, then $J(f) \cap A(f)$ has positive measure. Sixsmith remarked without proof that his result remains true for

$$f(z) = \sum_{j=1}^q a_j \exp(b_j z), \tag{1}$$

where $q \geq 3$, $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $\arg(b_j) < \arg(b_{j+1}) < \arg(b_j) + \pi$ for $j \in \{1, \dots, q - 1\}$, and $\arg(b_q) > \arg(b_1) + \pi$, with the argument chosen in $[0, 2\pi)$. Bergweiler and Chyzhykov [4] gave conditions ensuring that the Julia set and the escaping set of a transcendental entire function of completely regular growth have positive measure. These conditions are satisfied for the functions (1). In fact, they are also satisfied if one allows $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, \dots, q - 1\}$ or $\arg(b_q) = \arg(b_1) + \pi$. Further criteria for Julia sets and (fast) escaping sets to have positive measure are given in [1, 3].

For certain functions, it is possible to obtain stronger results in the sense that one can bound the size of the complement of the Julia set or (fast) escaping set. Schubert [13] used McMullen’s methods to prove that if $f(z) = \sinh(z)$, then the Lebesgue measure of $F(f)$ and $\mathbb{C} \setminus I(f)$ is finite in any horizontal strip of width 2π . In fact, the proof shows that one may replace $I(f)$ by the fast escaping set $A(f)$ here. Schubert’s result was generalized by Zhang and Yang [15] to functions of the form $f(z) = P(e^z)/e^z$, where P is a polynomial of degree at least two satisfying $P(0) \neq 0$.

There seem to be no papers whose main aim is to show that the Lebesgue measure of the Fatou set or the complement of the (fast) escaping set of certain transcendental entire functions is finite. However, there are some results presented in papers mainly treating a different subject. We mention two of them. Hemke [8, Theorem 5.1] showed that if

$$f(z) = Q_1(z) \exp(P(z)) + Q_2(z) \exp(-P(z)), \tag{2}$$

where P, Q_1, Q_2 are polynomials with $Q_1, Q_2 \not\equiv 0$ and $\deg(P) \geq 3$, then the Lebesgue measure of $\mathbb{C} \setminus I(f)$ is finite. One example of such a function is $f(z) = \sin(z^3)$. A result of Bock [6, Example 2] says that if $f(z) = \sin(\pi z)$, then $F(f) = \emptyset$, and $(f^n(z))$ tends to infinity for almost all $z \in \mathbb{C}$.

This is different for $f(z) = \sin(z)$, which is conjugate to the function $\sinh(z)$ considered by Schubert, and $f(z) = \sin(z^2)$.

Example 1.1. The Fatou set and the non-escaping set of $\sin(z)$ and $\sin(z^2)$ have infinite Lebesgue measure.

We will verify this in §5. See Figure 1 for an illustration of the non-escaping sets of $\sin(z)$, $\sin(z^2)$ and $\sin(z^3)$.

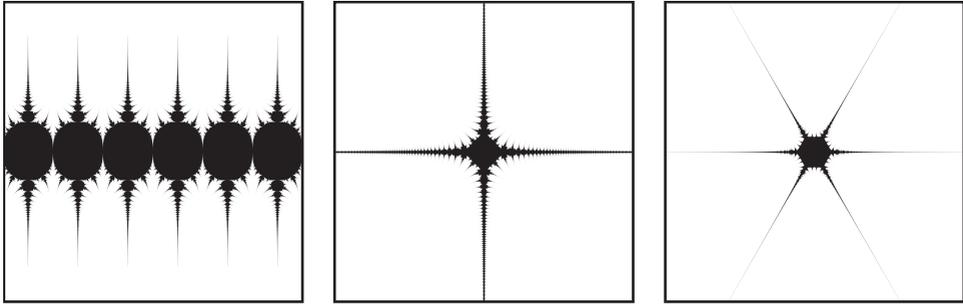


FIGURE 1. The non-escaping sets of $\sin(z)$, $\sin(z^2)$ and $\sin(z^3)$.

In this paper, we consider *exponential polynomials* of the form

$$f(z) = \sum_{j=1}^N Q_j(z) \exp(b_j z^d + P_j(z)),$$

where Q_j and P_j are polynomials with $\deg(P_j) < d$. We give conditions ensuring that the Lebesgue measure of the complement of $J(f) \cap A(f)$ is finite.

THEOREM 1.2. *Let*

$$f(z) := \sum_{j=1}^N Q_j(z) \exp(b_j z^d + P_j(z)),$$

where $d \in \mathbb{N}$ with $d \geq 3$, P_j and Q_j are polynomials with $Q_j \not\equiv 0$ and $\deg(P_j) < d$, and $b_j \in \mathbb{C} \setminus \{0\}$ are distinct numbers satisfying $\arg(b_j) \leq \arg(b_{j+1}) < \arg(b_j) + \pi$ for all $j \in \{1, \dots, N - 1\}$ and $\arg(b_N) > \arg(b_1) + \pi$, with the argument chosen in $[0, 2\pi)$. Then the Lebesgue measure of $\mathbb{C} \setminus (A(f) \cap J(f))$ is finite.

Note that the conditions on the b_j imply that $N \geq 3$. Recall that Sixsmith’s result for the functions (1) remains true if $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, \dots, q - 1\}$ or $\arg(b_q) = \arg(b_1) + \pi$. This is not true in general for Theorem 1.2, as the following example shows.

Example 1.3. Let

$$h(z) := \frac{1}{2} \exp(z^3 + iz) - \frac{1}{2} \exp(-z^3 + iz) = \exp(iz) \sinh(z^3).$$

Then h has a superattracting fixed point at zero, and the attractive basin of zero has infinite Lebesgue measure. In particular, the Lebesgue measure of $\mathbb{C} \setminus (A(h) \cap J(h))$ is infinite.

We will verify this in §5. However, the next theorem yields that under additional assumptions on the polynomials P_j , Theorem 1.2 can be extended to the case where $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, \dots, N - 1\}$ or $\arg(b_N) = \arg(b_1) + \pi$. These assumptions are satisfied if $\deg(P_j - (b_j/b_{j+1})P_{j+1}) \leq d - 3$ or $\deg(P_1 - (b_1/b_N)P_N) \leq d - 3$, respectively. We actually state a slightly more general condition, taking into account that several of the numbers b_j may have the same argument.

THEOREM 1.4. *Let*

$$f(z) := \sum_{j=1}^N Q_j(z) \exp(b_j z^d + P_j(z)),$$

where $d \in \mathbb{N}$ with $d \geq 3$, P_j and Q_j are polynomials with $Q_j \not\equiv 0$ and $\deg(P_j) < d$, and $b_j \in \mathbb{C} \setminus \{0\}$ are distinct numbers satisfying $\arg(b_j) \leq \arg(b_{j+1}) \leq \arg(b_j) + \pi$ for all $j \in \{1, \dots, N - 1\}$ and $\arg(b_N) \geq \arg(b_1) + \pi$, with the argument chosen in $[0, 2\pi)$.

If there exists $j \in \{1, \dots, N - 1\}$ such that $\arg(b_{j+1}) = \arg(b_j) + \pi$, or if $\arg(b_N) = \arg(b_1) + \pi$, in addition suppose that there are $k, l \in \{1, \dots, N\}$ with $\arg(b_k) = \arg(b_j)$ and $\arg(b_l) = \arg(b_{j+1})$, or $\arg(b_k) = \arg(b_1)$ and $\arg(b_l) = \arg(b_N)$, respectively, such that

$$\deg \left(P_k - \frac{b_k}{b_l} P_l \right) \leq d - 3. \tag{3}$$

Then the Lebesgue measure of $\mathbb{C} \setminus (A(f) \cap J(f))$ is finite.

Note that the conditions on the b_j imply that $N \geq 2$. Theorem 1.2 is a special case of Theorem 1.4. Also, the functions (2) considered by Hemke satisfy the assumptions of Theorem 1.4.

Throughout the rest of the paper, let f be an entire function satisfying the assumptions of Theorem 1.4. In §2, we will show that f can be approximated by simpler functions in large parts of the complex plane, and use this to prove that $|f(z)|$ is large outside a set of finite measure. Then, in §3, we show that f is injective in certain small disks. We finish the proof of Theorem 1.4 in §4, using a construction similar to one that was used in McMullen’s paper [9] and has since then been used by various authors. Finally, in §5, we verify the properties of Example 1.1 and Example 1.3.

2. *The behaviour of f*

In this section, we prove several properties of the function f . We first introduce some notation. For $j, k \in \{1, \dots, N\}$ with $j \neq k$, let

$$P_{j,k}(z) := (b_j z^d + P_j(z)) - (b_k z^d + P_k(z)) = (b_j - b_k)z^d + (P_j(z) - P_k(z)).$$

Let

$$v := d - \frac{5}{2},$$

and define the sets

$$U_1 := \{w \in \mathbb{C} : |\operatorname{Re}(w)| < |w|^{v/d}\}$$

and

$$U_2 := \{w \in \mathbb{C} : |\operatorname{Re}(w)| < 2|w|^{v/d}\}.$$

Moreover, define ‘exceptional sets’

$$E_l := \bigcup_{\substack{j,k=1 \\ j \neq k}}^N P_{j,k}^{-1}(U_l)$$

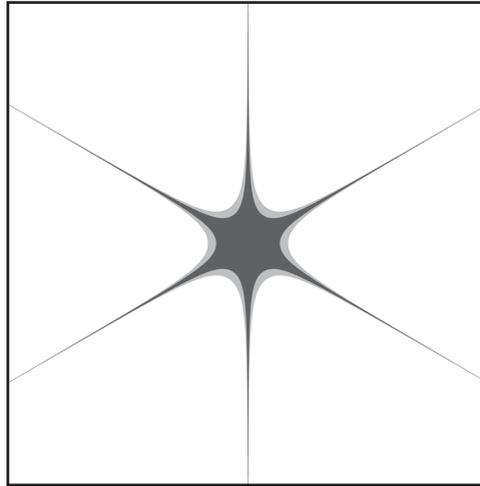


FIGURE 2. The sets E_1 (dark grey) and E_2 (light and dark grey) for $f(z) = Q_1(z) \exp(z^3) + Q_2(z) \exp(-z^3)$.

for $l \in \{1, 2\}$ (see Figure 2). We will later prove that the function f behaves ‘nicely’ outside E_1 .

For $z_0 \in \mathbb{C}$ and $r > 0$, we denote by $D(z_0, r) := \{z : |z - z_0| < r\}$ the open disk of radius r around z_0 . Moreover, let $\text{meas}(A)$ denote the Lebesgue measure of a measurable set $A \subset \mathbb{C}$.

LEMMA 2.1. *Let P be a polynomial of degree d , with d as before. Then the Lebesgue measure of $P^{-1}(U_2)$ is finite.*

Proof. Write

$$P(z) = \sum_{j=0}^d a_j z^j.$$

Fix $R > 2^{2d/5}$ such that all critical values of P are contained in $D(0, R)$, and let V be a component of $P^{-1}(\mathbb{C} \setminus (\overline{D(0, R)} \cup (-\infty, 0]))$. Then the restriction $P|_V : V \rightarrow \mathbb{C} \setminus (\overline{D(0, R)} \cup (-\infty, 0])$ is biholomorphic. Let φ denote the corresponding inverse function. Then $P^{-1}(U_2) \cap V = \varphi(U_2 \setminus \overline{D(0, R)})$. For $\eta(r) := \arcsin(2r^{-5/(2d)})$, let

$$W := U_2 \setminus \overline{D(0, R)} = \left\{ r e^{i\theta} : r > R, \min \left\{ \left| \theta - \frac{\pi}{2} \right|, \left| \theta - \frac{3\pi}{2} \right| \right\} < \eta(r) \right\}.$$

We have

$$\begin{aligned} \text{meas}(P^{-1}(U_2) \cap V) &= \int_W |\varphi'(z)|^2 dx dy = \int_W \frac{1}{|P'(\varphi(z))|^2} dx dy \\ &= \int_R^\infty r \left(\int_{\pi/2-\eta(r)}^{\pi/2+\eta(r)} \frac{1}{|P'(\varphi(re^{i\theta}))|^2} d\theta + \int_{3\pi/2-\eta(r)}^{3\pi/2+\eta(r)} \frac{1}{|P'(\varphi(re^{i\theta}))|^2} d\theta \right) dr. \end{aligned}$$

If r is sufficiently large, then

$$r = |P(\varphi(re^{i\theta}))| \leq 2^{d/(d-1)} |a_d| |\varphi(re^{i\theta})|^d.$$

Thus,

$$|\varphi(re^{i\theta})| \geq 2^{-1/(d-1)} |a_d|^{-1/d} \cdot r^{1/d}$$

and

$$|P'(\varphi(re^{i\theta}))| \geq \frac{1}{2} d |a_d| |\varphi(re^{i\theta})|^{d-1} \geq \frac{1}{4} d |a_d|^{1/d} \cdot r^{(d-1)/d}. \tag{4}$$

Since $\arcsin(x) \leq (\pi/2)x$ for $0 \leq x \leq 1$, we also have

$$\eta(r) = \arcsin(2r^{-5/(2d)}) \leq \pi r^{-5/(2d)}. \tag{5}$$

Using (4) and (5), we get

$$\begin{aligned} \text{meas}(P^{-1}(U_2) \cap V) &\leq \int_R^\infty r \cdot 4\eta(r) \cdot \frac{16}{d^2 |a_d|^{2/d} r^{2(d-1)/d}} dr \\ &\leq \frac{64\pi}{d^2} |a_d|^{-2/d} \int_R^\infty r \cdot r^{-5/(2d)} \cdot r^{-2(d-1)/d} dr \\ &= \frac{64\pi}{d^2} |a_d|^{-2/d} \int_R^\infty r^{-(1+1/(2d))} dr < \infty. \end{aligned}$$

Since there are only finitely many such components V , and also $P^{-1}(\overline{D(0, R)})$ has finite Lebesgue measure, the claim follows. □

The following corollary is an immediate consequence of Lemma 2.1.

COROLLARY 2.2. *The Lebesgue measure of E_1 and E_2 is finite.*

The next Lemma yields that if $R_0 > 0$ is large, then in each component of $\mathbb{C} \setminus (E_1 \cup D(0, R_0))$, f behaves like one of its summands $Q_m(z) \exp(b_m z^d + P_m(z))$.

LEMMA 2.3. *Let $\varepsilon > 0$. Then there exists $R_0 > 0$ such that for each connected component V of $\mathbb{C} \setminus (E_1 \cup D(0, R_0))$ there is an $m \in \{1, \dots, N\}$ such that for all $z \in V$ and all $j \in \{1, \dots, N\}$ with $j \neq m$, we have*

$$\text{Re}(b_m z^d + P_m(z)) > \text{Re}(b_j z^d + P_j(z))$$

and

$$\begin{aligned} \left| \frac{f(z)}{Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| &< \varepsilon, \\ \left| \frac{f'(z)}{d b_m z^{d-1} Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| &< \varepsilon, \\ \left| \frac{f''(z)}{d^2 b_m^2 z^{2d-2} Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| &< \varepsilon. \end{aligned}$$

Proof. Let $R_0 > 0$ such that all zeros of the polynomials $P_{j,k}$ are contained in $D(0, R_0)$, and let V be a component of $\mathbb{C} \setminus (E_1 \cup D(0, R_0))$. Let $z_0 \in V$, and let $m \in \{1, \dots, N\}$

such that

$$\operatorname{Re}(b_m z_0^d + P_m(z_0)) = \max_{1 \leq j \leq N} \operatorname{Re}(b_j z_0^d + P_j(z_0)). \tag{6}$$

Let $j \in \{1, \dots, N\}$ with $j \neq m$. For all $z \in V$, we have $P_{m,j}(z) \notin U_1$ and hence

$$|\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z))| = |\operatorname{Re}(P_{m,j}(z))| \geq |P_{m,j}(z)|^{v/d}. \tag{7}$$

In particular,

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) \neq 0$$

for all $z \in V$. By continuity and (6), we get

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) > 0$$

for all $z \in V$, and hence

$$\operatorname{Re}(b_m z^d + P_m(z)) - \operatorname{Re}(b_j z^d + P_j(z)) \geq |P_{m,j}(z)|^{v/d}$$

by (7). Thus, for all $z \in V$,

$$\begin{aligned} & \left| \frac{f(z)}{Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| \\ &= \left| \frac{\sum_{j=1}^N Q_j(z) \exp(b_j z^d + P_j(z)) - Q_m(z) \exp(b_m z^d + P_m(z))}{Q_m(z) \exp(b_m z^d + P_m(z))} \right| \\ &= \left| \sum_{\substack{j=1 \\ j \neq m}}^N \frac{Q_j(z)}{Q_m(z)} \exp(b_j z^d + P_j(z) - (b_m z^d + P_m(z))) \right| \\ &\leq \sum_{\substack{j=1 \\ j \neq m}}^N \left| \frac{Q_j(z)}{Q_m(z)} \right| \exp(\operatorname{Re}(b_j z^d + P_j(z)) - \operatorname{Re}(b_m z^d + P_m(z))) \\ &\leq \sum_{\substack{j=1 \\ j \neq m}}^N \left| \frac{Q_j(z)}{Q_m(z)} \right| \exp(-|P_{j,m}(z)|^{v/d}) < \varepsilon \end{aligned}$$

if R_0 , and hence $|z|$ is sufficiently large. This is the result for f .

Moreover,

$$f'(z) = \sum_{j=1}^N (db_j z^{d-1} Q_j(z) + P'_j(z) Q_j(z) + Q'_j(z)) \exp(b_j z^d + P_j(z)).$$

The result for f' now follows from similar estimates as above and the fact that

$$db_j z^{d-1} Q_j(z) + P'_j(z) Q_j(z) + Q'_j(z) = (1 + o(1)) db_j z^{d-1} Q_j(z)$$

as $|z| \rightarrow \infty$. Analogously, the result for f'' follows from

$$f''(z) = \sum_{j=1}^N (1 + o(1)) d^2 b_j^2 z^{2d-2} Q_j(z) \exp(b_j z^d + P_j(z))$$

as $|z| \rightarrow \infty$. □

Remark. In order to prove Lemma 2.3, we did not need any assumptions on the arguments of the b_j . In particular, the statement remains true without the additional condition (3) in the case where $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, \dots, N - 1\}$ or $\arg(b_N) = \arg(b_1) + \pi$. This is different for the next result.

LEMMA 2.4. *Let $\alpha \in (0, \nu)$. Then there exists $R_1 > 0$ such that for all $z \in \mathbb{C} \setminus E_1$ with $|z| \geq R_1$, we have*

$$|f(z)| \geq \exp(|z|^\alpha) \quad \text{and} \quad |f'(z)| \geq \exp(|z|^\alpha).$$

Remark. Without the additional condition (3) in the case where $\arg(b_{j+1}) = \arg(b_j) + \pi$ for some $j \in \{1, \dots, N\}$ or $\arg(b_N) = \arg(b_1) + \pi$, the statement of Lemma 2.4 is not true in general. We will prove in §5 that the function $h(z) = (1/2) \exp(z^3 + iz) - (1/2) \exp(-z^3 + iz)$ given in Example 1.3 is bounded in a set of infinite Lebesgue measure.

Proof of Lemma 2.4. We prove the statement for f . The proof for f' is analogous. Let $z \in \mathbb{C} \setminus E_1$. If $|z|$ is sufficiently large, then by Lemma 2.3 there is $m \in \{1, \dots, N\}$ such that

$$\operatorname{Re}(b_m z^d + P_m(z)) > \operatorname{Re}(b_j z^d + P_j(z))$$

for all $j \in \{1, \dots, N\}$ with $j \neq m$ and

$$\left| \frac{f(z)}{Q_m(z) \exp(b_m z^d + P_m(z))} - 1 \right| \leq \frac{1}{2}.$$

Then

$$|f(z)| \geq \frac{1}{2} |Q_m(z)| \exp(\operatorname{Re}(b_m z^d + P_m(z))).$$

Thus, it suffices to show that there exists $j \in \{1, \dots, N\}$ with

$$\operatorname{Re}(b_j z^d + P_j(z)) \geq 2|z|^\alpha.$$

We first consider the case where f satisfies the assumptions of Theorem 1.2, that is,

$$\arg(b_{j+1}) < \arg(b_j) + \pi \text{ for all } j \in \{1, \dots, N - 1\} \text{ and } \arg(b_N) > \arg(b_1) + \pi.$$

Then there is a constant $C > 0$ such that for all $w \in \mathbb{C}$ there exists $j \in \{1, \dots, N\}$ with $\operatorname{Re}(b_j w^d) \geq 2C|w|^d$. In particular, this applies to $w = z$. Since $\deg(P_j) < d$, we get

$$\operatorname{Re}(b_j z^d + P_j(z)) \geq 2C|z|^d - |P_j(z)| \geq C|z|^d > 2|z|^\alpha$$

if $|z|$ is sufficiently large.

Now suppose that f does not satisfy the assumptions of Theorem 1.2. Then the assumptions of Theorem 1.4 imply that there are $k, l \in \{1, \dots, N\}$ satisfying $|\arg(b_k) - \arg(b_l)| = \pi$, such that $\deg(P_k - (b_k/b_l)P_l) \leq d - 3$. Thus, there exist polynomials g, g_k, g_l with $\deg(g) \leq d - 1$ and $\max\{\deg(g_k), \deg(g_l)\} \leq d - 3$ such that

$$P_k = b_k g + g_k \quad \text{and} \quad P_l = b_l g + g_l.$$

With $\beta_k := \arg(b_k)$, we have

$$b_k = |b_k| e^{i\beta_k} \quad \text{and} \quad b_l = -|b_l| e^{i\beta_k}.$$

Thus,

$$b_k - b_l = (|b_k| + |b_l|)e^{i\beta_k}$$

and

$$P_{k,l}(z) = (|b_k| + |b_l|)e^{i\beta_k}(z^d + g(z)) + (g_k(z) - g_l(z)).$$

Moreover, we assume without loss of generality that

$$\operatorname{Re}(b_k(z^d + g(z))) \geq \operatorname{Re}(b_l(z^d + g(z))).$$

Then $\operatorname{Re}(b_k(z^d + g(z))) \geq 0$. Since $z \notin E_1$, we get

$$|P_{k,l}(z)|^{v/d} \leq |\operatorname{Re}(P_{k,l}(z))| \leq (|b_k| + |b_l|) \operatorname{Re}(e^{i\beta_k}(z^d + g(z))) + |g_k(z) - g_l(z)|.$$

Hence,

$$\begin{aligned} \operatorname{Re}(b_k z^d + P_k(z)) &= \operatorname{Re}(b_k(z^d + g(z)) + g_k(z)) \\ &= |b_k| \operatorname{Re}(e^{i\beta_k}(z^d + g(z))) + \operatorname{Re}(g_k(z)) \\ &\geq \frac{|b_k|}{|b_k| + |b_l|} (|P_{k,l}(z)|^{v/d} - |g_k(z) - g_l(z)|) - |g_k(z)|. \end{aligned}$$

Since

$$|P_{k,l}(z)| \geq 2^{-d/v} (|b_k| + |b_l|) |z|^d$$

if $|z|$ is large, and since $\max\{\deg(g_k), \deg(g_l)\} < v$, we deduce that

$$\operatorname{Re}(b_k z^d + P_k(z)) \geq \frac{1}{4} |b_k| (|b_k| + |b_l|)^{(v/d)-1} |z|^v > 2|z|^\alpha$$

if $|z|$ is sufficiently large. This completes the proof. □

3. Injectivity

The aim of this section is to prove that f is injective in certain disks contained in $\mathbb{C} \setminus E_1$. We start with a basic injectivity criterion (see, e.g., [11, Proposition 1.10]).

LEMMA 3.1. *Let $D \subset \mathbb{C}$ be a convex domain, and let $h : D \rightarrow \mathbb{C}$ be holomorphic. If $\operatorname{Re}(h'(z)) > 0$ for all $z \in D$, then h is injective in D .*

We also require the following criterion.

LEMMA 3.2. *Let $z_0 \in \mathbb{C}$ and $r > 0$. Let h be holomorphic in $D(z_0, r)$. Suppose that $h'(\zeta) \neq 0$ for all $\zeta \in D(z_0, r)$ and*

$$\sup_{|\zeta - z_0| < r} \left| \frac{h''(\zeta)}{h'(\zeta)} \right| < \frac{1}{r}.$$

Then h is injective in $D(z_0, r)$.

This follows directly from Becker’s univalence criterion (see, e.g., [10, Theorem 6.7]). However, Lemma 3.2 may also be proved by much more elementary arguments using Lemma 3.1. We sketch the proof here.

Sketch of proof. We may assume without loss of generality that $h'(z_0) = 1$. Let ψ be the branch of $\log h'$ in $D(z_0, r)$ satisfying $\psi(z_0) = 0$. Then, for all $z \in D(z_0, r)$,

$$|\psi(z)| = \left| \int_{z_0}^z \frac{h''(\zeta)}{h'(\zeta)} d\zeta \right| < \frac{1}{r} |z - z_0| < 1.$$

Thus, $\arg(h'(z)) = \text{Im}(\psi(z)) \in (-1, 1)$. In particular, $\text{Re}(h'(z)) > 0$. Hence, h is injective in $D(z_0, r)$ by Lemma 3.1. □

We now state the main result of this section.

LEMMA 3.3. *Let $\sigma \in (0, 1/(4d \max_j |b_j|))$. Then there exists $R_2 > 0$ such that the following holds. If $z \in \mathbb{C} \setminus E_1$ with $|z| \geq R_2$ is such that $D(z, 2\sigma|z|^{-(d-1)}) \subset \mathbb{C} \setminus E_1$, then f is injective in $D(z, 2\sigma|z|^{-(d-1)})$.*

Proof. Let $r := 2\sigma|z|^{-(d-1)}$. By Lemma 2.3, there exists $m \in \{1, \dots, N\}$ such that

$$\sup_{|\zeta-z|<r} \left| \frac{f''(\zeta)}{f'(\zeta)} \right| \leq \frac{3}{2}d|b_m| \sup_{|\zeta-z|<r} |\zeta|^{d-1} \leq 2d|b_m||z|^{d-1} < \frac{1}{r}$$

if $|z|$ is sufficiently large. Thus, f is injective in $D(z, r)$ by Lemma 3.2. □

For $z \in \mathbb{C}$ and $A \subset \mathbb{C}$, let $\text{dist}(z, A) := \inf\{|z - a| : a \in A\}$ denote the Euclidean distance between z and A .

LEMMA 3.4. *There are $C_1, R_3 > 0$ such that if $z \in \mathbb{C} \setminus E_2$ and $|z| \geq R_3$, then*

$$\text{dist}(z, E_1) \geq C_1|z|^{-3/2}.$$

The following corollary is an immediate consequence of Lemmas 3.4 and 3.3.

COROLLARY 3.5. *Let σ be as in Lemma 3.3. There is $R_4 > 0$ such that if $z \in \mathbb{C} \setminus E_2$ and $|z| \geq R_4$, then f is injective in $D(z, 2\sigma|z|^{-(d-1)})$.*

Proof of Lemma 3.4. Let $z \in \mathbb{C} \setminus E_2$. It suffices to show that if $|z|$ is sufficiently large, then

$$\text{dist}(z, E_1 \cap D(z, 1)) \geq C_1|z|^{-3/2}$$

for some constant $C_1 > 0$. Let $w \in E_1 \cap D(z, 1)$. Then there are $j, k \in \{1, \dots, N\}$ with $j \neq k$ such that $|\text{Re}(P_{j,k}(w))| \leq |P_{j,k}(w)|^{v/d}$. If $|z|$ is sufficiently large, then $|w| \leq (6/5)^{1/v}|z|$ and

$$\begin{aligned} |P_{j,k}(z) - P_{j,k}(w)| &\geq \text{Re}(P_{j,k}(z)) - \text{Re}(P_{j,k}(w)) \\ &\geq 2|P_{j,k}(z)|^{v/d} - |P_{j,k}(w)|^{v/d} \\ &\geq 2 \cdot \frac{4}{5}|b_j - b_k|^{v/d}|z|^v - \frac{6}{5}|b_j - b_k|^{v/d}|w|^v \\ &\geq \frac{8}{5}|b_j - b_k|^{v/d}|z|^v - \left(\frac{6}{5}\right)^2|b_j - b_k|^{v/d}|z|^v \\ &= \frac{4}{25}|b_j - b_k|^{v/d}|z|^{d-5/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |P_{j,k}(z) - P_{j,k}(w)| &= \left| \int_w^z P'_{j,k}(\zeta) d\zeta \right| \leq \sup_{\zeta \in D(z,1)} |P'_{j,k}(\zeta)| \cdot |z - w| \\ &\leq 2d|b_j - b_k|(|z| + 1)^{d-1}|z - w| \leq 4d|b_j - b_k||z|^{d-1}|z - w|. \end{aligned}$$

Thus,

$$|z - w| \geq \frac{|b_j - b_k|^{v/d-1}}{25d} |z|^{-3/2} \geq C_1 |z|^{-3/2}$$

for $C_1 = (\min_{l \neq n} |b_l - b_n|^{v/d-1})/25d$. □

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. First, we collect several results that we require. For $\alpha > 0$, consider the function

$$E_\alpha : [0, \infty) \rightarrow [0, \infty), \quad E_\alpha(x) = \exp(x^\alpha).$$

We will use the following result [3, Lemma 2.1].

LEMMA 4.1. *Let $\beta > \alpha > 0$. Then there exists $x_0 > 0$ such that*

$$E_\alpha^k(x) \geq E_\beta^{k-2}(x) \quad \text{for all } k \geq 4 \text{ and } x \geq x_0.$$

The next lemma is due to Sixsmith [14, Theorem 3.1].

LEMMA 4.2. *Let h be a transcendental entire function and $z_0 \in I(h)$. Let $z_n := h^n(z_0)$ for all $n \in \mathbb{N}$. Suppose that there exist $\lambda > 1$ and $N \geq 0$ such that*

$$h(z_n) \neq 0 \quad \text{and} \quad \left| z_n \frac{h'(z_n)}{h(z_n)} \right| \geq \lambda \quad \text{for all } n \geq N.$$

Then either z_0 is in a multiply connected Fatou component of h , or $z_0 \in J(h)$.

The following result is due to Zheng [16, Corollary 6 and Remark (J)].

LEMMA 4.3. *Let h be a transcendental entire function of the form*

$$h(z) = \sum_{j=1}^M q_j(z) \exp(p_j(z)),$$

with polynomials p_j and q_j . Then the Fatou set of h has no multiply connected components.

For a curve $\gamma \subset \mathbb{C}$ we denote by $\ell(\gamma)$ the Euclidean length of γ .

LEMMA 4.4. *Let $\gamma \subset \mathbb{C}$ be a curve of positive length, and let $s \in (0, \ell(\gamma))$. Then*

$$\text{meas}(\{z : \text{dist}(z, \gamma) \leq s\}) \leq \frac{9\pi}{2} s \cdot \ell(\gamma).$$

Proof. Let $L \in \mathbb{N}$ such that $L - 1 < \ell(\gamma)/s \leq L$. We divide γ into L subcurves $\gamma_1, \dots, \gamma_L$ satisfying $\ell(\gamma_j) \leq s$ for all $j \in \{1, \dots, L\}$. Then, for $j \in \{1, \dots, L\}$, there

is an $a_j \in \mathbb{C}$ such that $\gamma_j \subset \overline{D(a_j, s/2)}$. We have

$$\{z : \text{dist}(z, \gamma_j) \leq s\} \subset \overline{D(a_j, (3/2)s)}.$$

Thus,

$$\text{meas}(\{z : \text{dist}(z, \gamma_j) \leq s\}) \leq \frac{9\pi}{4}s^2.$$

Using $(\ell(\gamma)/s) + 1 \leq 2\ell(\gamma)/s$, we get

$$\text{meas}(\{z : \text{dist}(z, \gamma) \leq s\}) \leq L \frac{9\pi}{4}s^2 < \left(\frac{\ell(\gamma)}{s} + 1\right) \frac{9\pi}{4}s^2 \leq \frac{9\pi}{2}s \cdot \ell(\gamma). \quad \square$$

The next result is a direct consequence of the well-known Koebe distortion theorem (see, e.g., [10, Theorem 1.6]).

LEMMA 4.5. *Let $z_0 \in \mathbb{C}$, $r > 0$ and $\rho \in (0, 1)$. Suppose that $h : D(z_0, r) \rightarrow \mathbb{C}$ is holomorphic and injective. Then, for all $z \in D(z_0, \rho r)$,*

$$\frac{|h'(z_0)|}{(1 + \rho)^2} \leq \frac{|h(z) - h(z_0)|}{|z - z_0|} \leq \frac{|h'(z_0)|}{(1 - \rho)^2}.$$

Moreover,

$$\frac{\max_{|z| \leq \rho r} |h'(z)|}{\min_{|z| \leq \rho r} |h'(z)|} \leq \left(\frac{1 + \rho}{1 - \rho}\right)^4.$$

Proof of Theorem 1.4. Throughout the proof, write

$$J_A := J(f) \cap A(f).$$

The strategy of the proof is as follows. First, we construct a suitable collection \mathcal{S} of squares in $\mathbb{C} \setminus E_1$ such that $\mathbb{C} \setminus \bigcup_{S \in \mathcal{S}} S$ has finite Lebesgue measure. For $S_0 \in \mathcal{S}$, we then show that points in S_0 which stay in $\bigcup_{S \in \mathcal{S}} S$ under iteration lie in J_A , and that the set of such points has large density in S_0 . Finally, we prove that $\sum_{S \in \mathcal{S}} \text{meas}(S \setminus J_A) < \infty$.

Let $\varepsilon = 1/3$, $\alpha \in (0, \nu)$, $\sigma \in (0, 1/(4d \max_j |b_j|))$ and $\beta > d$, and let R_0, R_1, R_2, x_0 be the corresponding constants from Lemmas 2.3, 2.4, 3.3 and 4.1, respectively. Also, let C_1, R_3 be the constants from Lemma 3.4. Let B_0 be an open square centred at zero with sides parallel to the real and imaginary axes such that all $z \in \mathbb{C} \setminus B_0$ satisfy

$$|z| > \max\{R_0, \dots, R_3, x_0\},$$

$$\frac{C_1}{|z|^{3/2}} > \frac{3\sigma}{|z|^{d-1}}, \tag{8}$$

$$M(|z|, f) \leq E_\beta(|z|), \tag{9}$$

$$\frac{1}{2}d \min_k |b_k| |z|^d > 2, \tag{10}$$

$$\text{meas}(E_2) \leq \left(1 - \frac{18\pi\sigma}{\sqrt{2}|z|^{d-1}}\right) \exp(|z|^\alpha), \tag{11}$$

and

$$\left(\frac{5}{3}\right)^4 \frac{32|z|^{2(d-2)}}{\sigma^2 \exp(|z|^\alpha)} \leq \exp\left(-\frac{1}{2}|z|^\alpha\right). \tag{12}$$

For a compact set $X \subset \mathbb{C}$, let

$$\mathcal{M}(X) := \max_{z \in X} |z| \quad \text{and} \quad \mu(X) := \min_{z \in X} |z|.$$

More generally, for a continuous function $g : X \rightarrow \mathbb{C}$, let

$$\mathcal{M}(X, g) := \max_{z \in X} |g(z)| \quad \text{and} \quad \mu(X, g) := \min_{z \in X} |g(z)|.$$

Let $\tilde{\mathcal{S}}$ be a collection of closed squares in \mathbb{C} with sides parallel to the real and imaginary axes such that:

- $\bigcup_{S \in \tilde{\mathcal{S}}} S = \mathbb{C} \setminus B_0$;
- for all $S_1, S_2 \in \tilde{\mathcal{S}}$ with $S_1 \neq S_2$, we have $S_1^\circ \cap S_2^\circ = \emptyset$; and
- for all $S \in \tilde{\mathcal{S}}$, the side length s of S satisfies

$$\frac{\sigma}{4\sqrt{2}\mu(S)^{d-1}} \leq s \leq \frac{\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}}.$$

If the side length of B_0 is sufficiently large, this can be achieved as follows. First, divide $\mathbb{C} \setminus B_0$ into squares of a fixed size so that the side length of all squares satisfies the lower bound. If the side length s of a square does not satisfy the upper bound, divide it into four squares of side length $s/2$, and then continue this procedure until the side length of the squares satisfies the upper bound.

Let \mathcal{S} be the collection of all $S \in \tilde{\mathcal{S}}$ such that $\text{dist}(S, E_1) > 2\sigma/\mu(S)^{d-1}$. By Lemma 3.4, the definition of \mathcal{S} , and (8),

$$\mathbb{C} \setminus (E_2 \cup B_0) \subset \bigcup_{S \in \mathcal{S}} S \subset \mathbb{C} \setminus (E_1 \cup B_0). \tag{13}$$

Next, we construct a subset of J_A as an intersection of nested sets. Fix a square $S_0 \in \mathcal{S}$. Let

$$\mathcal{K}_0 := \{S_0\}$$

and, for $n \in \mathbb{N}$, let

$$\mathcal{K}_n := \{T_n \subset S_0 : f^n(T_n) \in \mathcal{S} \text{ and } T_n \subset T_{n-1} \text{ for some } T_{n-1} \in \mathcal{K}_{n-1}\}.$$

We first show that

$$T := \bigcap_{n=0}^{\infty} \left(\bigcup_{T_n \in \mathcal{K}_n} T_n \right) \subset J_A.$$

To do so, let $z \in T$. Then $f^n(z) \in \mathbb{C} \setminus E_1$ for all $n \geq 0$. Using Lemma 2.4, Lemma 4.1 and (9), we get

$$|f^n(z)| \geq E_\alpha^n(|z|) \geq E_\beta^{n-2}(|z|) \geq M^{n-2}(|z|, f)$$

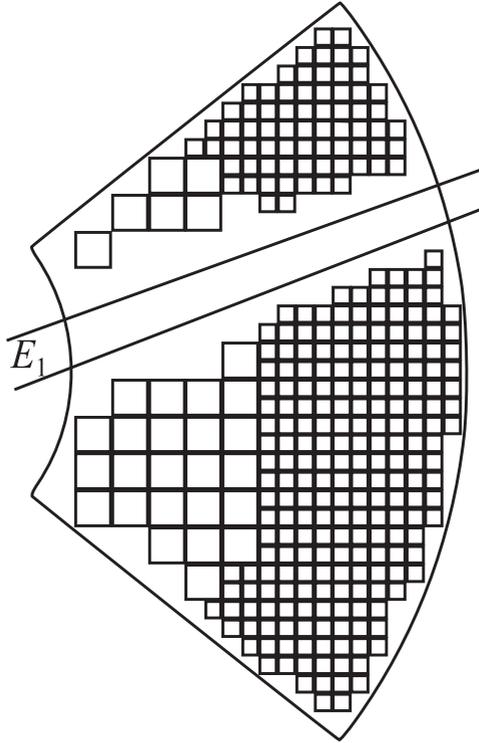


FIGURE 3. An illustration of $\text{pack}(f(S))$ (not to scale). It consists of squares contained in $f(S)$ which are not too close to E_1 . The size of the squares decreases as their distance from zero increases.

for all $n \geq 4$. This yields $z \in A(f)$. For $n \in \mathbb{N}$, let $z_n := f^n(z)$. By Lemma 2.3, Lemma 2.4 and (10),

$$\left| z_n \frac{f'(z_n)}{f(z_n)} \right| \geq \frac{1}{2} d \min_k |b_k| |z_n|^d > 2.$$

By Lemmas 4.2 and 4.3, $z \in J(f)$. Thus, $T \subset J_A$.

For $A \subset \mathbb{C}$, define

$$\text{pack}(A) := \{S \in \mathcal{S} : S \subset A\}.$$

Moreover, for measurable $A, B \subset \mathbb{C}$ with $\text{meas}(B) > 0$, let

$$\text{dens}(A, B) := \frac{\text{meas}(A \cap B)}{\text{meas}(B)}$$

denote the *density* of A in B . We will show that for any square $S \in \mathcal{S}$,

$$\text{dens}\left(f(S) \setminus \bigcup_{S' \in \text{pack}(f(S))} S', f(S)\right) \leq \exp\left(-\frac{1}{2} \mu(S)^\alpha\right).$$

See Figure 3 for an illustration of $\text{pack}(f(S))$.

By Lemma 3.3, f is injective in S . We have

$$\text{meas}(f(S)) = \int_S |f'(z)|^2 dx dy \geq \text{meas}(S) \cdot \mu(S, f')^2.$$

Moreover, $|f(z)| \geq \exp(|z|^\alpha)$ for all $z \in S$ by Lemma 2.4, and hence

$$f(S) \subset \mathbb{C} \setminus B_1 = \bigcup_{S' \in \tilde{\mathcal{S}}} S'.$$

By (13) and Corollary 2.2, the Lebesgue measure of the union of all squares $S' \in \tilde{\mathcal{S}} \setminus \mathcal{S}$ is at most $\text{meas}(E_2) < \infty$. We now consider the union of all squares $S' \in \mathcal{S}$ with $S' \cap \partial f(S) \neq \emptyset$. The length of $\partial f(S)$ satisfies

$$\ell(\partial f(S)) = \int_{\partial S} |f'(z)| |dz| \leq \mathcal{M}(S, f') \ell(\partial S) \leq \frac{4\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f').$$

Analogously,

$$\ell(\partial f(S)) \geq \frac{\sigma}{\sqrt{2}\mu(S)^{d-1}} \mu(S, f').$$

By Lemma 2.4, $|f'(z)| \geq \exp(|z|^\alpha)$ for all $z \in S$. In particular, $\ell(\partial f(S)) > 1$. For $S' \in \mathcal{S}$ with $S' \cap \partial f(S) \neq \emptyset$, we have

$$S' \subset \{z : \text{dist}(z, \partial f(S)) \leq 1\}.$$

By Lemma 4.4,

$$\text{meas}(\{z : \text{dist}(z, \partial f(S)) \leq 1\}) \leq \frac{9\pi}{2} \cdot \ell(\partial f(S)) \leq \frac{18\pi\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f').$$

Altogether, we get

$$\text{meas}\left(f(S) \setminus \bigcup_{S' \in \text{pack}(f(S))} S'\right) \leq \text{meas}(E_2) + \frac{18\pi\sigma}{\sqrt{2}\mathcal{M}(S)^{d-1}} \mathcal{M}(S, f') \leq \mathcal{M}(S, f'),$$

where we used (11) in the last step. Thus,

$$\text{dens}\left(f(S) \setminus \bigcup_{S' \in \text{pack}(f(S))} S', f(S)\right) \leq \frac{\mathcal{M}(S, f')}{\text{meas}(S) \cdot \mu(S, f')^2}.$$

Let z_0 be the centre of S . Then $S \subset D(z_0, (\sigma/2)|z_0|^{-(d-1)})$, and by Lemma 3.3, f is injective in $D(z_0, 2\sigma|z_0|^{-(d-1)})$. By the Koebe distortion theorem (Lemma 4.5), Lemma 2.4 and (12),

$$\begin{aligned} \text{dens}\left(f(S) \setminus \bigcup_{S' \in \text{pack}(f(S))} S', f(S)\right) &\leq \left(\frac{5}{3}\right)^4 \frac{1}{\text{meas}(S)\mu(S, f')} \\ &\leq \left(\frac{5}{3}\right)^4 \frac{32\mu(S)^{2(d-1)}}{\sigma^2 \exp(\mu(S)^\alpha)} \\ &\leq \exp\left(-\frac{1}{2}\mu(S)^\alpha\right). \end{aligned} \tag{14}$$

Recall that we want to show that the density of J_A in the square $S_0 \in \mathcal{S}$ is large by considering the subset $T = \bigcap_{n=0}^\infty \bigcup_{T_n \in \mathcal{K}_n} T_n \subset S_0 \cap J_A$. Let $n \geq 0$ and $T_n \in \mathcal{K}_n$. Then $f^n(T_n) \in \mathcal{S}$. By (14) applied to $S = f^n(T_n)$ and Lemma 2.4,

$$\begin{aligned} & \text{dens}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \text{pack}(f^{n+1}(T_n))} S', f^{n+1}(T_n)\right) \\ & \leq \exp\left(-\frac{1}{2}\mu(f^n(T_n))^\alpha\right) \leq \exp\left(-\frac{1}{2}(E_\alpha^n(\mu(S_0)))^\alpha\right). \end{aligned}$$

We use this to prove that the set $T_n \setminus (\bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1})$ has small density in T_n . For all $k \in \{1, \dots, n\}$, there is a square $S_k \in \mathcal{S}$ such that $f^k(T_n) \subset S_k$. In particular, f^{n+1} is injective in T_n . Thus,

$$\begin{aligned} & \text{meas}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}\right) \\ & \leq \frac{1}{\mu(T_n, (f^{n+1})')^2} \cdot \text{meas}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \text{pack}(f^{n+1}(T_n))} S'\right), \end{aligned}$$

and

$$\text{meas}(T_n) \geq \frac{1}{\mathcal{M}(T_n, (f^{n+1})')^2} \text{meas}(f^{n+1}(T_n)).$$

Hence,

$$\begin{aligned} & \text{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}, T_n\right) \\ & \leq \frac{\mathcal{M}(T_n, (f^{n+1})')^2}{\mu(T_n, (f^{n+1})')^2} \text{dens}\left(f^{n+1}(T_n) \setminus \bigcup_{S' \in \text{pack}(f^{n+1}(T_n))} S', f^{n+1}(T_n)\right) \\ & \leq \left(\frac{\mathcal{M}(T_n, (f^{n+1})')}{\mu(T_n, (f^{n+1})')}\right)^2 \exp\left(-\frac{1}{2}(E_\alpha^n(\mu(S_0)))^\alpha\right). \end{aligned} \tag{15}$$

To estimate $\mathcal{M}(T_n, (f^{n+1})')/\mu(T_n, (f^{n+1})')$, let $k \in \{0, \dots, n\}$ and $w_0 \in f^k(T_n)$. Then

$$f^k(T_n) \subset S_k \subset \overline{D(w_0, \sigma|w_0|^{-(d-1)})}.$$

By Lemma 3.3, f is injective in $D(w_0, 2\sigma|w_0|^{-(d-1)})$. The Koebe distortion theorem (Lemma 4.5) yields that for all $w \in f^k(T_n)$,

$$\frac{|f(w) - f(w_0)|}{|w - w_0|} \geq \frac{4}{9}|f'(w_0)|.$$

By Lemma 2.4, $|f'(w_0)| \geq 5$. Thus,

$$\text{diam}(f^{k+1}(T_n)) > 2 \text{diam}(f^k(T_n)).$$

Induction yields

$$\begin{aligned} \text{diam}(f^k(T_n)) &< \frac{1}{2^{n-k}} \text{diam}(f^n(T_n)) \leq \frac{1}{2^{n-k}} \cdot \frac{\sigma}{\mathcal{M}(f^n(S_0))^{d-1}} \\ &\leq \frac{1}{2^{n-k}} \cdot \frac{\sigma}{\mathcal{M}(f^k(S_0))^{d-1}}. \end{aligned}$$

In particular, for $z_k \in f^k(T_n)$,

$$f^k(T_n) \subset D\left(z_k, \frac{2^{k-n}\sigma}{\mathcal{M}(f^k(S_0))^{d-1}}\right) \subset D\left(z_k, 2^{k-n}\sigma|z_k|^{-(d-1)}\right).$$

Since f is injective in $D(z_k, 2\sigma|z_k|^{-(d-1)})$, the Koebe distortion theorem (Lemma 4.5) yields

$$\frac{\mathcal{M}(f^k(T_n), f')}{\mu(f^k(T_n), f')} \leq \left(\frac{1 + 2^{k-n-1}}{1 - 2^{k-n-1}}\right)^4.$$

Since $(f^{n+1})'(z) = \prod_{k=0}^n f'(f^k(z))$, we get

$$\begin{aligned} \frac{\mathcal{M}(T_n, (f^{n+1})')}{\mu(T_n, (f^{n+1})')} &\leq \prod_{k=0}^n \left(\frac{1 + 2^{k-n-1}}{1 - 2^{k-n-1}}\right)^4 = \left(\prod_{j=1}^{n+1} \frac{1 + 2^{-j}}{1 - 2^{-j}}\right)^4 \\ &\leq \left(\prod_{j=1}^{\infty} \frac{1 + 2^{-j}}{1 - 2^{-j}}\right)^4 =: C_2, \end{aligned}$$

where $C_2 \in (0, \infty)$. Together with (15), this implies

$$\text{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}, T_n\right) \leq C_2^2 \exp\left(-\frac{1}{2}(E_\alpha^n(\mu(S_0)))^\alpha\right).$$

Thus,

$$\begin{aligned} \text{meas}(S_0 \setminus T) &= \sum_{n=0}^{\infty} \sum_{T_n \in \mathcal{K}_n} \text{meas}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}\right) \\ &= \sum_{n=0}^{\infty} \sum_{T_n \in \mathcal{K}_n} \text{dens}\left(T_n \setminus \bigcup_{T_{n+1} \in \mathcal{K}_{n+1}} T_{n+1}, T_n\right) \cdot \text{meas}(T_n) \\ &\leq \sum_{n=0}^{\infty} C_2^2 \exp\left(-\frac{1}{2}(E_\alpha^n(\mu(S_0)))^\alpha\right) \sum_{T_n \in \mathcal{K}_n} \text{meas}(T_n) \\ &\leq C_2^2 \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}(E_\alpha^n(\mu(S_0)))^\alpha\right) \cdot \text{meas}(S_0). \end{aligned}$$

For all large x , $\exp(\alpha x^\alpha) \geq x^\alpha + 2 \log 2$, and thus

$$\exp\left(-\frac{1}{2}(E_\alpha(x))^\alpha\right) = \exp\left(-\frac{1}{2} \exp(\alpha x^\alpha)\right) \leq \frac{1}{2} \exp\left(-\frac{1}{2} x^\alpha\right).$$

Induction yields

$$\sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}(E_{\alpha}^n(\mu(S_0)))^{\alpha}\right) \leq \sum_{n=0}^{\infty} \frac{1}{2^n} \exp\left(-\frac{1}{2}\mu(S_0)^{\alpha}\right) = 2 \exp\left(-\frac{1}{2}\mu(S_0)^{\alpha}\right).$$

Thus,

$$\text{meas}(S_0 \setminus J_A) \leq \text{meas}(S_0 \setminus T) \leq 2C_2^2 \exp\left(-\frac{1}{2}\mu(S_0)^{\alpha}\right) \text{meas}(S_0).$$

To conclude that $\text{meas}(\mathbb{C} \setminus J_A) < \infty$, fix $R > 0$ such that $B_0 \subset D(0, R/2)$. For $r \geq R$, define

$$\text{ann}(r) := \{z : r \leq |z| \leq 2r\}.$$

Then

$$\bigcup_{\substack{S_0 \in \mathcal{S} \\ S_0 \cap \text{ann}(r) \neq \emptyset}} S_0 \subset \left\{z : \frac{r}{2} \leq |z| \leq 3r\right\} \subset \overline{D(0, 3r)}.$$

We get

$$\begin{aligned} \text{meas}(\text{ann}(r) \setminus (J_A \cup E_2)) &\leq \sum_{\substack{S_0 \in \mathcal{S} \\ S_0 \cap \text{ann}(r) \neq \emptyset}} 2C_2^2 \exp\left(-\frac{1}{2}\mu(S_0)^{\alpha}\right) \text{meas}(S_0) \\ &\leq 2C_2^2 \exp\left(-\frac{1}{2}\left(\frac{r}{2}\right)^{\alpha}\right) \sum_{\substack{S_0 \in \mathcal{S} \\ S_0 \cap \text{ann}(r) \neq \emptyset}} \text{meas}(S_0) \leq 2C_2^2 \exp\left(-\frac{r^{\alpha}}{2^{1+\alpha}}\right) \text{meas}(D(0, 3r)) \\ &= 18\pi C_2^2 r^2 \exp\left(-\frac{r^{\alpha}}{2^{1+\alpha}}\right) \leq \exp\left(-\frac{r^{\alpha}}{2^{2+\alpha}}\right) \end{aligned}$$

if r is sufficiently large. Applying this to the annuli $\text{ann}(2^n R)$, $n \geq 0$, yields

$$\begin{aligned} \text{meas}(\mathbb{C} \setminus J_A) &\leq \text{meas}(D(0, R)) + \text{meas}(E_2) + \sum_{n=0}^{\infty} \text{meas}(\text{ann}(2^n R) \setminus (J_A \cup E_2)) \\ &\leq \text{meas}(D(0, R)) + \text{meas}(E_2) + \sum_{n=0}^{\infty} \exp\left(-\frac{R^{\alpha}}{2^{2+\alpha}} 2^{n\alpha}\right) < \infty. \end{aligned}$$

This completes the proof. □

5. Verification of the properties of Examples 1.1 and 1.3

We first verify the properties of Example 1.1, that is, we prove that the Fatou set and the non-escaping set of $\sin(z)$ and $\sin(z^2)$ have infinite Lebesgue measure.

First, let $f(z) = \sin(z)$. From $f(0) = 0$ and $f'(0) = 1$, it follows that $F(f) \neq \emptyset$, and that there exists a component of $F(f)$ where $(f^n(z))$ tends to zero. Since $F(f)$ is open and 2π -periodic, the Lebesgue measure of $F(f)$ and $\mathbb{C} \setminus I(f)$ is infinite.

Let us now consider the function $f(z) = \sin(z^2)$. Note that f has a superattracting fixed point at zero. Let $\varepsilon > 0$ such that $D(0, \varepsilon)$ is contained in the attractive basin of zero. There

exists $\delta \in (0, \pi/2)$ such that $\sin(D(\pi k, \delta)) \subset D(0, \varepsilon)$ for all $k \in \mathbb{Z}$. Let $D_k := D(\pi k, \delta)$ and $p(z) = z^2$. Then $p^{-1}(D_k)$ is contained in the attractive basin of zero of the function f . We get

$$\text{meas}(p^{-1}(D_k)) = 2 \int_{D_k} \left(\frac{1}{2\sqrt{|z|}} \right)^2 dx dy \geq \frac{1}{2(|k|\pi + \delta)} \text{meas}(D_k) = \frac{\pi \delta^2}{2(|k|\pi + \delta)}.$$

Summing up over all k yields that the attractive basin of zero has infinite measure.

In the remaining part of this section, we consider the function

$$h(z) = \frac{1}{2} \exp(z^3 + iz) - \frac{1}{2} \exp(-z^3 + iz) = \exp(iz) \sinh(z^3)$$

given in Example 1.3. Note that h satisfies all assumptions of Theorem 1.4 except for condition (3). Moreover, function h has a superattracting fixed point at zero. Recall that we want to prove that the attractive basin of zero has infinite Lebesgue measure, so that in particular the Lebesgue measure of $\mathbb{C} \setminus (J(h) \cap A(h))$ is infinite.

To do so, fix $\varepsilon > 0$ such that $D(0, \varepsilon)$ is contained in the attractive basin of zero. For large $r_0 > 1$, let

$$B := \left\{ r e^{i\theta} : r \geq r_0, \left| \theta - \frac{\pi}{2} \right| \leq \frac{1}{r^2 \log r} \right\}.$$

We have

$$\text{meas}(B) = \int_{r_0}^{\infty} \int_{\pi/2-1/(r^2 \log r)}^{\pi/2+1/(r^2 \log r)} r d\theta dr = \int_{r_0}^{\infty} \frac{2}{r \log r} dr = 2 \int_{\log(r_0)}^{\infty} \frac{1}{u} du = \infty.$$

We now show that if r_0 is sufficiently large, then $h(B) \subset D(0, \varepsilon)$, and hence B is contained in the attractive basin of zero. Let $z = r e^{i\theta} \in B$. Then

$$\begin{aligned} |h(z)| &\leq |\exp(iz)| \cdot \frac{1}{2} (|\exp(z^3)| + |\exp(-z^3)|) \\ &= \exp(-r \sin(\theta)) \cdot \frac{1}{2} (\exp(r^3 \cos(3\theta)) + \exp(-r^3 \cos(3\theta))) \\ &\leq \exp(-r \sin(\theta)) \cdot \exp(r^3 |\cos(3\theta)|). \end{aligned}$$

We have

$$\sin(w) = 1 + O\left(\left(w - \frac{\pi}{2}\right)^2\right) \text{ as } w \rightarrow \frac{\pi}{2},$$

and

$$\cos(w) = \left(w - \frac{3\pi}{2}\right) + O\left(\left(w - \frac{3\pi}{2}\right)^3\right) \text{ as } w \rightarrow \frac{3\pi}{2}.$$

Since $|\theta - \pi/2| \leq 1/(r^2 \log r)$, this implies

$$-\sin(\theta) \leq -\frac{1}{2}$$

and

$$|\cos(3\theta)| \leq 2 \cdot \frac{3}{r^2 \log r}$$

if r is sufficiently large. Thus,

$$|h(z)| \leq \exp\left(-\frac{1}{2}r\right) \cdot \exp\left(6 \cdot \frac{r}{\log r}\right) \leq \exp\left(-\frac{1}{4}r\right) < \varepsilon$$

if r is sufficiently large. So $h(B) \subset D(0, \varepsilon) \subset F(h) \setminus I(h)$ if r_0 is sufficiently large.

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