

DIGIT PATTERNS AND TRANSCENDENTAL NUMBERS

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Abstract

We use a theorem of Loxton and van der Poorten to prove the transcendence of certain real numbers defined by digit patterns. Among the results we prove are the following. If k is an integer at least 2, P is any nonzero pattern of digits base k , and $e_p^{(r)}(n) \in [0, r-1]$ counts the number of occurrences (mod r) of P in the base k representation of n , then $\eta(e_p^{(r)}) = \sum_{n=0}^{\infty} e_p^{(r)}(n)/r^n$ is transcendental except when $k = 3$, $P = 1$ and $r = 2$. When $(r, k-1) = 1$ the linear span of the numbers $\eta(e_p^{(r)})$ has infinite dimension over \mathbb{Q} , where P ranges over all patterns base k without leading zeros.

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1. Introduction

This paper is concerned with proving the transcendence of certain real numbers defined using digit patterns. Specifically, let k be an integer at least 2, and let P be a pattern of digits base k . Further let $e_p(n)$ denote the number of occurrences of P in the base k representation of n , allowing for leading zeros as necessary (see [3], [5], or [13, Section 2]). If $r \geq 2$ is any integer, we define the sequence $e_p^{(r)}$ by

$$(1) \quad e_p^{(r)}(n) \equiv e_p(n) \pmod{r}, \quad \text{where } 0 \leq e_p^{(r)}(n) \leq r-1.$$

With these definitions the following results hold.

TRANSCENDENCE THEOREM. (a) *If P is any nonzero pattern of digits base k , the real number*

$$(2) \quad \eta(e_P^{(r)}) = \sum_{n=0}^{\infty} \frac{e_P^{(r)}(n)}{r^n}$$

is transcendental, unless $k = 3$, $P = 1$ and $r = 2$ (in which case $\eta(e_1^{(2)}) = 2/3$).

(b) *Let $\mathbb{P} = \{P\}$ be any set of patterns base k with the following properties:*

- (i) *if $P \in \mathbb{P}$, P has no leading zeros (e.g. $P \neq 0Q$ for some pattern Q);*
- (ii) *for any pattern Q , there is a digit i (base k) so that $Qi \notin \mathbb{P}$.*

If $(r, k - 1) = 1$, then the corresponding set of real numbers

$$\{\eta(e_P^{(r)}): P \in \mathbb{P}\} \cup \{1\}$$

is linearly independent over \mathbb{Q} . Moreover every finite sum

$$\sum_{P \in \mathbb{P}} c_P \eta(e_P^{(r)}), \quad c_P \in \mathbb{Q},$$

is transcendental.

For example, if $P = 1^j$ is the pattern consisting of the digit 1 repeated j times, the set $\mathbb{P} = \{1^j: j \geq 1\}$ satisfies (i) and (ii) for any $k \geq 2$. Hence the linear span of the numbers $\eta(e_P^{(r)})$ (for $P \in \mathbb{P}$) has infinite dimension over \mathbb{Q} for any $r \geq 2$ for which $(r, k - 1) = 1$.

Our proof is based on the following theorem of Loxton and van der Poorten (see [8, II, obs. 2.16], [11] and [12]).

THEOREM (Loxton and van der Poorten). *Let $r \geq 2$ be fixed and let $\{\alpha_n\}$ be a sequence of integers taken from a finite set. If $\{\alpha_n\}$ is generated by a finite automaton, then $\sum_{n=0}^{\infty} \alpha_n/r^n$ is either rational or transcendental and is rational if and only if the sequence $\{\alpha_n\}$ is ultimately periodic.*

A sequence $\{\alpha_n\}$, whose terms α_n are taken from a finite alphabet A , is said to be generated by a finite k -automaton, or more simply, is k -automatic, if the set of distinct sequences of the form

$$\{\alpha_{k^r n + j}\}, \quad 0 \leq j \leq k^r - 1,$$

is finite (see [2, Section 3]). This condition is conveniently expressed in terms of digit substitutions (see [2], [6], [7]). The sequence $\alpha = \{\alpha_n\}$ is said to be a fixed point of a uniform k -substitution if there is a rule which replaces each symbol of A by a string of k symbols of A , under which α is left fixed. Then α is k -automatic if and only if α is the image of a fixed point

of a k -substitution, that is, if and only if α is a sequence derived from a fixed point by renaming or identifying letters of A .

We apply the Loxton and van der Poorten theorem to certain groups of sequences that we investigated in [13]. These groups are defined as follows. Let G be an abelian group and $a = \{a(n)\}$ a sequence of elements in G . Associated to a are the vectors

$$(3) \quad X_n^q = (a(nk^q), a(nk^q + 1), \dots, a(nk^q + k^q - 1)),$$

for each $q \geq 0$, which we refer to as k^q -segments of a . The group $\Gamma_k(G)$ is defined to be the set of all sequences for which $a^{-1}(n)X_n^q$ is periodic, that is, for which there is a positive integer M satisfying

$$(4) \quad a^{-1}(n)X_n^q = a^{-1}(m)X_m^q, \quad \text{for } n \equiv m \pmod{M},$$

for all $q \geq 0$, where the scalar multiplication is performed by applying $a^{-1}(n)$ to each component of X_n^q using the operation in G . If G is an additive group we write this in the form

$$X_n^q - a(n) = X_m^q - a(m), \quad \text{for } n \equiv m \pmod{M}.$$

The least positive M for which (4) holds is called the conductor of a , and is denoted by $M(a)$. The moduli M for which (4) holds are then exactly the multiples of $M(a)$.

In [13] it is shown that $\Gamma_k(G)$ is a group under componentwise multiplication (addition, if G is additive), and that the sequences for which $M(a)$ divides a power of k form a subgroup $\Lambda_k(G)$.

An important example is the group $\Gamma_k(\mathbb{Z})$. Its subgroup $\Lambda_k(\mathbb{Z})$ is generated by the constant sequences and the pattern sequences e_p , where P is a pattern of digits base k (not all of which are zero), and $e_p(n)$ is defined as above. The related sequences $e_p^{(r)}$ and a_p defined by (1) and

$$(5) \quad a_p(n) = \zeta_r^{e_p(n)}, \quad \zeta_r \text{ a primitive } r\text{th root of unity,}$$

generate the respective groups $\Lambda_k(\mathbb{Z}_r)$ (additively) and $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ (multiplicatively). These are the sequences to which we apply the Loxton and van der Poorten theorem.

We first show (in Section 2) that for finite G the sequences in $\Gamma_k(G)$ are all k -automatic. This fact allows us to apply the Loxton and van der Poorten theorem to any sequence in $\Lambda_k(\mathbb{Z}_r)$: for a given sequence e in $\Lambda_k(\mathbb{Z}_r)$, the number

$$(6) \quad \eta(e) = \sum_{n=0}^{\infty} \frac{e(n)}{r^n}$$

is rational or transcendental. The sequences e for which $\eta(e)$ is rational are exactly the ultimately periodic sequences in $\Lambda_k(\mathbb{Z}_r)$, and these obviously form a subgroup $H_k(\mathbb{Z}_r)$ of $\Lambda_k(\mathbb{Z}_r)$. A large part of this paper is devoted to determining the subgroup $H_k(\mathbb{Z}_r)$. In Section 3 we prove several lemmas concerning the group of periodic sequences in $\Gamma_k(G)$, for arbitrary abelian groups G .

In Section 4 we determine all periodic sequences in $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ which have conductor 1 (Theorem 2). They form a cyclic subgroup of $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ which is generated by the sequence $\{\zeta_r^{\varepsilon\sigma(n)}\}$, where $\sigma(n)$ is the sum of the digits of n base k , and ε is the smallest divisor of r with the property that $r|\varepsilon(k-1)$. Using this and a certain endomorphism of $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ we show that any periodic sequence in $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ has a period of the form $bM(a)$, where $b|(r, k-1)$. If $(r, k-1) = 1$ this allows us to give a complete determination of the groups $H_k(\langle\langle\zeta_r\rangle\rangle)$ and $H_k(\mathbb{Z}_r)$.

The results of Section 4 lead directly to the transcendence results in Section 5, which are mostly concerned with the transcendence of the numbers in (6) defined using sequences in $\Lambda_k(\mathbb{Z}_r)$.

We obtain limited results for the group $\Lambda_k(\langle\langle\zeta_r\rangle\rangle)$. We show that when P is an arbitrary nonzero pattern base k with at least 2 digits, and $a_p(n)$ is defined in (5), then at least one of the series

$$\sum_{n=0}^{\infty} \frac{a_p^i(n)}{r^n}, \quad \text{for } 1 \leq i \leq r-1,$$

is transcendental.

Finally, we show that if $a \in \Lambda_k(\langle\langle\zeta_r\rangle\rangle)$ with $M(a) = 1$, so that a is k -multiplicative in Delange's sense [9], then

$$\sum_{n=0}^{\infty} \frac{a(n)}{r^n}$$

is transcendental if and only if a is not a power of the sequence $\{\zeta_r^{\varepsilon\sigma(n)}\}$ defined above.

We remark that our methods here, as in [13], are elementary. The Loxton and van der Poorten theorem is the only non-elementary component of our argument. We also refer the reader to Allouche [1], which contains results of a related nature, and to Allouche and Shallit [4], which generalizes the notion of an automatic sequence.

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2. Digit substitutions

Our first theorem shows that the sequences in $\Gamma_k(G)$ are k -automatic.

THEOREM 1. *Let G be a finite abelian group and let a be any sequence in $\Gamma_k(G)$. Then for suitable q , the sequence $\{X_n^q\}$ of k^q -segments of a is the fixed point of a k^r -substitution, for some $r \geq 1$. In particular, a is k -automatic.*

PROOF. Let $a \in \Gamma_k(G)$ with modulus M , let $q \geq 0$ and consider the vector

$$(X_0^q, X_1^q, \dots, X_{M-1}^q).$$

For each q we construct a code vector κ_q of length M which encodes relations among the X_n^q , for $n < M$. We do this as follows: each entry of κ_q will be an element (i, g) of $\mathbb{Z}_M \times G$, where $0 \leq i \leq M - 1$ and $g \in G$.

Letting $\kappa_q(n)$ denote the n th entry of κ_q , we set $\kappa_q(0) = (0, \iota)$ where ι is the identity in G and $\kappa_q(n) = (0, g)$ whenever $X_n^q = gX_0^q$. If m is the first index where $X_m^q \neq gX_0^q$ for any $g \in G$, set $\kappa_q(m) = (1, \iota)$, and set $\kappa_q(n) = (1, g)$ whenever $X_n^q = gX_m^q$. Continuing in this manner, we obtain a vector κ_q for which

$$(7) \quad \kappa_q(n) \in \mathbb{Z}_M \times G, \quad \text{for } 0 \leq n \leq M - 1$$

and

$$(8) \quad \kappa_q(n_1) = \kappa_q(n_2)(0, g) \quad \text{if and only if} \quad X_{n_1}^q = gX_{n_2}^q.$$

Since there are only finitely many vectors satisfying (7), there must be $q \geq 0$ and $r > 0$ for which $\kappa_q = \kappa_{q+r}$. For this q and r , (8) shows that

$$(9) \quad X_{n_1}^q = gX_{n_2}^q \quad \text{if and only if} \quad X_{n_1}^{q+r} = gX_{n_2}^{q+r},$$

for $n_1, n_2 < M$ and $g \in G$. We shall show that the map

$$X_n^q \rightarrow X_n^{q+r}, \quad n \geq 0,$$

is equivalent to a digit substitution. First we note that this map is well-defined. Suppose $X_m^q = X_n^q$. If $m \equiv m_1, n \equiv n_1 \pmod{M}$ and $m_1, n_1 < M$, then it is an easy exercise using (4), (9) and (4) again (with $q+r$ for q) to show that $X_{n_1}^{q+r} = X_{m_1}^{q+r}$.

Now number the $|G| = m$ elements of G by g_0, g_1, \dots, g_{m-1} and replace g_i in X_n^q and X_n^{q+r} by i . Taken together, the entries in X_n^q give a digit d_n base m^{k^q} , and the entries in X_n^{q+r} give a sequence of k^r digits $d_{n,1}, d_{n,2}, \dots, d_{n,k^r}$ base m^{k^q} . The sequence $\{d_n\}$ is therefore fixed by

the substitution

$$(10) \quad d_n \rightarrow d_{n,1}, d_{n,2}, \dots, d_{n,k^r}, \quad \text{for } n \geq 0.$$

The concatenated sequences $X_0^q, X_1^q, \dots, X_n^q, \dots = X_0^{q+r}, X_1^{q+r}, \dots, X_n^{q+r}, \dots$ coincide with the sequence a . Since the sequence a is the image of the fixed point $\{d_n\}$ under the map $d_n \rightarrow X_n^q$, a is k -automatic.

COROLLARY. *Let $G = \mathbb{Z}_m$ and let P be a nonzero pattern base k of length $d \geq 1$. If $e_p^{(m)}(n) \equiv e_p(n) \pmod{m}$, where $0 \leq e_p^{(m)}(n) < m$, then the sequence $\{X_n^{d-1}\}$ of k^{d-1} -segments of the sequence $e_p^{(m)}$ is fixed by a k -substitution.*

PROOF. By the proof of the theorem we only need to show for the sequence $e_p^{(m)}$ that $X_n^{d-1} \rightarrow X_n^d$ is well-defined, and for this it suffices to prove that

$$(11) \quad X_{n_1}^{d-1} \equiv X_{n_2}^{d-1} + \varepsilon \pmod{m} \text{ implies } X_{n_1}^d \equiv X_{n_2}^d + \varepsilon \pmod{m},$$

for all n_1 and n_2 , where $0 \leq \varepsilon \leq m - 1$. (We use additive notation since G is an additive group.) Note that (11) holds if and only if

$$(12) \quad e_p^{(m)}(n_1 k^{d-1} + j) \equiv e_p^{(m)}(n_2 k^{d-1} + j) + \varepsilon \pmod{m}$$

for all $j \in [0, k^{d-1})$ implies

$$(13) \quad e_p^{(m)}(n_1 k^d + j') \equiv e_p^{(m)}(n_2 k^d + j') + \varepsilon \pmod{m}$$

for all $j' \in [0, k^d)$. However the definition of $e_p^{(m)}$ implies easily that

$$e_p^{(m)}(n_i k^d + j') = e_p^{(m)}(n_i k^{d-1} + j) + e_p^{(m)}(j')$$

where $j' = jk + s$ and $0 \leq s \leq k - 1$. Thus (12) implies (13).

Alternate proofs of Theorem 1 have been given by Shallit (private communication) and Morton [14].

3. Properties of periodic sequences in $\Gamma_k(G)$

The lemmas in this section form a basis for the determination in Section 4 of the periodic sequences in $\Lambda_k(G)$ for various groups G .

Let G be any abelian group. It is easy to show that

LEMMA 1. *The set of ultimately periodic sequences in $\Gamma_k(G)$ is a subgroup of $\Gamma_k(G)$.*

LEMMA 2. *If $a \in \Gamma_k(G)$ is ultimately periodic with period ρ , then a is purely periodic with period ρ . Every period of a is also a period of the sequence $X^q = \{X_n^q\}$ of k^q -segments of a .*

PROOF. We have

$$(14) \quad a^{-1}(n)X_n^q = a^{-1}(m)X_m^q, \quad \text{for } n \equiv m \pmod{M}$$

and

$$(15) \quad a(n + \rho) = a(n), \quad \text{for } n \geq N, \text{ for some } N \geq 0.$$

It follows easily from (3) that

$$(16) \quad X_{n+\rho}^q = X_n^q \quad \text{if } nk^q \geq N.$$

Now choose q so that $k^q \geq N$. Then for $n, m \geq 1$, the equation

$$a^{-1}(n + \rho)X_{n+\rho}^q = a^{-1}(m + \rho)X_{m+\rho}^q, \quad \text{for } n \equiv m \pmod{M}$$

implies by (16) that

$$a^{-1}(n + \rho)X_n^q = a^{-1}(m + \rho)X_m^q, \quad \text{for } n \equiv m \pmod{M}.$$

Hence (14) shows that

$$a(n)a^{-1}(m) = a(n + \rho)a^{-1}(m + \rho)$$

or

$$(17) \quad a(m + \rho)a(n) = a(m)a(n + \rho),$$

for $n \equiv m \pmod{M}$ and $n, m \geq 1$.

Given $m \geq 1$, choose $n \geq N$ so that $n \equiv m \pmod{M}$. Then (15) and (17) imply $a(m) = a(m + \rho)$, if $m \geq 1$. It follows from (15) and (16) that

$$X_m^q = X_{m+\rho}^q, \quad \text{for } m \geq 1 \text{ and all } q \geq 0.$$

Finally, since $M \geq 1$ we see from what we have just shown that

$$a^{-1}(0)X_0^1 = a^{-1}(M)X_M^1 = a^{-1}(M + \rho)X_{M+\rho}^1 = a^{-1}(\rho)X_\rho^1.$$

Considering the second components in the first and last terms of this equality gives easily that $a(0) = a(\rho)$. This shows that (15) and (16) hold with $N = 0$ and proves Lemma 2.

LEMMA 3. If $a \in \Gamma_k(G)$ and a is periodic with least period ρ , then $\rho = bM(a)$ where $(b, k) = 1$.

PROOF. The conductor $M(a)$ is the least positive integer M for which (4) holds, that is, for which $n \equiv m \pmod{M}$ implies

$$(18) \quad a(nk^q + j)a(m) = a(mk^q + j)a(n), \quad \text{for all } j \text{ with } 0 \leq j \leq k^q - 1.$$

This does hold if $M = \rho$, and therefore $M(a) | \rho$.

Now let M be any modulus for a for which $\rho|Mk^q$ for some $q \geq 1$. Taking $j = 0$ and $m = n + M$ in (18) gives

$$a(nk^q)a(n + M) = a(nk^q + Mk^q)a(n) = a(nk^q)a(n),$$

which shows that $a(n + M) = a(n)$ for all $n \geq 0$. Therefore M is a period of a and $\rho|M$. Hence $\rho|Mk^q$ implies $\rho|M$, for any $q \geq 1$. If $\rho = bM(a)$, then a straightforward argument shows that $(b, k) = 1$.

LEMMA 4. *If $a \in \Gamma_k(G)$, then $X^1 = \{X_n^1\} \in \Gamma_k(G^k)$, with $M(X^1)|M(a)$.*

PROOF. Define $X_n = X_n^1$ and $\tilde{X}_n = (X_{nk}, X_{nk+1}, \dots, X_{nk+k-1})$. Then $X_n \in G^k$, and $X^1 \in \Gamma_k(G^k)$ if and only if

$$(19) \quad X_n^{-1}\tilde{X}_n = X_m^{-1}\tilde{X}_m \quad \text{for } n \equiv m \pmod{\tilde{M}}$$

for some \tilde{M} . (See [13, Lemma 2].) We show this holds with $\tilde{M} = M = M(a)$. By definition of $\Gamma_k(G)$, the congruence $n \equiv m \pmod{M}$ implies that

$$(20) \quad X_n = a(n)a^{-1}(m)X_m$$

and

$$X_{nk+j} = a(nk + j)a^{-1}(mk + j)X_{mk+j}, \quad \text{for } 0 \leq j \leq k - 1.$$

Hence

$$\begin{aligned} X_n^{-1}\tilde{X}_n &= a^{-1}(n)a(m)X_m^{-1}(\dots, a(nk + j)a^{-1}(mk + j)X_{mk+j}, \dots) \\ &= X_m^{-1}(\dots, a^{-1}(n)a(m)a(nk + j)a^{-1}(mk + j)X_{mk+j}, \dots) \\ &= X_m^{-1}(\dots, X_{mk+j}, \dots) = X_m^{-1}\tilde{X}_m, \quad \text{for } n \equiv m \pmod{M}, \end{aligned}$$

from (20). Since $M(X^1)$ is the least modulus \tilde{M} for which (19) holds, we have $M(X^1)|M(a)$.

COROLLARY. *If $a \in \Gamma_k(G)$, then $X^q = \{X_n^q\} \in \Gamma_k(G^{k^q})$ for $q \geq 1$, and $M(X^{q+1})|M(X^q)$ for $q \geq 0$. (Note that $X^0 = a$.)*

This follows by induction from Lemma 4.

Let $\rho_q = \rho(X^q)$ be the least period of the sequence $X^q = \{X_n^q\}$, if it exists, for $q \geq 0$.

LEMMA 5. If $a \in \Gamma_k(G)$ with least period $\rho = \rho_0$ then X^q is periodic for all $q \geq 0$ and

- (a) $\rho_q = (\rho_q, k)\rho_{q+1}$, for $q \geq 0$, and
- (b) $\rho = (\rho, k^q)\rho_q$.

PROOF. To prove $\rho = (\rho, k)\rho_1$, note that ρ_1 is the least integer for which $X_{n+\rho_1}^1 = X_n^1$ for $n \geq 0$. This holds if and only if

$$(21) \quad a((n + \rho_1)k + j) = a(nk + j), \quad \text{for } 0 \leq j \leq k - 1, n \geq 0.$$

Equation (21) is satisfied if and only if $\rho_1 k$ is a period of a . Hence ρ_1 is the least positive integer for which $\rho | \rho_1 k$. By Lemma 2, $\rho_1 | \rho$, so ρ_1 is the least positive integer for which $(\rho / \rho_1) | k$, that is, $\rho / \rho_1 = (\rho, k)$. Applying the same argument to X^q and X^{q+1} in place of a and X^1 proves (a). The proof of (b) is similar.

To summarize, if $a \in \Gamma_k(G)$ is a periodic sequence and we apply Lemma 3 to the sequence $X^q \in \Gamma_k(G^{k^q})$, we see that

$$\rho_q = b_q M_q, \quad q \geq 0,$$

where $(b_q, k) = 1$ and $M_q = M(X^q)$. (Note that $M_0 = M(a)$ and $b_0 = b$, as defined in Lemma 3.) Further, Lemma 5 shows that $\rho_{q+1} | \rho_q$ and $(\rho_q / \rho_{q+1}) | k$. By the Corollary to Lemma 4,

$$(22) \quad \rho_q / \rho_{q+1} = (b_q / b_{q+1})(M_q / M_{q+1}),$$

where the first and third quotients lie in \mathbb{Z} , and the second involves none of the prime factors of k .

Certainly each of the sequences ρ_q , b_q and M_q is constant past a certain point.

LEMMA 6. If $a \in \Gamma_k(G)$ is periodic and $M(a)$ divides a power of k (that is, $a \in \Lambda_k(G)$), then $b = b_0 = b_q$ for $q \geq 0$. If $\rho_{q+1} = \rho_q$ then the sequence X^q has conductor 1 in $\Gamma_k(G^{k^q})$, and $(\rho_q, k) = 1$.

PROOF. From (22), $b_q = b_{q+1}$, for $q \geq 0$, since ρ_q / ρ_{q+1} and M_q / M_{q+1} are products of prime factors of k and $(b_q, k) = (b_{q+1}, k) = 1$. Furthermore if $\rho_q = \rho_{q+1}$, then by Lemma 5, ρ_q and M_q are relatively prime to k . But M_q divides $M(a)$, which divides a power of k , so $M_q = 1$. Therefore the sequence X^q has conductor 1 in $\Gamma_k(G^{k^q})$ and period $\rho_q = b_q$ which is prime to k .

We now focus on sequences with conductor 1. Suppose $a \in \Lambda_k(G)$, where $M(a) = 1$. Then

$$X_n = a(n)a^{-1}(0)X_0, \quad \text{for } n \geq 0.$$

Multiplying a by $a^{-1}(0)$ we can assume $a(0) = 1$. This gives $X_n = a(n)X_0$. Writing out both sides of this equation we see that

$$(23) \quad a(nk + j) = a(n)a(j), \quad \text{for } n \geq 0, 0 \leq j \leq k - 1.$$

Now write $n = j_t k^t + j_{t-1} k^{t-1} + \dots + j_0$. Then using (23) repeatedly gives

$$a(n) = a(j_t k^{t-1} + \dots + j_1) a(j_0) = a(j_t) a(j_{t-1}) \dots a(j_1) a(j_0).$$

We have thus proved the following.

LEMMA 7. *If $a \in \Lambda_k(G)$, $M(a) = 1$, and $a(0) = 1$, then $a(n) = \prod_{i=0}^t a(j_i)$, where $n = j_t j_{t-1} \dots j_0$ (base k).*

The sequences described in Lemma 7 are called k -multiplicative sequences by Delange [9]. Combining Lemmas 6 and 7 we see that for every periodic sequence $a \in \Lambda_k(G)$ there is a q for which $(X_k^q)^{-1} X^q \in \Gamma_k(G^{k^q})$ is k -multiplicative.

4. Periodicity results

We first determine the periodicity of sequences defined by patterns consisting of single digits. We use this in Theorem 3 to describe the possible periods of periodic sequences in $\Lambda_k(\langle \zeta \rangle)$, where ζ is an r th root of unity.

THEOREM 2. *Let i be a digit base k and $a_i(n) = \zeta^{\varepsilon_i(n)}$, where $\zeta = \zeta_r$ is an r th root of unity. If S is a non-empty set of nonzero digits base k , and*

$$(24) \quad a_s = \prod_{i \in S} a_i^{\varepsilon_i},$$

where r does not divide ε_i , then a_s is not ultimately periodic, unless $a_s(n) = \zeta^{\varepsilon_1 \sigma(n)}$ where $\sigma = e_1 + 2e_2 + \dots + (k - 1)e_{k-1}$, and $r / (r, \varepsilon_1) | (k - 1)$. In this case a_s has least period $r / (r, \varepsilon_1)$, and $S = \{i | i \not\equiv 0 \pmod{r / (r, \varepsilon_1)}\}$.

PROOF. Assume a_s is ultimately periodic and that ρ is the least period of a_s . We know $M(a_s) = 1$ in $\Gamma_k(\langle \zeta \rangle)$ because the patterns in S all have length

1. Lemmas 2 and 3 show that a_s is purely periodic and that $(\rho, k) = 1$. Hence

$$(25) \quad a_s(n + m\rho) = a_s(n), \quad \text{for } m, n \geq 0.$$

I. Assume $m\rho$ has j as its last digit. If i is any digit for which $i + j < k$, then by Lemma 7,

$$a_s(i + m\rho) = a_s(m\rho)a_s^{-1}(j)a_s(i + j) = a_s^{-1}(j)a_s(i + j).$$

From (25) with $n = i$ it follows that

$$(26) \quad a_s(i + j) = a_s(i)a_s(j).$$

In fact this holds for any $i, j \in [0, k - 1]$ for which $i + j < k$. To see this we need only check that any j is the last digit of $m\rho$ for suitable m . This holds whenever $m\rho \equiv j \pmod{k}$ and since k and ρ are relatively prime, this congruence has a solution m for any $j \in [0, k - 1]$.

II. Assume $m\rho$ has $0j$ as its last two digits. If i is any digit for which $i + j \geq k$, Lemma 7 gives

$$\begin{aligned} a_s(i + m\rho) &= a_s(m\rho)a_s^{-1}(j)a_s(1)a_s(i + j - k) \\ &= a_s^{-1}(j)a_s(i + j - k)a_s(1). \end{aligned}$$

From (25) with $n = i$ we see that

$$(27) \quad a_s(i + j - k) = a_s(i)a_s(j)a_s^{-1}(1).$$

Again, (27) holds for any j , since the congruence $m\rho \equiv 0 \cdot k + j \pmod{k^2}$ is always solvable.

We now define $\alpha: \mathbb{Z}_k \rightarrow \langle \zeta \rangle$ by $\alpha(i) = a_s(i)$, $0 \leq i \leq k - 1$. There are two cases.

Case A: $1 \notin S$. By (24), $a_s(1) = 1$. Here (26) and (27) imply that α is a homomorphism. But $\alpha(1) = 1$, so the kernel of α contains all of \mathbb{Z}_k . Hence $a_s(i) = 1$ for all $i \in [0, k - 1]$, S is empty by (24), contradicting our assumption.

Case B: $1 \in S$. Here $a_s(1) = \zeta^{e_1} \neq 1$. Now (26) implies by induction that $a_s(j) = \zeta^{j e_1}$ for $j \in [0, k - 1]$. For general $n = j_t j_{t-1} \cdots j_0$ (base k), Lemma 7 gives

$$\begin{aligned} a_s(n) &= a_s(j_t)a_s(j_{t-1}) \cdots a_s(j_0) \\ &= \zeta^{(j_t + j_{t-1} + \cdots + j_0)e_1} = \zeta^{\sigma(n)e_1}, \end{aligned}$$

where $\sigma(n) = j_t + j_{t-1} + \cdots + j_0 = e_1(n) + 2e_2(n) + \cdots + (k - 1)e_{k-1}(n)$. By assumption, $\sigma(n)e_1$ has least period $\rho \pmod{r}$. With $g = (r, e_1)$, this says

$\sigma(n)$ has least period $\rho \pmod{r'}$, where $r' = r/g$. We write $a_s(n) = \zeta_1^{\sigma(n)}$, where $\zeta_1 = \zeta^{e_1}$ is a primitive r' th root of unity. Using (27), we have

$$1 = a_s(0) = a_s(k - 1 + 1 - k) \\ = a_s(k - 1)a_s(1)a_s^{-1}(1) = \zeta^{(k-1)e_1} = \zeta_1^{k-1}$$

which implies that $r' = r/(r, e_1)$ divides $k - 1$.

From the fact that $\sigma(n) \equiv n \pmod{r'}$, if r' divides $k - 1$, it follows that the least period of $\sigma(n) \pmod{r'}$ is r' and hence $\rho = r' = r/(r, e_1)$. The characterization of the set S follows from the formula

$$a_s(n) = \zeta^{e_1 \sigma(n)} = \prod_{i=1}^{k-1} a_i^{e_1 i}.$$

COROLLARY 1. *The only k, i , and $r \geq 2$ for which $a_i = \zeta^{e_i}$ can be periodic are $k = 3, i = 1$, and $r = 2$. In this case e_1 (base 3) is periodic $\pmod{2}$ with period 2.*

PROOF. If $a_i = \zeta^{e_i}$ is periodic, then Theorem 2 implies that $i = 1$ (so that $e_1 \neq 0$) and

$$e_1(n) \equiv \sigma(n) \pmod{r}, \quad \text{for } n \geq 0.$$

Thus we have

$$(28) \quad 2e_2(n) + \dots + (k - 1)e_{k-1}(n) \equiv 0 \pmod{r}, \quad \text{for } n \geq 0.$$

Note that $k \geq 3$, since $k = 2$ would give $r = 1$. Setting $n = 2$ in (28) gives $2 \equiv 0 \pmod{r}$, and so $r = 2$. It now follows easily that $k = 3$, for if $k \geq 4$, putting $n = 3$ in (28) gives $3 \equiv 0 \pmod{2}$.

The following additive form of Theorem 2 will be useful.

COROLLARY 2. *Let S be a non-empty set of nonzero digits base k , and*

$$e_s \equiv \sum_{i \in S} \varepsilon_i e_i \pmod{r}, \quad \text{for } n \geq 0,$$

where $\varepsilon_i \not\equiv 0 \pmod{r}$. Then the sequence e_s is not ultimately periodic \pmod{r} unless $e_s(n) \equiv \varepsilon_1 \sigma(n) \pmod{r}$ and $r/(r, \varepsilon_1) | (k - 1)$.

COROLLARY 3. *If S is a non-empty set of nonzero digits base k , and ε_i is a nonzero integer for $i \in S$, then the integer sequence*

$$e_s = \sum_{i \in S} \varepsilon_i e_i$$

is not ultimately periodic.

PROOF. Let r be a prime larger than $\max(k, \varepsilon_1, \dots, \varepsilon_{k-1})$. If e_s is ultimately periodic, then e_s is ultimately periodic (mod r). This is impossible by Corollary 2, since $r > k - 1$.

THEOREM 3. Let ζ be an r th root of unity. If $a \in \Lambda_k(\langle\zeta\rangle)$ is ultimately periodic, then a is purely periodic with period equal to $bM(a)$, where $b|(r, k - 1)$. If $(r, k - 1) = 1$, then a has period $M(a)$.

PROOF. Assume $a \in \Lambda_k(\langle\zeta\rangle)$ is ultimately periodic, and $a(0) = 1$. Then a is purely periodic, and for some q , $X^q = \{X_n^q \in \Lambda_k(\langle\zeta\rangle^{k^q})$ is periodic with period ρ_q , where $(\rho_q, k) = 1$, and $M_q = M(X^q) = 1$, by Lemma 6.

Set

$$(29) \quad Y_n = (X_0^q)^{-1} X_n^q.$$

Then $M(\{Y_n\}) = 1$, Y_0 is the identity in $G = \langle\zeta\rangle^{k^q}$, and $\{Y_n\}$ has least period ρ_q .

Now let $\varphi_i: G \rightarrow \langle\zeta\rangle$ be the projection homomorphism of G which takes a vector to its i th coordinate, and define the sequence α_i by

$$\alpha_i(n) = \varphi_i(Y_n).$$

Since $\varphi_i: G \rightarrow \langle\zeta\rangle$, $\alpha_i \in \Gamma_k(\langle\zeta\rangle)$ by results of [13] (equations (34) and (34')). It is easily checked that $M(\alpha_i) = 1$ and that α_i is periodic with $\rho(\alpha_i)|\rho_q$.

By results of [13] (see Theorem 7), $M(\alpha_i) = 1$ implies that α_i is a product of pattern sequences a_j , where j runs over a set S of digits base k :

$$\alpha_i = \prod_{j \in S} a_j^{\varepsilon_j}.$$

Since α_i is periodic, Theorem 2 now gives that $\rho(\alpha_i)|(r, k - 1)$. But since

$$Y_n = (\alpha_1(n), \alpha_2(n), \dots, \alpha_{k^q}(n)),$$

we have $\rho_q = \text{l.c.m.}_i \rho(\alpha_i)$, and therefore $\rho_q|(r, k - 1)$. It follows that $\rho_q = b_q = b$, by Lemma 6, and so Lemma 3 gives that $\rho(a) = bM(a)$, where $b|(r, k - 1)$.

REMARK. Exactly the same proof shows that any ultimately periodic sequence in $\Lambda_k(\mathbb{Z})$ is purely periodic with period equal to $M(a)$, a power of k , and that a periodic sequence a in $\Lambda_k(\mathbb{Z}_r)$ has a least period of the form $bM(a)$, where $b|(r, k - 1)$.

In [13, Theorem 8] it was proved that

$$(30) \quad e_P = e_{0P} + e_{1P} + \dots + e_{(k-1)P},$$

where P is a pattern base k and iP is the pattern consisting of the digit i followed by P . The following theorem complements this result and generalizes [13, Theorem 10]. We recall from [13] the notation $v(P)$, which denotes the value of P considered as an integer base k .

THEOREM 4. *If P is a nonzero pattern base k of length d and Pi is the pattern consisting of the pattern P followed by the digit i , then the sequence*

$$h_P = e_P - e_{P0} - e_{P1} - \dots - e_{P(k-1)}$$

in $\Lambda_k(\mathbb{Z})$ is periodic with period k^d and we have

$$(31) \quad h_P(n) = \begin{cases} 0 & \text{if } n \not\equiv v(P) \pmod{k^d}, \\ 1 & \text{if } n \equiv v(P) \pmod{k^d}. \end{cases}$$

The sequences h_P together with the constant sequences generate $H_k(\mathbb{Z})$, the subgroup of periodic sequences in $\Lambda_k(\mathbb{Z})$. If $(r, k - 1) = 1$, the sequences

$$h_P^{(r)} = e_P^{(r)} - e_{P0}^{(r)} - e_{P1}^{(r)} - \dots - e_{P(k-1)}^{(r)}$$

generate $H_k(\mathbb{Z}_r)$.

PROOF. Equation (31) follows from the fact that the occurrences of P in the k -adic expression for n coincide with the occurrences of Pi for $0 \leq i \leq k - 1$, with the possible exception of an occurrence of P at the end. The latter can occur if and only if $n \equiv v(P) \pmod{k^d}$.

Now let a be any sequence in $H_k(\mathbb{Z})$. By the remark following Theorem 3, a is purely periodic, with period equal to $M(a) = k^d$ for some d . It then follows easily from (31) that

$$a = a(0) + \sum (a(v(P)) - a(0))h_P,$$

where P runs over all nonzero patterns base k of length d . Exactly the same argument works for the group $H_k(\mathbb{Z}_r)$.

REMARK. If $P = 0^*$, the pattern consisting of d 0's, then in place of (31) we have

$$h_{0^*}(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod{k^d}, \\ -1 & \text{if } n \not\equiv 0 \pmod{k^d}. \end{cases}$$

Let $\zeta \neq 1$ be any r th root of unity for which $(r, k - 1) = 1$ and let $a_P(n) = \zeta^{e_P(n)}$. We define

$$(32) \quad b_P(n) = a_P(n)a_{P0}^{-1}(n)a_{P1}^{-1}(n)\dots a_{P(k-1)}^{-1}(n).$$

By (31)

$$b_P(n) = \begin{cases} 1 & \text{if } n \not\equiv v(P) \pmod{k^d}, \\ \zeta & \text{if } n \equiv v(P) \pmod{k^d}. \end{cases}$$

COROLLARY TO THEOREM 4. *If $(r, k - 1) = 1$ the sequences b_P , together with the constant sequences in $\Gamma_k(\langle\zeta\rangle)$, generate the subgroup $H_k(\langle\zeta\rangle)$ of periodic sequences in $\Lambda_k(\langle\zeta\rangle)$.*

The proof is analogous to the proof of Theorem 4.

The next theorem extends Corollary 1 of Theorem 2 to patterns of length greater than 1.

THEOREM 5. *Let P be a nonzero pattern of digits base k having length $d > 1$. If ζ is an r th root of unity then neither $e_P^{(r)}$ nor $a_P = (\zeta)^{e_P}$ is periodic.*

PROOF. We show that e_P is not periodic $(\text{mod } r)$, where P has d digits. If e_P is periodic $(\text{mod } r)$, Theorem 3 implies that $e_P^{(r)}$ is purely periodic with period $bM(e_P^{(r)})$, where $b|k-1$. From [13] we know that $M(e_P^{(r)})|k^{d-1}$. Therefore Lemma 5(b) implies that the period ρ_{d-1} of the sequence $X^{d-1} = \{X_n^{d-1}\}$ is b , and hence that $k-1$ is a period of X^{d-1} . But since

$$X_n^{d-1} = (e_P^{(r)}(nk^{d-1}), e_P^{(r)}(nk^{d-1} + 1), \dots, e_P^{(r)}(nk^{d-1} + k^{d-1} - 1)),$$

we have that

$$(33) \quad e_P^{(r)}(nk^{d-1} + j) = e_P^{(r)}((n+k-1)k^{d-1} + j),$$

for $n \geq 0$ and $0 \leq j \leq k^{d-1} - 1$.

Case 1. If the base k representation of P is $0p_2p_3 \cdots p_d$, choose n and j with base k representations

$$n = 11 \cdots 10, \text{ with } d+1 \text{ digits, and } j = p_2p_3 \cdots p_d.$$

Then the base k representations of $nk^{d-1} + j$ and $(n+k-1)k^{d-1} + j$ are, respectively,

$$11 \cdots 10p_2p_3 \cdots p_d \quad \text{and} \quad 11 \cdots 1(k-1)p_2p_3 \cdots p_d.$$

Thus

$$e_P(nk^{d-1} + j) = e_P((n+k-1)k^{d-1} + j) + 1.$$

Hence (33) cannot hold.

Case 2. If the base k representation of P is $p_1p_2p_3 \cdots p_d$, with $p_1 \geq 1$, choose n and j with base k representations

$$n = 1\ 0 \cdots 00p_1, \text{ with } d + 2 \text{ digits, and } j = p_2p_3 \cdots p_d.$$

Then the base k representations of $nk^{d-1} + j$ and $(n + k - 1)k^{d-1} + j$ are, respectively,

$$1\ 0 \cdots 00p_1p_2p_3 \cdots p_d \text{ and } 1\ 0 \cdots 01(p_1 - 1)p_2p_3 \cdots p_d.$$

The same conclusion holds as in Case 1 unless $1(p_1 - 1)p_2p_3 \cdots p_{d-1} = P$. In this case replace j by $j' = j + 1$ if $p_d = 0$ or by $j' = j - 1$ if $p_d \geq 1$. Then clearly

$$e_p(nk^{d-1} + j') = e_p((n + k - 1)k^{d-1} + j') - 1.$$

Hence (33) cannot hold in this case either, and e_p cannot be periodic (mod r).

LEMMA 8. *If P is any pattern base k . Then*

$$h_{0P} = h_P - h_{1P} - h_{2P} - \cdots - h_{(k-1)P}.$$

The same relationship exists between $h_{iP}^{(r)}$ and $h_P^{(r)}$.

This is immediate from (30) and the definition of h_P (see Theorem 4).

LEMMA 9. *The sequences $e_P^{(r)}$, for patterns P (base k) with no leading zeros, are independent in $\Lambda_k(\mathbb{Z}_r)$.*

PROOF. We have to show that

$$\sum_P c_P e_P^{(r)} = 0 \text{ in } \Lambda_k(\mathbb{Z}_r), \quad c_P \in \mathbb{Z}_r,$$

implies all $c_P \equiv 0 \pmod{r}$. But this follows easily from the fact that $e_P(n) = 1$ if $n = v(P)$ and $e_P(n) = 0$ if $n < v(P)$. (See also the argument in the proof of Theorem 9 in [13].)

DEFINITION. Fix k and let $\mathbb{P} = \{P\}$ be a set of nonzero patterns base k with no leading zeros. Then \mathbb{P} is called a *set of independence* if it has the property that for all patterns P base k there exists a digit i (base k) so that $Pi \notin \mathbb{P}$.

THEOREM 6. *If $(r, k - 1) = 1$ and \mathbb{P} is a set of independence then $\{e_P^{(r)}; P \in \mathbb{P}\}$ is a set of independent elements in the quotient group*

$$\Lambda_k(\mathbb{Z}_r)/H_k(\mathbb{Z}_r).$$

PROOF. We have to show that

$$(34) \quad \sum_{P \in \mathbb{P}} c_P e_P^{(r)} \in H_k(\mathbb{Z}_r), \quad c_P \in \mathbb{Z}_r,$$

implies $c_P \equiv 0 \pmod r$ for all $P \in \mathbb{P}$. Assume (34) holds. Then by Theorem 4 and Lemma 8 we can write

$$(35) \quad \sum_{P \in \mathbb{P}} c_P e_P^{(r)} = \sum_Q d_Q h_Q^{(r)},$$

where the sum is over all patterns Q base k with no leading zeros, and only a finite number of d_Q are nonzero $\pmod r$. Rewriting the right side of (35) gives

$$(36) \quad \sum_{P \in \mathbb{P}} c_P e_P^{(r)} = \sum_Q d_Q (e_Q^{(r)} - e_{Q0}^{(r)} - \dots - e_{Q(k-1)}^{(r)}) = \sum_Q (d_Q - d_{Q'}) e_Q^{(r)},$$

where $Q = Q'i$ for some i . Suppose some $d_Q \not\equiv 0 \pmod r$. By assumption on \mathbb{P} there is a digit j base k such that $Qj \notin \mathbb{P}$. Moreover there exist digits $j_1 (= j), j_2, \dots, j_n$, for all $n \geq 0$, such that $Q_n = Qj_1 j_2 \dots j_n \notin \mathbb{P}$. Thus Lemma 9 gives that $d_{Q_n} - d_{Q_{n-1}} \equiv 0 \pmod r$, for all $n \geq 1$. Therefore $d_{Q_n} \equiv d_{Q_{n-1}} \equiv \dots \equiv d_Q \not\equiv 0 \pmod r$ for all n , and so there are infinitely many nonzero d_Q 's. This contradiction shows that all $d_Q \equiv 0 \pmod r$ in (35), which implies all $c_P \equiv 0 \pmod r$.

5. Transcendence results

Theorems 7 and 8 in this section are immediate consequences of Theorem 1, the results of Section 4, and the Loxton and van der Poorten theorem. Theorem 7 is concerned with sequences defined by single digit patterns, while Theorem 8 gives a necessary and sufficient condition for the number $\eta(e)$ in (6) to be transcendental, where e is a sequence in $\Lambda_k(\mathbb{Z}_r)$ with $(r, k-1) = 1$. From Corollary 2 to Theorem 2 we have

THEOREM 7. *Let S be a non-empty set of nonzero digits base k , and*

$$e_s(n) \equiv \sum_{i \in S} \varepsilon_i e_i(n) \pmod r,$$

where $e_s(n) \in [0, r-1]$ and $\varepsilon_i \not\equiv 0 \pmod r$. If either

(A) $(r, k-1) > 1$ and $e_s(n) \not\equiv \varepsilon \sigma(n) \pmod r$ for any integer $\varepsilon \not\equiv 0 \pmod r$ (with σ as in Theorem 2), or

(B) $(r, k-1) = 1$, then

$$\sum_{n=0}^{\infty} \frac{e_s(n)}{r^n}$$

is a transcendental number.

REMARK. If $1 \notin S$, then condition (A) holds.

COROLLARY. If $e_i^{(r)}(n)$ is defined as in (1) with $P = i$, a digit base k , then unless $(k, i, r) = (3, 1, 2)$ the number

$$(37) \quad \eta(e_i^{(r)}) = \sum_{n=0}^{\infty} \frac{e_i^{(r)}(n)}{r^n}$$

is transcendental. If $(k, i, r) = (3, 1, 2)$, then

$$\eta(e_1^{(2)}) = \sum_{n=0}^{\infty} \frac{e_1^{(2)}(n)}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{2}{3}.$$

THEOREM 8. If $e \in \Lambda_k(\mathbb{Z}_r)$ where $e(n) \in [0, r - 1]$, then the number $\eta(e)$ defined as in (6) is transcendental if $e \notin H_k(\mathbb{Z}_r)$ and rational if $e \in H_k(\mathbb{Z}_r)$. If $(r, k - 1) = 1$ then $e \in H_k(\mathbb{Z}_r)$ if and only if e can be expressed as

$$e \equiv e(0) + \sum_Q c_Q h_Q \pmod{r},$$

where $h_Q = e_Q - e_{Q0} - e_{Q1} - \dots - e_{Q(k-1)}$, for patterns Q base k , all having the same length.

COROLLARY. If P is any nonzero pattern of digits base k having length at least 2, and if $e_P^{(r)}(n)$ is defined as in (1), then the number

$$\eta(e_P^{(r)}) = \sum_{n=0}^{\infty} \frac{e_P^{(r)}(n)}{r^n}$$

is transcendental.

This corollary follows immediately from Theorem 5.

THEOREM 9. If \mathbb{P} is a set of independence for the base k , and $(r, k - 1) = 1$, then the set

$$\{\eta(e_P^{(r)}): P \in \mathbb{P}\} \cup \{1\}$$

is linearly independent over \mathbb{Q} . Moreover every finite sum

$$\sum_{P \in \mathbb{P}} c_P \eta(e_P^{(r)}), \quad c_P \in \mathbb{Q},$$

is transcendental.

PROOF. Assume there is a dependence among 1 and the $\eta(e_P^{(r)})$:

$$(38) \quad \sum_{P \in \mathbb{P}} c_P \eta(e_P^{(r)}) = c,$$

where, without loss of generality, $c_P \in \mathbb{Z}$ for $P \in \mathbb{P}$, not all $c_P \equiv 0 \pmod{r}$, and $c \in \mathbb{Q}$. Rewriting (38) gives

$$(39) \quad \sum_{n=0}^{\infty} \frac{\sum_{P \in \mathbb{P}} c_P e_P^{(r)}(n)}{r^n} = c.$$

By Theorem 1, each of the sequences $e_P^{(r)}$ is k -automatic, and so the sequence

$$a = \sum_{P \in \mathbb{P}} c_P e_P^{(r)}$$

is also k -automatic. Note that the values $a(n)$ do not necessarily lie in $[0, r - 1]$, but that they are from a finite set. By Theorem 6, a is not periodic \pmod{r} and therefore is not a periodic sequence. The Loxton and van der Poorten theorem now shows that (39) is impossible. The transcendence statement follows in the same way.

We now prove the following multiplicative result.

THEOREM 10. *Let P be an arbitrary nonzero pattern base k with at least 2 digits, and let $a_P(n) = (\zeta)^{e_P(n)}$, as before, where ζ is a primitive r th root of unity. Then at least one of the series*

$$(40) \quad \sum_{n=0}^{\infty} \frac{a_P^i(n)}{r^n}, \quad \text{for } 1 \leq i \leq r - 1,$$

is transcendental.

PROOF. The key to the proof lies in relating (40) to the number

$$\sum_{n=0}^{\infty} \frac{e_P^{(r)}(n)}{r^n}$$

and appealing to the corollary to Theorem 8. In general, suppose $a(n) = \zeta^{e(n)}$, where $e(n) \in \{0, 1, \dots, r - 1\}$. Then

$$e(n) = \sum_{i=0}^{r-1} c_i a^i(n)$$

for suitable constants $c_i \in \mathbb{Q}(\zeta)$ (independent of the particular sequence $e(n)$), as is easily seen by replacing $e(n)$ by its r possible values $0, \dots, r - 1$ and solving the resulting linear system, whose determinant is a nonzero

Vandermonde determinant. Hence

$$\sum_{n=0}^{\infty} \frac{e_p^{(r)}(n)}{r^n} = c_0 \frac{r}{r-1} + \sum_{i=1}^{r-1} c_i \sum_{n=0}^{\infty} \frac{a_p^i(n)}{r^n}.$$

Since the number on the left is transcendental, at least one of the numbers on the right is transcendental. This proves Theorem 10.

COROLLARY. *If $r = 2$ or 3 in Theorem 10, then $\sum_{n=0}^{\infty} a_p(n)/r^n$ is transcendental.*

Our last result deals with sequences of the kind considered in Theorem 2.

THEOREM 11. *If a_s is defined by (24), then*

$$(41) \quad \sum_{n=0}^{\infty} \frac{a_s(n)}{r^n}$$

is transcendental if and only if a_s does not have the form $a_s = \zeta^{\varepsilon_1 \sigma}$, where $\varepsilon_1 \not\equiv 0 \pmod{r}$, $r/(r, \varepsilon_1) | (k-1)$, and σ is the sequence defined in Theorem 2. Otherwise (41) represents a number in $\mathbb{Q}(\zeta)$.

PROOF. This is a consequence of [10, Theorem 1]. To see this, let

$$f(z) = \sum_{n=0}^{\infty} a_s(n)z^n.$$

Since $M(a_s) = 1$, it is clear that $a_s(nk + j) = a_s(n)a_s(j)$, for $0 \leq j \leq k-1$, by Lemma 7. Thus

$$f(z) = \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} a_s(nk + j)z^{nk+j} = \sum_{j=0}^{k-1} a_s(j)z^j \sum_{n=0}^{\infty} a_s(n)(z^k)^n = p(z)f(z^k),$$

where $p(z)$ is a polynomial of degree $k-1$. Since the constant term of $p(z)$ is $a_s(0) = 1$, $p(0) \neq 0$. Further, the leading term of $p(z)$ is $a_s(k-1)$, so $p(z)$ has no rational roots other than ± 1 , since its constant and leading terms are roots of unity and therefore units in the ring $\mathbb{Z}[\zeta]$. This shows that any positive rational number $\alpha \leq 1/2$ is admissible in the sense of [10] (taking $Tz = z^k$ and $A(z) = p(z)$ in [10]). By Theorem 1 of [10], $f(1/r)$ is transcendental if and only if $f(z)$ is a transcendental function. Since the coefficients of $f(z)$ are from a finite set, Szegő's theorem (see Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 2, page 315) implies that $f(z)$ is transcendental if and only if the coefficient sequence $\{a_s(n)\}$ is not ultimately periodic. Theorem 11 now follows from Theorem 2.

References

- [1] J. P. Allouche, 'Somme de chiffres et transcendance', *Bull. Soc. Math. France* **110** (1982), 279–295.
- [2] J. P. Allouche, 'Automates finis en théorie des nombres', *Expo. Math.* **5** (1987), 239–266.
- [3] J. P. Allouche and J. O. Shallit, 'Infinite products associated with counting blocks in binary strings', *J. London Math. Soc.*, to appear.
- [4] J. P. Allouche and J. Shallit, 'The ring of k -regular sequences', preprint.
- [5] D. W. Boyd, J. Cook and P. Morton, *On sequences of ± 1 's defined by binary patterns*, Dissertationes Mathematicae 283, Polish Scientific Publishers, Warszawa, 1989.
- [6] G. Christol, T. Kamae, M. Mendès France and G. Rauzy, 'Suites algébriques, automates et substitutions', *Bull. Soc. Math. France* **108** (1980), 401–419.
- [7] A. Cobham, 'Uniform tag sequences', *Math. Systems Theory* **6** (1972), 164–192.
- [8] M. Dekking, M. Mendès France and A. van der Poorten, 'Folds! I, II, III', *Math. Intelligencer* **4** (1982), 130–138, 173–181, 190–195.
- [9] H. Delange, 'Sur la fonction sommatoire de la fonction \ll somme des chiffres \gg ', *Enseign. Math.* (2) **21** (1975), 31–47.
- [10] J. H. Loxton and A. van der Poorten, 'Arithmetic properties of the solutions of a class of functional equations', *J. Reine Angew. Math.* **330** (1982), 159–172.
- [11] J. H. Loxton and A. J. van der Poorten, 'Arithmetic properties of automata: regular sequences', *J. Reine Angew. Math.* **392** (1988), 57–69.
- [12] M. Mendès France, 'Nombres algébriques et théorie des automates', *Enseign. Math.* (2) **26** (1980), 193–199.
- [13] P. Morton and W. J. Mourant, 'Paper folding, digit patterns and groups of arithmetic fractals', *Proc. London Math. Soc.* (3) **59** (1989), 253–293.
- [14] P. Morton, 'Connections between binary patterns and paperfolding', *Séminaire de Théorie des Nombres, Bordeaux* **2** (1990), 1–12.

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