1

Introduction

This introduction aims to present the principal actors of the book and to explain the main results of our monograph. We begin with the question about transcendence of periods of integrals of 1-forms over closed or non-closed paths. Historically integrals over non-closed paths were not considered as periods. The change came from looking at them in the relative cohomology. This leads us to distinguish between complete and incomplete periods.

The order of topics presented here does not follow the order in the main text but is, we hope, designed to help those readers without a background in transcendence.

1.1 Transcendence

The vector space $\mathcal{P}^1$ over $\overline{\mathbb{Q}}$ of one-dimensional periods, complete or incomplete, has a number of different descriptions. In the most elementary situation its elements are given by the period integrals

$$\alpha = \int_\sigma \omega,$$

where

- X is a smooth projective curve over $\overline{\mathbb{Q}}$;
- ω is a rational differential form on X;
- $\sigma = \sum_{i=1}^n a_i\gamma_i$ is a chain in the Riemann surface X^{an} defined by X which avoids the singularities of ω and has boundary divisor $\partial\sigma$ in $X(\overline{\mathbb{Q}})$; in particular $\gamma_i\colon [0,1] \to X^{\mathrm{an}}$ is a path and $a_i \in \mathbb{Z}$.

This set includes many interesting numbers like $2\pi i$, $\log\alpha$ for algebraic α and the periods of elliptic curves over $\overline{\mathbb{Q}}$. We study their transcendence properties.

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The case of *complete* periods in the general case, i.e. X and ω arbitrary, γ closed, was settled in 1986 by the second author in [Wüs87]: if a period is non-zero, it is transcendental. Both cases can arise. A simple example is a hyperelliptic curve whose Jacobian is isogenous to a product of two elliptic curves. Then 8 of the 16 standard periods are 0. The others are transcendental.

When X is an elliptic curve we refer the reader to [BW07, Section 6.2] for the case of *incomplete* periods. The general case has been described as an open problem in [Wüs84a]. Often the values are transcendental, e.g. $\int_1^2 dz/z = \log 2$, but certainly not always, e.g. $\int_0^2 dz = 2$. Again, it is not difficult to write down a list of simple cases in which the period is a non-zero algebraic number. However, it was not at all clear whether the list was complete and what the structure behind the examples was; see [Wüs12]. The answer that we give now is surprisingly simple:

Theorem 1.1 (Theorem 13.9)**.** *Let $\alpha = \int_\sigma \omega$ be a one-dimensional period on X. Then α is algebraic if and only if*

$$\omega = df + \omega',$$

where $f \in \overline{\mathbb{Q}}(X)^*$ and $\int_\sigma \omega' = 0$ with ω' a form with no extra poles.

The condition is clearly sufficient because the integral evaluates to

$$\sum_i a_i(f(\gamma_i(1)) - f(\gamma_i(0))) \in \overline{\mathbb{Q}}$$

in this case.

Theorem 13.9 gives a complete answer to two of the seven problems listed in Schneider's book [Sch57, p. 138], [1] [2] open for more than 60 years. We even include periods of abelian integrals of the third kind.

1.2 Relations Between Periods

Questions on transcendence can be viewed as a very special case of the question on $\overline{\mathbb{Q}}$-linear relations between 1-periods: a complex number is transcendental if it is $\overline{\mathbb{Q}}$-linearly independent of 1. The most general problem of this kind is to determine the dimension of the period space generated over $\overline{\mathbb{Q}}$ by the periods of all rational 1-forms of an algebraic variety. It is easy to give an upper bound for this dimension in terms of cohomological data. The problem

[1] Problem 3. Es ist zu versuchen, Transzendenzresultate über elliptische Integrale dritter Gattung zu beweisen.

[2] Problem 4. Die Transzendenzsätze über elliptische Integrale erster und zweiter Gattung sind in weitestmöglichem Umfang auf analoge Sätze über abelsche Integrale zu verallgemeinern.

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is then to decide whether the upper bound is the correct number or whether there are linear relations between periods.

This fundamental question will be one of the central topics in this monograph. We establish a complete description of the linear relations between (not necessarily complete) periods for all rational differential forms of degree 1. It is crucial to use here the more conceptual descriptions of $\mathcal{P}^1$ either as periods in cohomological degree 1 or as cohomological periods of curves, or even better periods of 1-motives.

The following theorem gives a first answer. It establishes Kontsevich's version of the Period Conjecture for $\mathcal{P}^1$ and furnishes a qualitative description of the period relations.

Theorem 1.2 (Kontsevich's Period Conjecture for $\mathcal{P}^1$, Theorem 13.3)**.** *All $\overline{\mathbb{Q}}$-linear relations between elements of $\mathcal{P}^1$ are induced by bilinearity and functoriality of pairs (C, D) where C is a smooth affine curve over $\overline{\mathbb{Q}}$ and $D \subset C$ a finite set of points over $\overline{\mathbb{Q}}$.*

The conjecture has an alternative formulation in terms of motives. In fact, we deduce Theorem 1.2 from the motivic version below, together with the result of Ayoub and Barbieri-Viale in [ABV15] which says that the subcategory of $\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}$ generated by $H^*(C, D)$ with C of dimension at most 1 agrees with Deligne's much older category of 1-motives; see [Del74].

Every 1-motive M has a singular realisation $V_{\mathrm{sing}}(M)$ and a de Rham realisation $V_{\mathrm{dR}}(M)$. They are linked via a period isomorphism

$$V_{\mathrm{sing}}(M) \otimes_{\mathbb{Q}} \mathbb{C} \cong V_{\mathrm{dR}}(M) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

There is a well-known relation between curves and 1-motives provided by the theory of generalised Jacobians. From this fact we see that the set $\mathcal{P}^1$ has another alternative description as the union of the images of the period pairings

$$V_{\mathrm{sing}}(M) \times V_{\mathrm{dR}}^{\vee}(M) \to \mathbb{C}$$

for all 1-motives M over $\overline{\mathbb{Q}}$.

Theorem 1.3 (Period Conjecture for 1-motives, Theorem 9.10)**.** *All $\overline{\mathbb{Q}}$-linear relations between elements of $\mathcal{P}^1$ are induced by bilinearity and functoriality for morphisms of iso-1-motives over $\overline{\mathbb{Q}}$.*

This theorem does not say anything about the actual dimension of the period space. We need a quantitative answer. In other words the space of relations has to be determined. It turns out that finding the answer is rather difficult in some cases.

1.3 Dimensions of Period Spaces

The above qualitative theorems can be refined into an explicit computation of the dimension $\delta(M)$ of the $\overline{\mathbb{Q}}$-vector space generated by the periods of a given 1-motive M. The result depends on the subtle and very unexpected interplay between the constituents of M.

Not only for the proofs, but also for the very formulation of the dimension formulas, we rely on the theory of 1-motives introduced by Deligne; see [Del74]. They form an abelian category that captures all cohomological properties of algebraic varieties in degree 1, including all one-dimensional periods.

We review the basics: a 1-motive over $\overline{\mathbb{Q}}$ is a complex $M = [L \to G]$, where G is a semi-abelian variety over $\overline{\mathbb{Q}}$ and L is a free abelian group of finite rank. The map is a group homomorphism. As mentioned earlier, every 1-motive has de Rham and singular realisations, and a period isomorphism between them after extension of scalars to $\mathbb{C}$.

If C is a smooth curve over k, $D \subset C$ a finite set of $\overline{\mathbb{Q}}$-points, then there is a 1-motive $M_1(C)$ such that $H_1^{\mathrm{sing}}(C^{\mathrm{an}}, D; \mathbb{Q})$ agrees with the singular realisation of $M_1(C)$, and $H^1_{\mathrm{dR}}(C, D)^\vee$ agrees with the de Rham realisation of $M_1(C)$. Hence the periods of the pair (C, D) agree with the periods of $M_1(C)$. Explicitly, $M_1(C) = [\mathbb{Z}[D]^0 \to J(C)]$, where $J(C)$ is the generalised Jacobian of C and $\mathbb{Z}[D]^0$ means divisor of degree 0 supported on D.

We denote by $\mathcal{P}(M)$ the image of the period pairing for M and by $\mathcal{P}\langle M\rangle$ the abelian group (or, equivalently, $\overline{\mathbb{Q}}$-vector space) generated by $\mathcal{P}(M) \subset \mathbb{C}$.

We fix a 1-motive $M = [L \to G]$, with G an extension of an abelian variety A by a torus T and L a free abelian group of finite rank. For the definition of its singular realisation $V_{\mathrm{sing}}(M)$ and its de Rham realisation $V^\vee_{\mathrm{dR}}(M)$, we refer the reader to Chapter 8.

The weight filtration on M, explicitly given by

$$[0 \to T] \subset [0 \to G] \subset [L \to G],$$

induces

$$V_{\mathrm{sing}}(T) \hookrightarrow V_{\mathrm{sing}}(G) \hookrightarrow V_{\mathrm{sing}}(M)$$

and dually

$$V^\vee_{\mathrm{dR}}(M) \twoheadleftarrow V^\vee_{\mathrm{dR}}([L \to A]) \twoheadleftarrow V^\vee_{\mathrm{dR}}([L \to 0]).$$

Together, they introduce a bifiltration

$$\begin{array}{ccccc}
\mathcal{P}\langle T\rangle & \hookrightarrow & \mathcal{P}\langle G\rangle & \hookrightarrow & \mathcal{P}\langle M\rangle \\
 & & \uparrow & & \uparrow \\
 & & \mathcal{P}\langle A\rangle & \hookrightarrow & \mathcal{P}\big\langle [L\to A]\big\rangle \\
 & & & & \uparrow \\
 & & & & \mathcal{P}\big\langle [L\to 0]\big\rangle
\end{array}$$

on $\mathcal{P}\langle M\rangle$.

We introduce the following notation and terminology:

$$\begin{array}{rll}
\mathcal{P}_{\mathrm{Ta}}(M) &= \mathcal{P}\langle T\rangle & \text{Tate periods,} \\
\mathcal{P}_2(M) &= \mathcal{P}\langle A\rangle & \text{2nd kind wrt closed paths,} \\
\mathcal{P}_{\mathrm{alg}}(M) &= \mathcal{P}\big\langle [L\to 0]\big\rangle & \text{algebraic periods,} \\
\mathcal{P}_3(M) &= \mathcal{P}\langle G\rangle/(\mathcal{P}_{\mathrm{Ta}}(M)+\mathcal{P}_2(M)) & \text{3rd kind wrt closed paths,} \\
\mathcal{P}_{\mathrm{inc2}}(M) &= \mathcal{P}\big\langle [L\to A]\big\rangle/(\mathcal{P}_2(M)+\mathcal{P}_{\mathrm{alg}}(M)) & \text{2nd kind wrt non-cl. paths,} \\
\mathcal{P}_{\mathrm{inc3}}(M) &= \mathcal{P}\langle M\rangle/(\mathcal{P}_3(M)+\mathcal{P}_{\mathrm{inc2}}(M)) & \text{3rd kind wrt non-cl. paths,}
\end{array}$$

where wrt and non-cl. are abbreviations for ‘with respect to’ and ‘non-closed’. After choosing bases, we can organise the periods into a period matrix of the form

$$\begin{pmatrix} \mathcal{P}_{\mathrm{Ta}}(M) & \mathcal{P}_3(M) & \mathcal{P}_{\mathrm{inc3}}(M) \\ 0 & \mathcal{P}_2(M) & \mathcal{P}_{\mathrm{inc2}}(M) \\ 0 & 0 & \mathcal{P}_{\mathrm{alg}}(M) \end{pmatrix}.$$

The contribution of $\mathcal{P}_{\mathrm{Ta}}(M)$ (multiples of $2\pi i$) and $\mathcal{P}_{\mathrm{alg}}(M)$ (algebraic numbers) is readily understood. Note that the off-diagonal entries are only well defined up to periods on the diagonal. This can also be seen in the case of Baker periods, which are contained in $\mathcal{P}_{\mathrm{inc3}}(M)$ for special M. The value of $\log\alpha$ depends on the chosen path and is only well defined up to multiples of $2\pi i$. The total dimension is obtained by adding up these dimensions. In particular, we have, for example,

$$\mathcal{P}\big\langle [L\to A]\big\rangle \cap \mathcal{P}\big\langle [0\to G]\big\rangle = \mathcal{P}\big\langle [0\to A]\big\rangle.$$

The complete result takes a rather complicated form. In order to state it we write $\delta(M) = \dim\mathcal{P}\langle M\rangle$ and $\delta_?(M) = \dim\mathcal{P}_?(M)$ for the different entries of the period matrix. If B is a simple abelian variety, $g(B)$ will be its genus and

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$e(B)$ the $\mathbb{Q}$-dimension of $\mathrm{End}(B)_{\mathbb{Q}}$. We also need the invariants $\mathrm{rk}_B(L, M)$, $\mathrm{rk}_B(T, M)$ as introduced in Notation 15.2.

Theorem 1.4 (Corollary 16.4, Proposition 16.5)**.** *The following always holds:*

$$\delta(M) = \delta_{\mathrm{Ta}}(M) + \delta_2(M) + \delta_{\mathrm{alg}}(M) + \delta_3(M) + \delta_{\mathrm{inc2}}(M) + \delta_{\mathrm{inc3}}(M).$$

1. *All Tate periods are $\overline{\mathbb{Q}}$-multiples of $2\pi i$. All algebraic periods are in $\overline{\mathbb{Q}}$. In particular, $\delta_{\mathrm{Ta}}(M)$ and $\delta_{\mathrm{alg}}(M)$ take the values* 0 *or* 1*, depending on the (non)-vanishing of T and L.*
2. *We have*
$$\delta_2(M) = \sum_B \frac{4g(B)^2}{e(B)},$$
where the sum is taken over all simple factors of A, without multiplicities.
3. *We have*
$$\delta_3(M) = \sum_B 2g(B)\mathrm{rk}_B(L, M),$$
$$\delta_{\mathrm{inc2}}(M) = \sum_B 2g(B)\mathrm{rk}_B(T, M).$$

The special case $A = 0$ gives Baker's Theorem. The most interesting and hardest contribution is $\mathcal{P}_{\mathrm{inc3}}(M)$. The computation of this contribution was not possible without the methods that we develop here. Up to particular cases the formulas for the other contributions were not in the literature either. For an overview see, for example [BW07, Section 6.2], [Wüs84a, Wüs12] and [Wüs21].

The formula for $\mathcal{P}_{\mathrm{inc3}}(M)$ simplifies in the case of motives that we call saturated; see Definition 15.1.

Theorem 1.5 (Theorem 15.3)**.** *If $M = M_0 \times M_1$ is the product of a Baker motive $M_0 = [L_0 \to T_0]$, i.e. with vanishing abelian part, and a saturated motive $M_1 = [L_1 \to G_1]$, then*

$$\delta_{\mathrm{inc3}}(M) = \mathrm{rk}_{\mathrm{gm}}(L, M_1) + \sum_B e(B)\mathrm{rk}_B(G_1, M_1)\mathrm{rk}_B(L_1, M_1).$$

Fortunately, by Theorem 15.3 (2) the periods of a general motive are always included in the period space of $M_0 \times M_{\mathrm{sat}}$ with M_0 of Baker type ($A_0 = 0$) and M_{sat} saturated.

There is a precise recipe for $\delta_{\mathrm{inc3}}(M)$ for any 1-motive M. It is spelt out in Chapter 17, in particular Theorem 17.8. See also Chapter 11 for examples of elliptic curves without and with CM.

1.4 Method of Proof

As in the case of closed paths, the main ingredient of our proof (and the only input from transcendence theory) is the Analytic Subgroup Theorem of [Wüs89]. We give a reformulation as Theorem 6.2: given a smooth connected commutative algebraic group over $\overline{\mathbb{Q}}$ and $u \in \mathrm{Lie}(G^{\mathrm{an}})$ such that $\exp_G(u) \in G(\overline{\mathbb{Q}})$, there is a canonical short exact sequence

$$0 \to G_1 \to G \xrightarrow{\pi} G_2 \to 0$$

of algebraic groups over $\overline{\mathbb{Q}}$ such that $\mathrm{Ann}(u) = \pi^*(\mathrm{coLie}(G_2))$ and $u \in \mathrm{Lie}(G_1^{\mathrm{an}})$. Here $\mathrm{Ann}(u) \subset \mathrm{coLie}(G)$ is the largest subspace such that $\langle \mathrm{Ann}(u), u \rangle = 0$ under the canonical pairing.

Given a 1-motive M, Deligne constructed a vector extension $M^\natural$ of G such that $V_{\mathrm{dR}}(M) = \mathrm{Lie}(M^\natural)$. This is the group to which we apply the Subgroup Theorem.

Theorem 1.6 (Subgroup Theorem for 1-motives, Theorem 9.7)**.** *Given a* 1*-motive M over $\overline{\mathbb{Q}}$ and $u \in V_{\mathrm{sing}}(M)$, there is a short exact sequence of* 1*-motives up to isogeny*

$$0 \to M_1 \xrightarrow{i} M \xrightarrow{p} M_2 \to 0,$$

such that $\mathrm{Ann}(u) = p^* V_{\mathrm{dR}}^\vee(M_2)$ *and* $u \in i_* V_{\mathrm{sing}}(M_1)$*. Here* $\mathrm{Ann}(u) \subset V_{\mathrm{dR}}^\vee(M)$ *is the left kernel under the period pairing. The sequence is uniquely determined by these properties.*

Given a pair of non-zero $u \in V_{\mathrm{sing}}(M)$ and $\omega \in V_{\mathrm{dR}}^\vee(M)$ with vanishing period, the theorem provides a proper submotive M_1 of M such that $u = i_* u_1$ for $u_1 \in V_{\mathrm{sing}}(M_1)$ and $\omega = p^*\omega_2$ for $\omega_2 \in V_{\mathrm{dR}}^\vee(M_2)$. Any $\overline{\mathbb{Q}}$-linear relation between periods can be translated into the vanishing of a period. Then the Subgroup Theorem for 1-motives is applied.

As a by-product, we also get a couple of new results on 1-motives over $\overline{\mathbb{Q}}$: they are a full subcategory of the category of $\mathbb{Q}$-Hodge structures over $\overline{\mathbb{Q}}$ (see Proposition 8.17) and of the category of (non-effective) Nori motives (see Theorem 13.5) and of the category $(\mathbb{Q}, \overline{\mathbb{Q}})$-Vect of pairs of vector spaces together with a period matrix. The last statement was also obtained independently by Andreatta, Barbieri-Viale and Bertapelle; see [ABVB20]. The case of Hodge structures has just recently been considered by André in [And21]. He proves that the functor from 1-motives into $\mathbb{Q}$-Hodge structures is fully faithful for all algebraically closed fields $k \subset \mathbb{C}$.

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1.5 Why 1-Motives?

This seems the right moment to address the question of whether our emphasis on 1-motives is necessary. We think that the answer is yes.

Obviously, all proofs using 1-motives could be rewritten in terms of commutative algebraic groups because this is how the Subgroup Theorem for 1-Motives itself is deduced. However, the dimension formulas depend on the constituents of the 1-motive and do not admit a transparent formulation in terms of the constituents of the algebraic group.

More generally, 1-motives are the link between the classical objects of transcendence theory à la Lindemann, Schneider or Baker and the structural predictions linked with Grothendieck, André or Kontsevich.

1.6 The Case of Elliptic Curves

The above results are very general and depend on a subtle interplay between the data. It is a non-trivial task to make them explicit in particular examples. We have carried this out to some extent in the case of an elliptic curve E defined over $\overline{\mathbb{Q}}$.

Recall the Weierstraß $\wp$-, ζ- and σ-functions for E. We obtain the following result.

Theorem 1.7 (Theorem 18.6)**.** *Let $u \in \mathbb{C}$ be such that $\wp(u) \in \overline{\mathbb{Q}}$ and $\exp_E(u)$ is non-torsion in $E(\overline{\mathbb{Q}})$. Then*

$$u\zeta(u) - 2\log\sigma(u)$$

is transcendental.

This is an incomplete period integral of the third kind. The proof of the above result is actually not a direct consequence of Theorem 1.1 but rather uses the insights of our dimension computations.

We also carry out the dimension computation in this case: let $M = [L \to G]$ with $L \cong \mathbb{Z}$, G an extension of E by $\mathbb{G}_m$ that is non-split up to isogeny, $L_{\mathbb{Q}} \to E_{\mathbb{Q}}$ injective. Then by Propositions 11.1 and 11.3,

$$\dim \mathcal{P}\langle M\rangle = \begin{cases} 11 & E \text{ without CM,} \\ 9 & E \text{ CM.} \end{cases}$$

The incomplete periods of the third kind become more difficult already if we consider $M = [L \to G]$ with $L \cong \mathbb{Z}^2$, G an extension of E by $\mathbb{G}_m^2$, again $L_{\mathbb{Q}} \to E_{\mathbb{Q}}$ injective and G completely non-split up to isogeny. If E does not have CM, then

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$$\dim \mathcal{P}\langle M\rangle = 18.$$

If E is CM, then

$$\dim \mathcal{P}\langle M\rangle = 16, 14, 12, 10,$$

depending on the interplay of the complex multiplication and L and G. The extreme case occurs when $\mathrm{End}(M)$ is the CM-field. Then the resulting dimension is 10.

1.7 Values of Hypergeometric Functions

Euler had already known that the hypergeometric function $F(a,b,c;z)$ can be written as a quotient of two integrals. If a,b,c are rational numbers, these integrals can be regarded as periods on certain explicit algebraic curves. Knowledge about $\overline{\mathbb{Q}}$-linear indepdendence of periods then translates into transcendence statements for the values $F(a,b,c;\lambda)$ for $\lambda \in \overline{\mathbb{Q}} \smallsetminus \{0,1\}$. This insight is exploited by Wolfart in [Wol88] and by Chudnovsky–Chudnovsky in [CC88]. We explain the argument in detail for $a = b = 1/2$ and $c = 1$:

Proposition 1.8 (Proposition 19.3). *The value $F(1/2, 1/2, 1; z)$ of the hypergeometric function is transcendental for $z \in \overline{\mathbb{Q}} \smallsetminus \{0,1\}$.*

The proposition follows from the $\overline{\mathbb{Q}}$-linear independence of π and the complete periods of elliptic curves established first by Schneider in 1936; see [Sch37, Satz IIIa].

In the case of general $a,b,c \in \mathbb{Q}$ with least common denominator N, the Euler integrals can be seen as periods for the algebraic curve with affine equation

$$y^N = x^r(1-x)^s(1-\lambda x)^t$$

for suitable r,s,t. For the formula in the case of $N = p$ a prime, see Proposition 19.19. These curves have been intensely studied. Using results of Gross–Rohrlich [GR78], Archinard [Arc03b] and Asakura–Otsubo [AO18], we work out another example.

Theorem 1.9 (Corollary 19.22). *Let p be a prime such that $p \not\equiv 1 \mod 3$, $1 \le n \le p-1$. Let $0 < r, s < p$ such that p does not divide $r+s$, put $t = p-s$ and*

$$u = \left[\frac{nr}{p}\right], \quad v = \left[\frac{ns}{p}\right], \quad w = \left[\frac{nt}{p}\right].$$

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We further assume

$$\left\langle \frac{nr}{p} \right\rangle + \left\langle \frac{ns}{p} \right\rangle - \left\langle \frac{n(r+s)}{p} \right\rangle \neq 1.$$

Then, for all $\lambda \in \overline{\mathbb{Q}} \smallsetminus \{0,1\}$, the corresponding value $F(a,b,c;\lambda)$ is zero or transcendental and transcendental if $\lambda \in (0,1)$.

An explict example where the assumptions are satisfied is $p = 11$, $r = s = 2$, $n = 1,2,6,7,8$. We deduce, for example, that the numbers $F(6/11, 6/11, 12/11; \lambda)$ are zero or transcendental, provided $\lambda \in \overline{\mathbb{Q}} \smallsetminus \{0,1\}$.

We should stress that this application relies only on complete periods on abelian varieties and not on the more general theory developed in our monograph. It should be seen as a proof of concept: the same argument can be applied to other geometric families of curves, allowing families of differential forms of the third kind and non-closed paths. The hyergeometric function would be replaced by the solutions of differential equations defined by the Gauss–Manin connection.

1.8 Structure of the Monograph

We have tried to make the monograph accessible to readers who are not familiar with either motives or periods.

The first part provides foundational material that will be used throughout, for example terminology from category theory, a review of the theory of the generalised Jacobian and the basics on singular homology and de Rham cohomology. We provide precise references for the facts that we need later. Along the way we also fix notation and normalisations. Depending on their background, readers are invited to skip some or all of these chapters and use them only for reference.

Chapters 6 and 7 address less classical material. The first deduces a reformulation of our key tool, the Analytic Subgroup Theorem. We apply it to the comparison between analytic and algebraic homomorphisms between connected commutative algebraic groups.

Chapter 7 presents an abstract formulation of the theory of periods and the Period Conjecture for abelian categories without a tensor structure.

Part II is the heart of the monograph and presents our main result. It addresses periods of 1-motives. After settling some notation, Chapter 8 starts by reviewing Deligne's category of 1-motives and its properties. We then establish auxiliary results that are needed in the next chapter.

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Chapter 9 discusses periods of 1-motives and proves the version of the Period Conjecture purely in terms of 1-motives. We then consider examples: in Chapter 10 we treat the classical cases like the transcendence of π and values of log in our language. In Chapter 11 we apply the general results in the case of a 1-motive whose constituents are as small as possible without being trivial and compute the dimensions of their period spaces.

In Part III we turn to periods of algebraic varieties. Chapter 12 clarifies the notion of a cohomological period. After defining $\mathcal{P}^1$ in a down-to-earth way, the interpretation of cohomological periods as the periods of 1-motives is explained. Finally, we explain the interpretation as periods of Nori or Voevodsky motives.

In Chapter 13 we use the results on periods of 1-motives to deduce the qualitative results on $\mathcal{P}^1$ and periods of curves: the criterion on transcendence and the Period Conjecture. The results are made explicit in the classical terms of differential forms of the first, second and third kind on an algebraic curve in Chapter 14.

Part IV aims at a dimension formula for the space of periods of a 1-motive in terms of its data. Chapter 15 treats mainly the saturated case. This can be applied to deduce complete structural results in Chapter 16. Finally, Chapter 17 is devoted to an explicit dimension computation for the space of incomplete periods of the third kind, which is very involved. In this rather complicated case the results were unexpected.

In Chapter 18 we deal with the case of elliptic curves and make our results explicit in terms of the classical Weierstraß functions $\wp, \zeta, \sigma$.

We explain in Chapter 19 how transcendence results on special values of the hypergeometric function can be deduced from $\overline{\mathbb{Q}}$-linear independence of 1-periods.

There are three appendices: the first two sketch the theories of Nori and Voevodsky motives to the extent used in the proof of Theorem 13.3.

The last appendix is of a technical nature: we need to verify that the singular and de Rham realisations of a 1-motive agree with the realisation of the attached geometric motive.