Econometric Theory, **0**, 2025, 1–32.
doi:10.1017/S0266466625100066

TESTING FOR STRUCTURAL CHANGE BY ISOTONIC REGRESSION

BIN CHEN
University of Rochester

ROBERT DE JONG
Ohio State University

Detecting structural changes in economic relationships has been a longstanding challenge in econometrics. Most of the literature on structural breaks has considered abrupt structural breaks. Existing tests for detecting smooth structural change typically rely on kernel estimation. In this article, we introduce a novel tuning-parameter-free test that minimizes a criterion function over all possible nondecreasing or nonincreasing structural change functions. This test is pivotal (after appropriate scaling) in the scalar case and remains computationally simple even in multivariate settings. Compared to existing nonparametric tests, our method offers superior power against local monotonic structural changes and does not involve the choice of a bandwidth parameter. A simulation study and two empirical examples highlight the merits of the proposed test relative to some popular tests for structural changes in the literature.

1. INTRODUCTION

Detecting structural changes in economic relationships has been a persistent challenge in econometrics. Historically, most existing tests have focused on identifying abrupt structural breaks. However, as emphasized by Hansen (2001), structural changes may not occur instantaneously; rather, they often emerge gradually over time. Factors such as technological progress, shifts in preferences, and policy adjustments, which are often key drivers of structural changes, tend to undergo gradual and evolutionary transformations over the long term.

The study of structural breaks began with the seminar work of Chow (1960). This test relies primarily on the assumption of error normality and is designed to detect a single structural break with a known break date. Andrews (1993) further advances the field by assuming that the structural break occurs at a fixed fraction of the sample size, enabling asymptotic analysis. Andrews (1993) explores the optimization of various test statistics across a range of potential change points, typically within the interval $[\eta n,\ 1-\eta n]$, where n denotes the sample size and

We thank the editor, Peter C. B. Phillips, the co-editor, Anna Mikusheva, and two referees for careful and constructive comments. Any remaining errors are solely ours. The second author gratefully acknowledges financial support from the Jubiläumsfonds of the Austrian Central Bank (Grant No. 15334). Address correspondence to Bin Chen, Department of Economics, University of Rochester, Rochester, New York, NY, USA, e-mail: binchen@rochester.edu

η is some small number, such as 0.15 or 0.20. Building on Andrews' work, Bai and Perron (1998) extend the methodology to accommodate multiple structural breaks. More recently, modeling and testing for smooth structural changes have attracted more attention in the literature. Chen and Hong (2012), Kristensen (2012) and Zhang and Wu (2012) introduce novel approaches by considering structural change functions of the form $f(t/n)$ and propose various tests capable of detecting both smooth structural changes and abrupt breaks. Expanding upon these ideas, Li, Phillips, and Gao (2020) extend the framework to accommodate nonstationary regressors and Su and Wang (2017) consider models with latent factors. All these tests rely on nonparametric estimation of the structural change function $f(\cdot)$.[1]

This article addresses settings where the direction of the potential change is clear and proposes a novel test based on isotonic regression to detect both smooth structural changes and abrupt structural breaks. Our test complements existing methods for abrupt structural breaks, eliminating the complexities associated with identifying multiple breaks or unknown break-points.

In contrast to nonparametric tests in the literature designed for smooth structural changes, our test exhibits significant power against local smooth structural changes with a rate of $n^{-1/2}$ and does not require the selection of a tuning parameter. The foundation of our test lies in isotonic regression, which minimizes a stochastic function within a domain of monotonic functions. While the requirement of monotonicity may initially appear stringent, it aligns naturally with many real-world applications, particularly when structural changes result from factors like population growth or technological advancements.[2] The monotonicity assumption has been imposed implicitly in some existing works in the structural break literature as well. An indicator function is a monotonic function. Hence, if the true DGP has a single break, the monotonicity assumption is automatically satisfied. Alternatively, if the true is a first-order logistic function, as the smooth transition regression suggested in Lin and Teräsvirta (1994), monotonicity holds as well.[3]

Our test can be applied to assess the presence of monotonic trends or monotonic structural changes within linear regression. Unlike many tests that optimize over a function space, our statistic is asymptotically pivotal after scaling in the scalar case and remains computationally simple even in multivariate scenarios.

The remainder of the article is organized as follows. In Section 2, we introduce our test and establish its asymptotic distribution and consistency. Section 3 outlines

[1] The literature on structural changes is huge and still growing. For brevity, we focus on methods most relevant to our approach.

[2] One example is the trend regression considered in Section 4. With global warming, technological progress, or population growth, the monotonicity assumption seems reasonable in many applications of trend regression. Another example is the expectations-augmented Phillips curve studied by Alogoskoufis and Smith (1991), Bai and Perron (2003), and Blanchard and Bernanke (2024). The parameter of interest is the coefficient that measures the persistence of inflation, which is expected to increase over time over the past 20 years.

[3] We conjecture that if the true structural change function is well approximated by a monotonic function, our results can still go through. However, the theoretical extension is very challenging and we would like to leave it for future research. Nevertheless, we explore the power of our test against non-monotonic structural changes via a comprehensive simulation study in Session 4.

the computation of the test statistic. In Section 4, we conduct a simulation study to validate the reliability of the asymptotic theory in finite samples. Additionally, we apply our test to two practical examples: global warming and the impact of personal tax exemptions on fertility rates. Section 5 concludes. All mathematical proofs are collected in the "Appendix" Section for reference.

2. SETUP AND MAIN RESULT

2.1. Setup

Consider the data generating process (DGP)

$$y_t = \theta_0' x_t + \beta_0 (t/n)' z_t + \varepsilon_t, \qquad t = 1, .., n, \tag{1}$$

where y_t is a dependent variable, x_t is an $m \times 1$ vector of explanatory variables, z_t is the same as x_t or part of x_t, $\beta_0 : [0,1] \to \mathbb{R}^k$ $(k \leq m)$ is a $k \times 1$ possibly time-varying parameter vector, ε_t is an unobservable disturbance with $E(\varepsilon_t | x_t) = 0$ almost surely (a.s.). The null hypothesis of interest is

$$\mathbb{H}_0 : \beta_0 (t/n) = 0 \text{ for all } t.$$

The alternative hypothesis is

$$\mathbb{H}_A : \beta_0 (r) \text{ is a nondecreasing function of } r, \text{ where } r \in [0,1].$$

The case where $\beta_0 (r)$ is a nonincreasing function is analogous. If z_t is the same as x_t, we have pure structural changes and the whole parameter vector is subject to change under the alternative hypothesis; if z_t is part of x_t, we have partial structural changes (Andrews, 1993).

We consider a test by minimizing the distance between the sums of squared residuals over a space of functions that is monotone on [0, 1]. To be precise, let $B = B(\eta)$ denote the class of functions on [0, 1] such that $\beta_j (r)$ is nondecreasing in r for every $j \in \{1, ..., k\}$ and constant on $[0, \eta]$ and on $[1 - \eta, 1]$, where η is a small constant. In the literature, a similar constant occurs in works such as Andrews (1993) and Bai and Perron (1998). When one wants to test for structural change that is initiated by some political or institutional change, the prior information can be used to choose η. When no information is available, the common choice of η is 0.15 in the literature, as suggested by Andrews (1993), and hence we follow this tradition.[4] In our tables below, we will list critical values for $\eta = 0.10, 0.15, 0.20$, and 0.25. While we will prove our results for B, the class of functions such that $\beta_i(r)$ is nondecreasing in r, the results remain valid as long as it is a priori known which $\beta_i(r)$ are increasing and which are decreasing.

[4]Note that the choice of η and the choice of the bandwidth h in those existing nonparametric tests (e.g., Chen and Hong, 2012; Kristensen, 2012) are very different. The choice of h has an impact on the convergence rate of the test statistic and the local alternative the test can detect is $n^{-1/2}h^{-1/4}$. In contrast, the convergence rate of our test defined below does not depend on the choice of η as long as η is bounded away from zero and one. Our additional simulation results reported in the appendix show that the test is not sensitive to the choice of η.

Our test statistic is defined as

$$\inf_{\beta \in B} A_n(\beta) \equiv \inf_{\beta \in B} \sum_{t=1}^{n} \left[\left(\hat{\varepsilon}_t - \beta(t/n)' z_t \right)^2 - \hat{\varepsilon}_t^2 \right], \tag{2}$$

where $\hat{\varepsilon}_t = y_t - \hat{\theta}' x_t$ and $\hat{\theta}$ is the OLS estimator of θ in the model $y_t = \theta' x_t + \varepsilon_t$. As z_t is a subset of x_t, we can write $z_t = S x_t$ where S is a $(k \times m)$ selection matrix that contains zeros and ones and satisfies $SS' = I_k$. For any function $\beta(\cdot)$, we have

$$\begin{aligned} A_n(\beta) &= \sum_{t=1}^{n} ((\beta(t/n)' z_t)^2 - 2\hat{\varepsilon}_t \beta(t/n)' z_t) \\ &= \sum_{t=1}^{n} \beta(t/n)' S x_t x_t' S' \beta(t/n) - 2 \sum_{t=1}^{n} \varepsilon_t x_t' S' \beta(t/n) \\ &\quad - 2 \sum_{t=1}^{n} \beta_0(t/n)' S x_t x_t' S' \beta(t/n) + 2 \sum_{t=1}^{n} \beta_0(t/n)' S x_t x_t' (\sum_{t=1}^{n} x_t x_t')^{-1} \sum_{t=1}^{n} x_t x_t' S' \beta(t/n) \\ &\quad + 2 \sum_{t=1}^{n} \varepsilon_t x_t' (\sum_{t=1}^{n} x_t x_t')^{-1} \sum_{t=1}^{n} x_t x_t' S' \beta(t/n), \end{aligned} \tag{3}$$

where we have used the fact that $y_t = \theta_0' x_t + \beta_0(t/n)' z_t + \varepsilon_t$.

The situation of minimizing an objective function

$$\sum_{t=1}^{n} w_t (v_t - \beta(t/n))^2$$

over all nondecreasing $\beta(\cdot)$ for a sequence of positive weights w_t is referred to as *isotonic regression*. For an overview of isotonic regression, we refer to Robertson, Wright, and Dykstra (1988). As explained in Wu, Woodroofe, and Mentz (2001), in general, isotonic regression will not generate a limit distribution for statistics such as $\inf_\beta A_n(\beta)$, where the "inf" is taken over the space B. This is due to a "spiking problem." This problem concerns the limit behavior of the endpoints of the $\beta(\cdot)$ functions that minimize $A_n(\beta)$. In their paper, Wu *et al.* (2001) consider limiting the distance between the endpoints of the isotonic regression function. However, their approach does not deal with a regression setting and we suggest an alternative modification of the isotonic regression setting, namely, fixing the $\beta(\cdot)$ function in the neighborhoods of 0 and of 1. This modification is very much in the spirit of what is done in Andrews (1993) and has the feature of generating an asymptotically pivotal statistic $\inf A_n(\beta)$ for the case of scalar z_t.

Let $\| \beta \| = \sup_{r \in [0,1]} |\beta(r)|$ where $|.|$ denotes the Euclidean norm. Also, for any $K > 0$ define $B_K = \{\beta \in B : \| \beta \| \leq K\}$. Throughout this article, the maintained assumption is the following.

Assumption 1. There exists a positive definite matrix Ω such that

$$W_n(r) = \Omega^{-1/2} n^{-1/2} \sum_{t=1}^{[rn]} \varepsilon_t x_t$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

satisfies $W_n(r) \Rightarrow W(r)$. Furthermore, for

$$V_n(r) = n^{-1}\sum_{t=1}^{[rn]} x_t x_t'$$

and $V(r) = rQ_{xx}$ we have

$$\sup_{r\in[0,1]} |V_n(r) - V(r)| \xrightarrow{p} 0,$$

and Q_{xx} is positive definite.

Noting that

$$x_t x_t' = n(V_n(t/n) - V_n((t-1)/n))$$

and

$$\varepsilon_t x_t = n^{1/2}\Omega^{1/2}(W_n(t/n) - W_n((t-1)/n)),$$

we can now write

$$\begin{aligned}
A_n(\beta) = {} & n\sum_{t=1}^{n} \beta(t/n)'S(V_n(t/n) - V_n((t-1)/n))S'\beta(t/n) \\
& - 2n^{1/2}\sum_{t=1}^{n}(W_n(t/n) - W_n((t-1)/n))'\Omega^{1/2}S'\beta(t/n) \\
& - 2n\sum_{t=1}^{n}\beta_0(t/n)'S(V_n(t/n) - V_n((t-1)/n))S'\beta(t/n) \\
& + 2n\sum_{t=1}^{n}\beta_0(t/n)'S(V_n(t/n) - V_n((t-1)/n))V_n(1)^{-1} \\
& \times \sum_{t=1}^{n}(V_n(t/n) - V_n((t-1)/n))S'\beta(t/n) \\
& + 2n^{1/2}W_n(1)'\Omega^{1/2}V_n(1)^{-1}\sum_{t=1}^{n}(V_n(t/n) - V_n((t-1)/n))S'\beta(t/n). \qquad (4)
\end{aligned}$$

This five-term representation will be key to establishing our results under both the null and alternative hypothesis.

2.2. Limit Distribution Under the Null Hypothesis

The representation of Equation (4) suggests that if $\beta_0(\cdot) = 0$, the third and fourth terms will disappear in the expression of Equation (4) and hence $A_n(n^{-1/2}\beta)$ will asymptotically resemble

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$A^1(\beta) = \int_0^1 \beta(r)' S Q_{xx} S' \beta(r) dr - 2\int_0^1 \beta(r)' S \Omega^{1/2} dW(r) + 2W(1)'\Omega^{1/2} S' \bar{\beta}, \quad \textbf{(5)}$$

where $\bar{\beta} = \int_0^1 \beta(r)dr$.

Formalizing this intuition gives our main result, which is the following theorem.

THEOREM 1. *Under Assumption 1, if* $\beta_0(\cdot) = 0$*,*

$$\inf_{\beta \in B} A_n(\beta) \xrightarrow{d} \inf_{\beta \in B} A^1(\beta).$$

Furthermore,

$$\inf_{\beta \in B} A^1(\beta) = \inf_{\beta \in B} (\int_0^1 (\beta(r) - \bar{\beta})' S Q_{xx} S' (\beta(r) - \bar{\beta}) dr - 2\int_0^1 (\beta(r) - \bar{\beta})' S \Omega^{1/2} dW(r)).$$

The proofs of the theorems are provided in the Mathematical Appendix.

2.3. A Pivotal Test for Structural Change

It seems hard to generate a pivotal statistic in general from the result of Theorem 1, except for the case where $k = 1$ and $z_t \in \mathbb{R}$. For that case, define $Q_{zz} = S Q_{xx} S'$, $\lambda^2 = S\Omega S'$ and $\tilde{W}(r) = \lambda^{-1} S \Omega^{1/2} W(r)$, and note that $\tilde{W}(r)$ is a scalar Brownian motion process. We then have

$$\begin{aligned} \inf_{\beta \in B} A_n(\beta) &= \inf_{\beta \in B} A_n(\lambda\beta/Q_{zz}) \\ &\xrightarrow{d} \inf_{\beta \in B} (\int_0^1 ((\lambda/Q_{zz})(\beta(r) - \bar{\beta}))' S Q_{xx} S' ((\beta(r) - \bar{\beta})(\lambda/Q_{zz})) dr \\ &\quad - 2\int_0^1 (\lambda/Q_{zz})(\beta(r) - \bar{\beta})' S \Omega^{1/2} dW(r)) \\ &= (\lambda^2/Q_{zz}) \inf_{\beta \in B} (\int_0^1 (\beta(r) - \bar{\beta})^2 dr - 2\int_0^1 (\beta(r) - \bar{\beta}) d\tilde{W}(r)), \end{aligned}$$

and the expression

$$\inf_{\beta \in B} A_1^1(\beta) = \inf_{\beta \in B} (\int_0^1 (\beta(r) - \bar{\beta})^2 dr - 2\int_0^1 (\beta(r) - \bar{\beta}) d\tilde{W}(r)$$

is pivotal. Since B depends on the choice of η and on whether B denotes the set of (1) all nonincreasing or (2) all nondecreasing functions, different critical values are obtained for each case. Table 1 lists the critical values of the $\inf_{\beta \in B} A_1^1(\beta)$ statistic for $\eta = 0.10$, 0.15, 0.20, and 0.25 for the case where B contains nondecreasing functions; these values were obtained by simulation using 10,000 replications. For the case when B contains nonincreasing functions the critical values are identical, because of the distributional equivalence of $W(\cdot)$ to $-W(\cdot)$.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

TABLE 1. Critical values of $\inf_{\beta\in B} A^1(\beta)$.

η	0.10		0.15		0.20		0.25	
n	10%	5%	10%	5%	10%	5%	10%	5%
100	−6.42	−8.10	−5.78	−7.48	−5.21	−6.78	−4.72	−6.27
250	−6.62	−8.40	−5.96	−7.63	−5.38	−6.95	−4.89	−6.39
500	−6.90	−8.67	−6.22	−7.91	−5.63	−7.32	−5.12	−6.74
1,000	−7.03	−9.15	−6.28	−8.13	−5.70	−7.52	−5.21	−6.90
2,500	−7.22	−9.11	−6.35	−8.39	−5.83	−7.58	−5.25	−6.83
5,000	−7.15	−9.08	−6.33	−8.18	−5.77	−7.60	−5.27	−7.03

2.4. Limit Behavior Under the Alternative

Under the alternative hypothesis that $\beta_0 \in B$, but $\beta_0(\cdot) \neq 0$, we should consider $n^{-1}A_n(\beta)$. In this case, it can be shown that the second and fifth terms of Equation (4) vanish, and this observation gives the following consistency result.

THEOREM 2. *Assume that Assumption 1 holds. Then*

$$n^{-1}\inf_{\beta\in B} A_n(\beta) \xrightarrow{p} -\int_0^1 (\beta_0(r) - \int_0^1 \beta_0(r)dr)' S Q_{xx} S' (\beta_0(r) - \int_0^1 \beta_0(r)dr)dr.$$

The above theorem is based on showing that $n^{-1}A_n(\beta)$ approaches $A^2(\beta)$, where

$$A^2(\beta) = \int_0^1 \beta(r)' S Q_{xx} S' \beta(r)dr - 2\int_0^1 \beta_0(r)' S Q_{xx} S' \beta(r)dr + 2\int_0^1 \beta_0(r)' dr S Q_{xx} S' \int_0^1 \beta(r)dr.$$

The expression $A^2(\beta)$ is minimal for $\beta(r) = \beta_0(r) - \int_0^1 \beta_0(r)dr$ because

$$\begin{aligned}
0 &\leq \int_0^1 (\beta(r) - \beta_0(r) + \int_0^1 \beta_0(r)dr)' S Q_{xx} S' (\beta(r) - \beta_0(r) + \int_0^1 \beta_0(r)dr)dr \\
&= \int_0^1 \beta(r)' S Q_{xx} S' \beta(r)dr + \int_0^1 (\beta_0(r) - \int_0^1 \beta_0(r)dr)' S Q_{xx} S' (\beta_0(r) - \int_0^1 \beta_0(r)dr)dr \\
&\qquad - 2\int_0^1 (\beta_0(r) - \int_0^1 \beta_0(r)dr)' S Q_{xx} S' \\
&= A^2(\beta) + \int_0^1 (\beta_0(r) - \int_0^1 \beta_0(r)dr)' S Q_{xx} S' (\beta_0(r) - \int_0^1 \beta_0(r)dr)dr,
\end{aligned}$$

and therefore,

$$\begin{aligned}
A^2(\beta) &\geq -\int_0^1 (\beta_0(r) - \int_0^1 \beta_0(r)dr)' S Q_{xx} S' (\beta_0(r) - \int_0^1 \beta_0(r)dr)dr \\
&= A^2(\beta_0(r) - \int_0^1 \beta_0(r)dr).
\end{aligned}$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

It is notable that $A^2(\cdot)$ is minimized at $\beta_0(r) - \int_0^1 \beta_0(r)dr$, instead of at $\beta_0(r)$. Since z_t is included in x_t, adding a constant to $\beta(r)$ and subtracting the same constant from the corresponding element of θ_0 will not alter the objective function. Therefore, we can view the subtraction of $\int_0^1 \beta_0(r)dr = 0$ as a necessary normalization.

To gain more insight into the power property of $\inf_{\beta\in B} A_n(\beta)$, we consider the following sequence of local alternatives:

$$\mathbb{H}_A(n) : \beta_0(r) = \frac{g(r)}{\sqrt{n}}, r \in [0,1],$$

where $g : [0,1] \to \mathbb{R}^k$ is a monotonic vector function. Following the proof of Theorem 1, we can verify that under $\mathbb{H}_A(n)$, $\inf_{\beta\in B} A_n(\beta) \xrightarrow{d} \inf_{\beta\in B} A^1(\beta) - \int_0^1 (g(r) - \bar{g})' SQ_{xx}S'\beta(r)dr$, where $A^1(\beta)$ is defined in Equation (5) and $\bar{g} = \int_0^1 g(r)dr$. This suggests that our test has nontrivial power against the class of smooth monotonic alternatives with rate $n^{-1/2}$, which is faster than the nonparametric rate $n^{-1/2}h^{-1/4}$, where h is the bandwidth, obtained in Chen and Hong (2012) and Kristensen (2012).

3. CALCULATING THE STATISTIC

Considering the DGP in Equation (1), we conduct our test via the following steps:

1. Run OLS regression of y_t on x_t and get the estimated residual $\hat{\varepsilon}_t$.
2. Estimate $\beta(t)$ via isotonic regression

$$\hat{\beta}(t/n) = \arg\min_{\beta\in B} \sum_{t=1}^{n} (\hat{\varepsilon}_t - \beta(t/n)' z_t)^2.$$

3. Compute the test statistic

$$\inf_{\beta\in B} A_n(\beta) = \sum_{t=1}^{n} \left[\left(\hat{\varepsilon}_t - \hat{\beta}(t/n)' z_t \right)^2 - \hat{\varepsilon}_t^2 \right]$$

and compare it with the critical values.

Our algorithm covers pure and partial structural changes in a unified framework, where z_t is the same as x_t for pure changes and a subset of x_t for partial changes. Step 2 in our algorithm involves isotonic regression and minimizing a random function over a function space may seem cumbersome at first sight. However, the computational problem turns out to be surprisingly simple. In the literature on isotonic regression, such as Wu et al. (2001), or Robertson et al. (1988), a common formulation is

$$\min \sum_{t=1}^{n} w_t (v_t - \mu_t)^2, \quad \textbf{(6)}$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

where $\{v_t\}_{t=1}^n$ are the observed time series, $\{w_t\}_{t=1}^n$ are prespecified positive weights and $\{\mu\}_t$ are the parameters to be estimated, and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. Therefore, from a perspective of calculating our statistic, the proposal of this article focuses on the class of nondecreasing functions that are constant on $[0,\eta]$ and on $[1-\eta,1]$, which amounts to forcing $\mu_1 = \cdots = \mu_{[\eta n]}$ and $\mu_{[(1-\eta)n]+1} = \cdots = \mu_n$. The maximization of this objective function is a strictly convex quadratic programming problem, and its solution is unique (Robertson et al., 1988)

In the case of scalar $\beta(.)$ and z_t, under the assumptions $\beta(1/n) = \cdots = \beta([\eta n]/n)$ and $\beta(([(1-\eta)n]+1)/n) = \cdots = \beta(1)$, Step 2 in our algorithm can be written as

$$\begin{aligned}
&\min \sum_{t=1}^{n} (\hat{\varepsilon}_t - \beta(t/n) z_t)^2 \\
&= \min \sum_{t=[n\eta]+1}^{n-[n\eta]} z_t^2 (\hat{\varepsilon}_t / z_t - \beta(t/n))^2 + \sum_{t=1}^{[n\eta]} z_t^2 (\hat{\varepsilon}_t / z_t - \beta(t/n))^2 + \sum_{t=n-[n\eta]+1}^{n} z_t^2 (\hat{\varepsilon}_t / z_t - \beta(t/n))^2 \\
&= C + \min \sum_{t=[n\eta]+1}^{n-[n\eta]} z_t^2 (\hat{\varepsilon}_t / z_t - \beta(t/n))^2 + (\sum_{t=1}^{[n\eta]} z_t^2)(\frac{\sum_{t=1}^{[n\eta]} \hat{\varepsilon}_t z_t}{\sum_{t=1}^{[n\eta]} z_t^2} - \beta(1/n))^2 \\
&\quad + (\sum_{t=[(1-\eta)n]+1}^{n} z_t^2)(\frac{\sum_{t=[(1-\eta)n]+1}^{n} \hat{\varepsilon}_t z_t}{\sum_{t=[(1-\eta)n]+1}^{n} z_t^2} - \beta(1))^2,
\end{aligned}$$

which implies that Step 2 is equivalent to carrying out the isotonic regression in Equation (6) on a dataset with $n - 2[\eta n] + 2$ observations: $\{v_t\}_{t=1}^{n-2[\eta n]+2}$ for which the first observation $v_1 = \frac{\sum_{t=1}^{[n\eta]} \hat{\varepsilon}_t z_t}{\sum_{t=1}^{[n\eta]} z_t^2}$ has weight $w_1 = \sum_{t=1}^{[n\eta]} z_t^2$, the second one $v_2 = \hat{\varepsilon}_{[n\eta]+1}/z_{[n\eta]+1}$ has weight $w_2 = z_{[n\eta]+1}^2$, etc., up to $v_{n-2[n\eta]+1} = \hat{\varepsilon}_{n-[n\eta]}/z_{n-[n\eta]}$ which has weight $w_{n-2[n\eta]+1} = z_{n-[n\eta]}^2$, and the last observation $v_{n-2[n\eta]+2} = \frac{\sum_{t=[(1-\eta)n]+1}^{n} \hat{\varepsilon}_t z_t}{\sum_{t=[(1-\eta)n]+1}^{n} z_t^2}$ has weight $w_{n-2[n\eta]+1} = \sum_{t=[(1-\eta)n]+1}^{n} z_t^2$.

As shown in Robertson et al. (1988), the explicit formula of the values $\hat{\beta}(t/n)$ for isotonic regression with weights w_t can be found as:

$$\hat{\beta}(i/n) = \max_{j:j\leq i} \min_{h:h\geq i} \mathrm{Av}(\hat{\varepsilon}_j, \hat{\varepsilon}_{j+1}, \ldots, \hat{\varepsilon}_h),$$

where

$$\mathrm{Av}(\hat{\varepsilon}_j, \hat{\varepsilon}_{j+1}, \ldots, \hat{\varepsilon}_h) = \frac{\sum_{t=j}^{h} w_t \hat{\varepsilon}_t}{\sum_{t=j}^{h} w_t}.$$

For weights w_t all equal to 1 this simplifies to

$$\hat{\beta}(i/n) = \max_{j:j\leq i} \min_{h:h\geq i} \frac{\hat{\varepsilon}_j + \cdots + \hat{\varepsilon}_h}{h-j+1}.$$

Therefore, the test statistic is straightforward to calculate from the data.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

4. MONTE CARLO AND APPLICATION

4.1. Monte Carlo Simulations

In the simulations below, we used the value $\eta = 0.15$, $\nu_t \sim i.i.d.N(0,1)$ and ε_t and ν_t are mutually independent.[5] To examine the size of all tests under $\mathbb{H}_0$, we considered the following DGP, which was also used in Chen and Hong (2012):

- DGP0: No Structural Change

$$y_t = 1 + 0.5x_t + \varepsilon_t,$$
$$x_t = 0.5x_{t-1} + \nu_t.$$

To examine the robustness of tests, we consider two cases for $\{\varepsilon_t\}$: (i) $\varepsilon_t \sim i.i.d.N(0,1)$; (ii) $\varepsilon_t = 0.5\varepsilon_{t-1} + u_t, u_t \sim i.i.d.N(0,1)$. We generated 5,000 data sets of the random sample $\{x_t, y_t\}_{t=1}^n$ for $n = 100, 250$ and 500 respectively. We compared our test with a variety of popular tests, including: Andrews' (1993) supremum *LM* test; Lin and Teräsvirta's (1994) LM test test based on the first-order Taylor expansion; Bai and Perron's (1998) *UD*max test; Elliott and Müller's (2006) *qLL* test, and Chen and Hong's (2012) generalized Hausman test. Following Andrews (1993), we chose the trimming region $\Pi = [0.15, 0.85]$ for the tests of Andrews (1993) and Bai and Perron (1998). For Bai and Perron's (2003) test, the maximum number of breaks is set to five. For the generalized Hausman test, we adopted the rule-of-thumb bandwidth $h = \frac{1}{\sqrt{12}}n^{-1/5}$, as suggested by Chen and Hong (2012).

To investigate the power of all tests in detecting structural changes, we considered six alternatives: (i) a single break, (ii) monotonic multiple breaks, (iii) non-monotonic multiple breaks, (iv) monotonic smooth structural changes, (v) non-monotonic smooth structural changes, and (vi) non-persistent temporary breaks, respectively:

- DGP1: Single Structural Break

$$y_t = \begin{cases} 1 + 0.5x_t + \varepsilon_t, & \text{if } t \le 0.3n, \\ 1.2 + x_t + \varepsilon_t, & \text{otherwise.} \end{cases}$$

- DGP2: Monotonic Multiple Structural Breaks

$$y_t = \begin{cases} 1 + 0.5x_t + \varepsilon_t, & \text{if } t \le 0.2n, \\ 1.2 + 0.7x_t + \varepsilon_t, & \text{if } 0.2n < t < 0.6n, \\ 1.4 + 0.9x_t + \varepsilon_t, & \text{otherwise.} \end{cases}$$

- DGP P.3: Non-monotonic Multiple Structural Breaks

$$y_t = \begin{cases} 1 + 0.5x_t + \varepsilon_t, & \text{if } t \le 0.3n, \\ 1 + x_t + \varepsilon_t, & \text{if } 0.3n \le t \le 0.7n, \\ 1 + 0.8x_t + \varepsilon_t, & \text{otherwise.} \end{cases}$$

[5] Simulation results with different η can be found in Table A.1 in the Appendix.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

TABLE 2. Empirical size of tests.

	DGP S.1 *i.i.d.* error			DGP S.2 serially correlated error		
n	100	250	500	100	250	500
$\inf_{\beta\in B} A_n(\beta)$	.047	.052	.048	.032	.049	.047
$\hat{H}$	.095	.078	.053	.204	.177	.156
LM	.044	.054	.052	.081	.070	.065
SupLM	.049	.048	.052	.197	.109	.072
UDMax	.051	.052	.050	.375	.208	.125
qLL	.065	.055	.052	.042	.057	.060

Note: (1) 5% significance level; (2) $\inf_{\beta\in B} A_n(\beta)$ is our test based on isotonic regression; $\hat{H}$ is Chen and Hong's (2012) generalized Hausman test; LM is Lin and Teräsvirta's (1994) LM test based on the first-order Taylor expansion; SupLM is Andrews' (1993) supremum LM test; UDMax is Bai and Perron's (1998) double maximum test; qLL is Elliott and Müller's (2006) efficient test based on a quasilocal level model.

- DGP4: Monotonic Smooth Structural Changes

  $$y_t = F(r)(1+0.5x_t)+\varepsilon_t,$$

  where $r=\frac{t}{n}$ and $F(r)=0.2\exp(-0.7+3.5r)$.
- DGP P.5: Non-monotonic Smooth Structural Changes

  $$y_t = 1+0.5F(r)x_t+\varepsilon_t,$$

  where $r=\frac{t}{n}$ and $F(r)=r+\exp[\,-4+(r-0.5)^2]-1$.
- DGP P.6: Non-persistent Temporary Breaks

  $$y_t = \begin{cases} 1+0.5x_t+\varepsilon_t, & \text{if } t\le 0.3n \text{ or } t\ge 0.7n,\\ 1+x_t+\varepsilon_t, & \text{otherwise.}\end{cases}$$

For each of DGPs 1–6, we generated 1,000 data sets of the random sample $\{y_t,x_t\}_{t=1}^n$ for $n=100, 250$, and 500. Table 2 reports the rejection rates of all tests under DGP0 using asymptotic critical values at the 5% significance level. Under i.i.d. and serially correlated errors, our $\inf_{\beta\in B} A_n(\beta)$ test underrejected H_0 when $n=100$, but not excessively and improved as n increases. For other tests, under i.i.d. errors, Chen and Hong's (2012) test and Elliott and Müller's (2006) *qLL* test have some overrejection when n is small, but they improve as n increases. Under serially correlated errors, Chen and Hong's (2012) Hausman test, Andrews' (1993) SupLm test and Bai and Perron's (1998) double maximum test have rather large overrejection although the overrejection becomes smaller with the increase of sample sizes. Overall, our test displays the most robust size although Elliott and Müller's (2006) *qLL* test also has good size control.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Table 3. Empirical power of tests under *i.i.d.* errors.

	DGP P.1 single break			DGP P.2 monotonic multiple breaks			DGP P.3 non-monotonic multiple breaks		
n	100	250	500	100	250	500	100	250	500
$\inf_{\beta\in B} A_n(\beta)$	.646	.958	1.00	.458	.770	.972	.439	.821	.987
$\hat{H}$	.396	.790	.990	.220	.424	. 797	.303	.660	.960
LM	.422	.850	.993	.312	.654	.937	.204	.410	.703
SupLM	.446	.893	.994	.225	.556	.885	.264	.697	.968
UDMax	.493	.935	.999	.258	.629	.929	.326	.774	.988
qLL	.436	.897	1.00	.240	.635	.917	.311	.805	.984
	DGP P.4 monotonic smooth changes			DGP P.5 non-monotonic smooth changes			DGP P.6 temporary break		
n	100	250	500	100	250	500	100	250	500
$\inf_{\beta\in B} A_n(\beta)$	.681	.949	1.00	.487	.837	.990	.149	.378	.744
$\hat{H}$	.345	.748	.981	.229	.548	.820	.381	.796	.994
LM	.472	.876	.995	.256	.604	.918	.067	.063	.066
SupLM	.414	. 839	.993	.273	.666	.944	.178	.518	.903
UDMax	.477	.885	.996	.298	.727	.971	.332	.831	.998
qLL	.446	.888	.995	.261	.724	.967	.404	.884	.998

Note: (1) 5% significance level; (2) $\inf_{\beta\in B} A_n(\beta)$ is our test based on isotonic regression; $\hat{H}$ is Chen and Hong's (2012) generalized Hausman test; LM is Lin and Teräsvirta's (1994) LM test based on the first-order Taylor expansion; SupLM is Andrews' (1993) supremum LM test; UDMax is Bai and Perron's (1998) double maximum test; qLL is Elliott and Müller's (2006) efficient test based on a quasilocal level model.

Next, we consider power. Tables 3 and 4 report the rejection rates of all tests using empirical critical values, which are size-adjusted critical values, at the 5% level under i.i.d. errors and serially correlated errors, respectively. Under i.i.d. errors, our test compares favorably to other competing tests across DGPs P.1-P.5. In particular, our test is more powerful than the kernel-based generalized Hausman test, confirming our theoretical comparison. Notably, even when $\beta(r)$ exhibits non-monotonic changes under DGPs P.3 and P.5, the proposed test outperforms all competing tests. Under DGP P.6, where the break persists only for a limited period, Lin and Teräsvirta's LM test has no power even at $n = 500$. While our test $\inf_{\beta\in B} A_n(\beta)$ is less powerful than $\hat{H}$, SupLM, UDMax, and qLL tests in this setting, its rejection rate still increases with sample size. Overall, $\inf_{\beta\in B} A_n(\beta)$ has comparable power against non-monotonic structure breaks, except when those breaks are temporary. The rankings of all tests under serially correlated errors are consistent with those observed under i.i.d. errors.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Table 4. Empirical power of tests under serially correlated errors.

	DGP P.1 single break			DGP P.2 monotonic multiple breaks			DGP P.3 non-monotonic multiple breaks		
n	100	250	500	100	250	500	100	250	500
$\inf_{\beta \in B} A_n(\beta)$	.367	.714	.968	.459	.731	.932	.411	.691	.928
$\hat{H}$	.166	.628	.882	.329	.441	.670	.455	.690	.920
LM	.207	.479	.789	.174	.361	.619	.115	.219	.368
SupLM	.229	.532	.915	.424	.565	.785	.340	.621	.856
UDMax	.164	.513	.885	.438	.559	.756	.440	.780	.908
qLL	.231	.541	.890	.136	.329	.575	.138	.405	.762
	DGP P.4 monotonic smooth changes			DGP P.5 non-monotonic smooth changes			DGP P.6 temporary break		
n	100	250	500	100	250	500	100	250	500
$\inf_{\beta \in B} A_n(\beta)$	.607	.864	.989	.288	.550	.844	.208	.355	.634
$\hat{H}$	.536	.805	.963	.458	.706	.910	.289	.446	.804
LM	.839	.989	1.00	.169	.347	.577	.079	.063	.066
SupLM	.566	.769	.934	.151	.351	.753	.214	.385	.670
UDMax	.700	.823	.942	.260	.719	.885	.229	.447	.900
qLL	.251	.563	.868	.149	.369	.720	.181	.483	.851

Note: (1) 5% significance level; (2) $\inf_{\beta \in B} A_n(\beta)$ is our test based on isotonic regression; $\hat{H}$ is Chen and Hong's (2012) generalized Hausman test; Lin and Teräsvirta's (1994) LM test based on the first-order Taylor expansion; SupLM is Andrews' (1993) supremum LM test; UDMax is Bai and Perron's (1998) double maximum test; qLL is Elliott and Müller's (2006) efficient test based on a quasilocal level model.

4.2. Application to Data

The issue of global warming has received considerable attention for more than two decades, as evidenced by studies such as those by Melillo (1999), Delworth and Knutson (2000) and Nordhaus (2019). In our research, we apply our test to assess the significance of the increasing global temperature anomalies over time. Annual temperature anomaly data, spanning from 1850 to 2023, were obtained from the National Centers for Environmental Information. Figure 1 displays the time series plot. The estimated $\inf_{\beta \in B} A_n(\beta)$ statistic with $\eta = 0.15$ is -150.42, which is highly significant at any conventional significance level. The strong rejection is echoed by all other tests statistics: Andrews' (1993) SupLM test is 28.29, Bai and Perron's (1998) test is 37.04, Elliott and Müller's (2006) *qLL* test is -29.041, Chen and Hong's (2012) Hausman test is 409.69.

As another application, we examine the effect of personal tax exemption on fertility rates studied by Wooldridge (2008). The yearly data, covering 1913–1984,

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

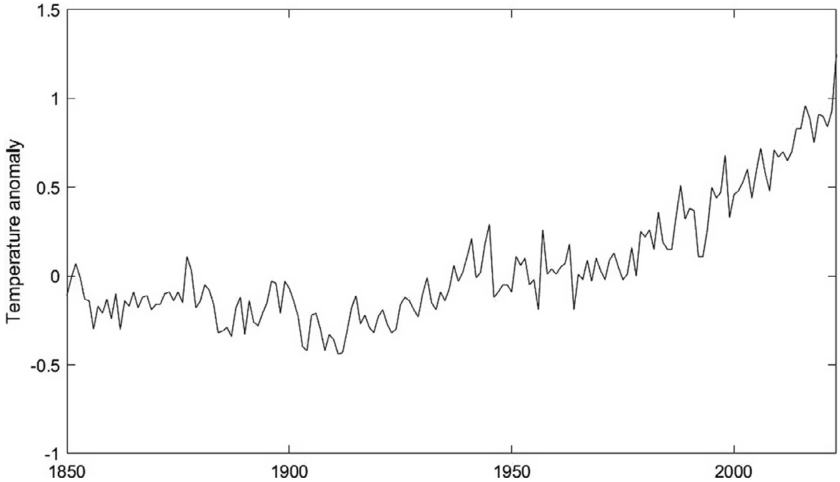

FIGURE 1. Global warming data.

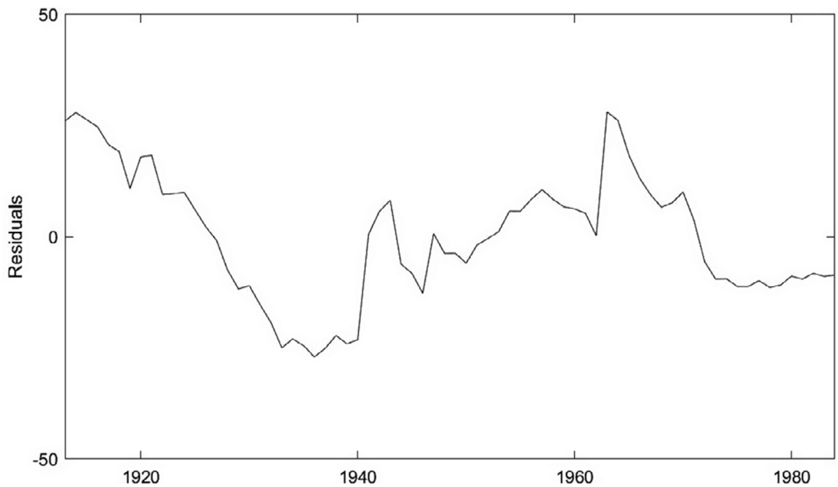

FIGURE 2. The estimated residuals of Equation (7).

were sourced from Whittington, Alm, and Peters (1990). Following Wooldridge (2008), we consider the linear regression model:

$$gfr_t = \beta_0 + \beta_1 pe_t + \beta_2 ww2_t + \beta_3 pill_t + u_t, \tag{7}$$

where gfr_t is the general fertility rate, pe_t is the average real dollar value of the personal tax exemption, and $ww2_t$ and $pill_t$ are dummy variables. The dummy variable $ww2_t$ equals 1 during the years 1941 through 1945 and 0 otherwise, while

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$pill_t$ equals 1 from 1963 onward and 0 otherwise. Equation (7) is estimated via OLS, and the estimated residuals are plotted in Figure 2.

The intercept captures the expected general fertility rate with 0 personal tax exemption, controlling the effect of World War II and the introduction of the birth control pill. To test whether β_0 is changing over time, we applied our $\inf_{\beta \in B} A_n(\beta)$ test. Using $\eta = 0.15$, the estimated statistic is -17.76, which strongly rejects the null hypothesis that β_0 is a constant over time. Similarly, Andrews' (1993) SupLM test and Bai and Perron's (1998) UDmax test are 16.98 and 19.95, respectively, which also reject the null. The *qLL* test of Elliott and Müller's (2006) rejects the null hypothesis at the 10% significance level but not at the 5% level. In contrast, Chen and Hong's (2012) Hausman test fails to reject the null hypothesis, with the statistic value -0.87.

5. CONCLUSION

Detection of structural changes has been a long-standing interest in econometrics, and in this article, we have introduced a novel tuning-parameter-free test that is designed to detect both smooth structural changes and abrupt structural breaks. While existing tests rely on kernel estimation, our test is based on isotonic regression. This approach allows us to detect monotonic trends or structural changes, making it well-suited for scenarios where prior information might suggest such patterns.

Features of our test are (1) our test minimizes an objective function over a space of functions; (2) improved power against monotonic smooth structural changes, as compared to approaches that use a tuning parameter; and (3) simplicity, as a pivotal limit distribution is obtained after scaling in the scalar case. Our simulation study and application underscore the decent power properties of our test and the pragmatic value of our approach.

A. Appendix

A.1. Mathematical Appendix

The main result is based on the following lemma, which holds for general $\bar{A}_n(\cdot)$. Everywhere in this Appendix, we write $B_K = \{\beta \in B : \| \beta \| \leq K\}$, and s_j denote a k-vector of all zeros, except for an entry of 1 at spot j.

LEMMA A.1. *Assume*

1. *For* $\bar{A}_n : B \to \mathbb{R}$ *and* $\bar{A}^1 : B \to \mathbb{R}$*, for all* $K > 0$, $\inf_{\beta \in B_K} \bar{A}_n(n^{-1/2}\beta)$, $\inf_{\beta \in B} \bar{A}_n(\beta)$, $\inf_{\beta \in B_K} \bar{A}^1(\beta)$*, and* $\inf_{\beta \in B} \bar{A}^1(\beta)$ *are proper random variables;*
2. *For all* $K > 0$,

$$\inf_{\beta \in B_K} \bar{A}_n(n^{-1/2}\beta) \xrightarrow{a.s.} \inf_{\beta \in B_K} \bar{A}^1(\beta);$$

3. $\limsup_{K \to \infty} \limsup_{n \to \infty} P(\inf_{\beta \in B} \bar{A}_n(n^{-1/2}\beta) \neq \inf_{\beta \in B_K} \bar{A}_n(n^{-1/2}\beta)) = 0.$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Then

$$\inf_{\beta\in B}\bar{A}_n(\beta)\xrightarrow{d}\inf_{\beta\in B}\bar{A}^1(\beta).$$

Proof of Lemma A.1. Define $Y_n=\inf_{\beta\in B}\bar{A}_n(\beta)=\inf_{\beta\in B}\bar{A}_n(n^{-1/2}\beta)$, $Y_{nK}=\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta)$, $Y^K=\inf_{\beta\in B_K}\bar{A}^1(\beta)$, and $Y=\inf_{\beta\in B}\bar{A}^1(\beta)$. The lemma asserts that $Y_n\xrightarrow{d}Y$, and because

$$|E\exp(irY_n)-E\exp(irY)|\le|E\exp(irY_n)-E\exp(irY_{nK})|$$
$$+|E\exp(irY_{nK})-E\exp(irY^K)|+|E\exp(irY^K)-E\exp(irY)|,$$

it suffices to show that

$$\limsup_{K\to\infty}\limsup_{n\to\infty}|E\exp(irY_n)-E\exp(irY_{nK})|=0,$$
$$\limsup_{n\to\infty}|E\exp(irY_{nK})-E\exp(irY^K)|=0,$$

and

$$\limsup_{K\to\infty}|E\exp(irY^K)-E\exp(irY)|=0.$$

The first result follows because

$$\limsup_{K\to\infty}\limsup_{n\to\infty}|E\exp(irY_n)-E\exp(irY_{nK})|\le 2\limsup_{K\to\infty}\limsup_{n\to\infty}P(Y_n\ne Y_{nK})=0$$

by Assumption 3. Also, because $Y_{nK}=\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta)\xrightarrow{a.s.}\inf_{\beta\in B_K}\bar{A}^1(\beta)=Y^K$ by Assumption 2,

$$\limsup_{n\to\infty}|E\exp(irY_{nK})-E\exp(irY^K)|=0.$$

Finally, since Y^K is decreasing in K, $\lim_{K\to\infty}Y^K$ exists a.s. and equals $Y=\inf_{\beta\in B}\bar{A}^1(\beta)$. Therefore, the result is now proven. □

For our result under the alternative, we need a result similar to Lemma A.1.

LEMMA A.2. *Assume*

1. *For* $\bar{A}_n:B\to\mathbb{R}$ *and* $\bar{A}^2:B\to\mathbb{R}$, *for all* $K>0$, $\inf_{\beta\in B}\bar{A}_n(\beta)$, $\inf_{\beta\in B_K}\bar{A}_n(\beta)$, $\inf_{\beta\in B_K}\bar{A}^2(\beta)$, *and* $\inf_{\beta\in B}\bar{A}^2(\beta)$ *are proper random variables;*
2. *For all* $K>0$,

$$\inf_{\beta\in B_K}n^{-1}\bar{A}_n(\beta)\xrightarrow{a.s.}\inf_{\beta\in B_K}\bar{A}^2(\beta);$$

3. $\limsup_{K\to\infty}\limsup_{n\to\infty}P(\inf_{\beta\in B}n^{-1}\bar{A}_n(\beta)\ne\inf_{\beta\in B_K}n^{-1}\bar{A}_n(\beta))=0.$

Then

$$n^{-1}\inf_{\beta\in B}\bar{A}_n(\beta)\xrightarrow{d}\inf_{\beta\in B}\bar{A}^2(\beta).$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Proof of Lemma A.2. Define $Y_n = \inf_{\beta\in B} n^{-1}\bar{A}_n(\beta)$, $Y_{nK} = \inf_{\beta\in K} n^{-1}\bar{A}_n(\beta)$, $Y^K = \inf_{\beta\in B_K}\bar{A}^2(\beta)$, and $Y = \inf_{\beta\in B}\bar{A}^2(\beta)$. Using these alternative definitions, the proof of Lemma A.1 is now identical to the proof of Lemma A.2. □

Next, note that under Assumption 1, we have $(V_n, W_n) \Rightarrow (V, W)$, since $V(r)$ is deterministic. By the Skorokhod representation theorem, because $(V_n, W_n) \Rightarrow (V, W)$ in the product space $D\big([0,1],\mathbb{R}^{m\times m}\big)\times D\big([0,1],\mathbb{R}^m\big)$ under the product J_1-topology, there exist in a suitably expanded probability space, $(\bar{V}_n, \bar{W}_n)$ and $(\bar{V}, \bar{W})$ such that $(\bar{V}_n, \bar{W}_n) \overset{d}{=} (V_n, W_n)$, $(\bar{V}, \bar{W}) \overset{d}{=} (V, W)$, and $\sup_{j,l,r\in[0,1]} |s_j'(\bar{V}_n(r) - \bar{V}(r))s_l| + \sup_{j,r\in[0,1]} |s_j'(\bar{W}_n(r)) - \bar{W}(r))| \xrightarrow{a.s.} 0$. We now define versions of $A_n(\beta)$ and $A^1(\beta)$ defined on this Skorokhod space analogous to the expression of Equation (4), viz.

$$
\begin{aligned}
\bar{A}_n(\beta) = {} & n\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)\\
& -2n^{1/2}\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(t/n)\\
& -2n\sum_{t=1}^{n}\beta_0(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)\\
& +2n\sum_{t=1}^{n}\beta_0(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))V_n(1)^{-1}\\
& \times\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)\\
& +2n^{1/2}\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n),
\end{aligned}
\tag{A.1}
$$

and

$$
\bar{A}^1(\beta) = \int_0^1 \beta(r)'SQ_{xx}S'\beta(r)dr - 2\int_0^1 \beta(r)'S\Omega^{1/2}d\bar{W}(r) + 2\bar{W}(1)'\Omega^{1/2}S'\bar{\beta}, \tag{A.2}
$$

$$
\bar{A}^2(\beta) = \int_0^1 \beta(r)'SQ_{xx}S'\beta(r)dr - 2\int_0^1 \beta_0(r)'SQ_{xx}S'\beta(r)dr + 2\int_0^1 \beta_0(r)'drSQ_{xx}S'\int_0^1 \beta(r)dr. \tag{A.3}
$$

This definition is such that $A_n(\beta) \overset{d}{=} \bar{A}_n(\beta)$ and $A^1(\beta) \overset{d}{=} \bar{A}^1(\beta)$, and $A^2(\beta) \overset{d}{=} \bar{A}^2(\beta)$.

Our strategy now is to rewrite the five terms of $\bar{A}_n(\cdot)$ in a way that allows us to find their limits uniformly of $\beta \in B_K$ relatively easily. The following five lemmas give such results.

LEMMA A.3.

$$
\begin{aligned}
&\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)\\
&\quad=\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)\beta_j(0)s_j'S\bar{V}_n(1)S's_l
\end{aligned}
$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_j(r)$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_l(r)$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^1 s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(\max(r,s)-1/n))S's_l d\beta_j(r)d\beta_l(s)$$

and

$$\int_0^1\beta(r)'SQ_{xx}S'\beta(r)dr$$
$$=\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)\beta_j(0)s_j'S\bar{V}(1)S's_l$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)s_j'S\int_0^1(\bar{V}(1)-\bar{V}(r))d\beta_j(r)S's_l$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_0^1(\bar{V}(1)-\bar{V}(r))d\beta_l(r)S's_l$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^1 s_j'S\int_0^1(\bar{V}(1)-\bar{V}(\max(r,s)))d\beta_j(r)d\beta_l(s)S's_l.$$

Proof of Lemma A.3. To show the results for $\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$, note that

$$\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$
$$=\sum_{j=1}^{k}\sum_{l=1}^{k}\sum_{t=1}^{n}\beta_j(t/n)\beta_l(t/n)s_j'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_l$$
$$=\sum_{j=1}^{k}\sum_{l=1}^{k}\sum_{t=1}^{n}(\beta_j(0)+\int_0^{t/n}d\beta_j(r))(\beta_l(0)+\int_0^{t/n}d\beta_l(r))s_j'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_l$$
$$=\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)\beta_j(0)s_j'S\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_l$$
$$+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)s_j'S\int_0^1\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_l I(r\leq t/n)d\beta_j(r)$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$
\begin{aligned}
&+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_0^1\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_lI(r\le t/n)d\beta_l(r)\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'S\int_0^1\int_0^1\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S's_lI(r\le t/n)I(s\le t/n)d\beta_l(r)d\beta_j(s)\\
&=\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)\beta_j(0)s_j'S\bar{V}_n(1)S's_l\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_l(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_ld\beta_j(r)\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_ld\beta_l(r)\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^1 s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(\max(r,s)-1/n))S's_ld\beta_j(r)d\beta_l(s). \qquad \textbf{(A.4)}
\end{aligned}
$$

To show the result for $\int_0^1\beta(r)'SQ_{xx}S'\beta(r)dr$, note that

$$
\begin{aligned}
&\int_0^1\beta(r)'SQ_{xx}S'\beta(r)dr\\
&=\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\beta_j(r)\beta_l(r)dr\\
&=\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\left(\int_0^r d\beta_j(x)+\beta_j(0)\right)\left(\int_0^r d\beta_l(y)+\beta_l(0)\right)dr\\
&=\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\int_0^1\beta_j(0)\beta_l(0)dr\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\beta_l(0)\int_0^r d\beta_j(x)dr\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\beta_j(0)\int_0^r d\beta_l(x)dr\\
&+\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\int_0^1\int_0^r d\beta_j(x)\int_0^r d\beta_l(y)dr\\
&=\sum_{j=1}^{k}\sum_{l=1}^{k}s_j'SQ_{xx}S's_l\beta_j(0)\beta_l(0)
\end{aligned}
$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$
\begin{aligned}
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \beta_l(0) \int_0^1 \int_0^1 I(0 \le x \le r) dr d\beta_j(x) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \beta_j(0) \int_0^1 \int_0^1 I(0 \le x \le r) dr d\beta_l(x) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \int_0^1 \int_0^1 \int_0^1 I(0 \le x \le r) I(0 \le y \le r) dr d\beta_j(x) d\beta_l(y) \\
=&\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \beta_j(0)\beta_l(0) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \beta_l(0) \int_0^1 (1-x) d\beta_j(x) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \beta_j(0) \int_0^1 (1-x) d\beta_l(x) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} s_j' S Q_{xx} S' s_l \int_0^1 \int_0^1 (1-\max(x,y)) d\beta_j(x) d\beta_l(y) \\
=&\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_l(0)\beta_j(0) s_j' S \bar{V}(1) S' s_l \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_l(0) \int_0^1 s_j' S(\bar{V}(1)-\bar{V}(r)) S' s_l d\beta_j(r) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_j(0) \int_0^1 s_j' S(\bar{V}(1)-\bar{V}(r)) S' s_l d\beta_l(r) \\
&+\sum_{j=1}^{k}\sum_{l=1}^{k} \int_0^1 \int_0^1 s_j' S(\bar{V}(1)-\bar{V}(\max(r,s))) S' s_l d\beta_j(r) d\beta_l(s). \qquad \textbf{(A.5)}
\end{aligned}
$$

□

LEMMA A.4.

$$
\begin{aligned}
&\sum_{t=1}^{n} (\bar{W}_n(t/n) - \bar{W}_n((t-1)/n))' \Omega^{1/2} S' \beta(t/n) \\
&= \bar{W}_n(1)' \Omega^{1/2} S' \beta(1) - \int_0^1 \bar{W}_n(r)' \Omega^{1/2} S' d\beta(r)
\end{aligned}
$$

and

$$
\int_0^1 \beta(r)' S \Omega^{1/2} d\bar{W}(r) = \beta(1)' S \Omega^{1/2} \bar{W}(1) - \int_0^1 \bar{W}(r)' \Omega^{1/2} S' d\beta(r).
$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Proof of Lemma A.4. First, consider $\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(t/n)$. This term equals

$$\begin{aligned}
&\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\sum_{j=1}^{t}(\beta(j/n)-\beta((j-1)/n))\\
&\quad+\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(0)\\
&\quad=\sum_{j=1}^{n}\sum_{t=j}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\int_{(j-1)/n}^{j/n}d\beta(r)\\
&\quad+\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(0)\\
&\quad=\sum_{j=1}^{n}\int_{(j-1)/n}^{j/n}(\bar{W}_n(1)-\bar{W}_n((j-1)/n))'\Omega^{1/2}S'd\beta(r)+\bar{W}_n(1)'\Omega^{1/2}S'\beta(0)\\
&\quad=\int_{0}^{1}(\bar{W}_n(1)-\bar{W}_n(r))'\Omega^{1/2}S'd\beta(r)+\bar{W}_n(1)'\Omega^{1/2}S'\beta(0)\\
&\quad=\bar{W}_n(1)'\Omega^{1/2}S'\beta(1)-\int_{0}^{1}\bar{W}_n(r)'\Omega^{1/2}S'd\beta(r)
\end{aligned}\tag{A.6}$$

because $\bar{W}_n((j-1)/n)=\bar{W}_n(r)$ for $r\in((j-1)/n,j/n)$. Also, by integration by parts,

$$\begin{aligned}
&\int_{0}^{1}\beta(r)'S\Omega^{1/2}d\bar{W}(r)=[\beta(r)'S\Omega^{1/2}\bar{W}(r)]_0^1-\int_{0}^{1}\bar{W}(r)'\Omega^{1/2}S'd\beta(r)\\
&\quad=\beta(1)'S\Omega^{1/2}\bar{W}(1)-\int_{0}^{1}\bar{W}(r)'\Omega^{1/2}S'd\beta(r).
\end{aligned}$$

□

LEMMA A.5.

$$\begin{aligned}
&\sum_{t=1}^{n}\beta_0(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)\\
&\quad=\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_{0l}(0)\beta_j(0)s_j'S\bar{V}_n(1)S's_l\\
&\quad+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_{0l}(0)s_j'S\int_{0}^{1}(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_j(r)\\
&\quad+\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_{0}^{1}(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_{l0}(r)\\
&\quad+\sum_{j=1}^{k}\sum_{l=1}^{k}\int_{0}^{1}s_j'S\int_{0}^{1}(\bar{V}_n(1)-\bar{V}_n(\max(r,s)-1/n))S's_l d\beta_j(r)d\beta_{l0}(s)
\end{aligned}$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

and

$$\int_0^1 \beta_0(r)' S Q_{xx} S' \beta(r) dr$$
$$= \sum_{j=1}^{k}\sum_{l=1}^{k} \beta_{0l}(0)\beta_j(0) s_j' S \bar{V}(1) S' s_l$$
$$+ \sum_{j=1}^{k}\sum_{l=1}^{k} \beta_{0l}(0) s_j' S \int_0^1 (\bar{V}(1) - \bar{V}(r)) d\beta_j(r) S' s_l$$
$$+ \sum_{j=1}^{k}\sum_{l=1}^{k} \beta_j(0) s_j' S \int_0^1 (\bar{V}(1) - \bar{V}(r)) d\beta_{0l}(r) S' s_l$$
$$+ \sum_{j=1}^{k}\sum_{l=1}^{k} \int_0^1 s_j' S \int_0^1 (\bar{V}(1) - \bar{V}(\max(r,s))) d\beta_j(r) d\beta_{0l}(s) S' s_l.$$

Proof of Lemma A.5. This proof is nearly identical to the proof of Lemma A.3 and is therefore omitted. □

LEMMA A.6.

$$\sum_{t=1}^{n} \beta_0(t/n)' S(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n)) V_n(1)^{-1} \sum_{t=1}^{n} (\bar{V}_n(t/n) - \bar{V}_n((t-1)/n)) S' \beta(t/n)$$
$$= (\bar{V}_n(1)' S' \beta_0(1) - \int_0^1 \bar{V}_n(r) S' d\beta_0(r))' V_n(1)^{-1} (\bar{V}_n(1)' S' \beta(1) - \int_0^1 \bar{V}_n(r) S' d\beta(r))$$

and

$$\int_0^1 \beta_0(r)' dr S Q_{xx} S' \int_0^1 \beta(r) dr$$
$$= (\bar{V}(1)' S' \beta_0(1) - \int_0^1 \bar{V}(r) S' d\beta_0(r))' V(1)^{-1} (\bar{V}(1)' S' \beta(1) - \int_0^1 \bar{V}(r) S' d\beta(r)).$$

Proof of Lemma A.6. To start the proof, note that

$$\sum_{t=1}^{n} (\bar{V}_n(t/n) - \bar{V}_n((t-1)/n)) S' \beta(t/n)$$
$$= \sum_{t=1}^{n} (\bar{V}_n(t/n) - \bar{V}_n((t-1)/n)) S' \sum_{j=1}^{t} (\beta(j/n) - \beta((j-1)/n))$$
$$+ \sum_{t=1}^{n} (\bar{V}_n(t/n) - \bar{V}_n((t-1)/n)) S' \beta(0)$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$= \sum_{j=1}^{n}\sum_{t=j}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\int_{(j-1)/n}^{j/n} d\beta(r)$$

$$+ \sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(0)$$

$$= \sum_{j=1}^{n}\int_{(j-1)/n}^{j/n}(\bar{V}_n(1) - \bar{V}_n((j-1)/n))S'd\beta(r) + \bar{V}_n(1)S'\beta(0)$$

$$= \int_0^1 (\bar{V}_n(1) - \bar{V}_n(r))S'd\beta(r) + \bar{V}_n(1)S'\beta(0)$$

$$= \bar{V}_n(1)'S'\beta(1) - \int_0^1 \bar{V}_n(r)S'd\beta(r), \qquad \textbf{(A.7)}$$

and the first part of the lemma now follows. To deal with the second part, note that because $V(r) = rQ_{xx}$,

$$(\bar{V}(1)'S'\beta_0(1) - \int_0^1 \bar{V}(r)S'd\beta_0(r))'V(1)^{-1}(\bar{V}(1)'S'\beta(1) - \int_0^1 \bar{V}(r)S'd\beta(r))$$

$$= (Q_{xx}S'\beta_0(1) - Q_{xx}S'\int_0^1 rd\beta_0(r))'Q_{xx}^{-1}(Q_{xx}S'\beta(1) - Q_{xx}S'\int_0^1 rd\beta(r))$$

$$= (S'\beta_0(1) - S'\int_0^1 rd\beta_0(r))'Q_{xx}(S'\beta(1) - S'\int_0^1 rd\beta(r))$$

$$= \int_0^1 \beta_0(r)'drSQ_{xx}S'\int_0^1 \beta(r)dr.$$

□

LEMMA A.7.

$$\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(t/n)$$

$$= \bar{W}_n(1)'\Omega^{1/2}S'\beta(1) - \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1 \bar{V}_n(r)S'd\beta(r)$$

and

$$\bar{W}(1)'\Omega^{1/2}S'\bar{\beta} = \bar{W}(1)'\Omega^{1/2}S'\beta(1) - \bar{W}(1)'\Omega^{1/2}S'\int_0^1 rd\beta(r).$$

Proof of Lemma A.7. Note that

$$\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(t/n)$$

$$= \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\sum_{j=1}^{t}(\beta(j/n) - \beta((j-1)/n))$$

$$+ \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(0)$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$= \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{j=1}^{n}\sum_{t=j}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\int_{(j-1)/n}^{j/n} d\beta(r)$$

$$+\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(0)$$

$$= \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{j=1}^{n}\int_{(j-1)/n}^{j/n}(\bar{V}_n(1)-\bar{V}_n((j-1)/n))S'd\beta(r)$$

$$+W_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(1)S'\beta(0)$$

$$= \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1(\bar{V}_n(1)-\bar{V}_n(r))S'd\beta(r)+\bar{W}_n(1)'\Omega^{1/2}S'\beta(0)$$

$$= \bar{W}_n(1)'\Omega^{1/2}S'\beta(1)-\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1\bar{V}_n(r)S'd\beta(r) \qquad \textbf{(A.8)}$$

because $V_n((j-1)/n) = V_n(r)$ for $r \in ((j-1)/n, j/n)$. In addition, to show the second part of the lemma, note that

$$\bar{W}(1)'\Omega^{1/2}S'\bar{\beta} = \bar{W}(1)'\Omega^{1/2}S'([r\beta(r)]_0^1 - \int_0^1 rd\beta(r))$$

$$= \bar{W}(1)'\Omega^{1/2}S'\beta(1) - \bar{W}(1)'\Omega^{1/2}S'\int_0^1 rd\beta(r), \qquad \textbf{(A.9)}$$

which completes the proof of the lemma. □

Using the above five lemmas, we can now establish the following two lemmas that are key to the proof of Theorem 1.

LEMMA A.8. *Under Assumption 1, if* $\beta_0(\cdot)=0$,

$$\limsup_{K\to\infty}\limsup_{n\to\infty}P(\inf_{\beta\in B}\bar{A}_n(n^{-1/2}\beta)\neq\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta))=0,$$

$\inf_{\beta\in B}\bar{A}_n(\beta)=O_p(1)$, *and* $\inf_{\beta\in B}\bar{A}^1(\beta)$ *is a proper random variable.*

Proof of Lemma A.8. Note that for $\beta(\cdot)\in B$, because $\beta(\cdot)$ is constant on $[0,\eta]$ and $[1-\eta,1]$ and $\bar{V}_n(t/n)-\bar{V}_n((t-1)/n)$ is positive semidefinite,

$$\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$

$$\geq \sum_{t=1}^{[\eta n]}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$

$$+\sum_{t=[n(1-\eta)]+1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$

$$
\begin{aligned}
&= \beta(0)' \sum_{t=1}^{[\eta n]} S(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(0) + \beta(1)' \\
&\quad \times \sum_{t=[n(1-\eta)]+1}^{n} S(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(1) \\
&= \beta(0)'S\bar{V}_n([\eta n]/n)S'\beta(0) + \beta(1)'S(\bar{V}_n(1) - \bar{V}_n([n(1-\eta)]/n))S'\beta(1) \\
&\geq |\beta(0)|^2 \lambda_{min}(\bar{V}_n([\eta n]/n)) + |\beta(1)|^2 \lambda_{min}(\bar{V}_n(1) - \bar{V}_n(n(1-\eta)]/n)),
\end{aligned}
$$

where λ_{min} denotes the minimal eigenvalue. Also, by the result of Equation (A.6),

$$
\begin{aligned}
&|\sum_{t=1}^{n} (\bar{W}_n(t/n) - \bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(t/n)| \\
&\quad = |\int_0^1 \bar{W}_n(r)'\Omega^{1/2}S'd\beta(r) - \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1 \bar{V}_n(r)S'd\beta(r)| \\
&\quad = |\sum_{j=1}^{k}\int_0^1 \bar{W}_n(r)'\Omega^{1/2}S's_j d\beta_j(r) - \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1 \bar{V}_n(r)S's_j d\beta_j(r)| \\
&\quad \leq (\sup_{r,j}|\bar{W}_n(r)'\Omega^{1/2}S's_j| + \sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|)\sum_{j=1}^{k}\int_0^1 d\beta_j(r) \\
&\quad \leq (\sup_{r,j}|\bar{W}_n(r)'\Omega^{1/2}S's_j| + \sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|)\sum_{j=1}^{k}(\beta_j(1) - \beta_j(0)) \\
&\quad \leq (\sup_{r,j}|\bar{W}_n(r)'\Omega^{1/2}S's_j| + \sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|)k(|\beta(1)| + |\beta(0)|)
\end{aligned}
\tag{A.10}
$$

because $\beta_j(r) \leq (\sum_{j=1}^{k}\beta_j(r)^2)^{1/2} = |\beta(r)|$. And by the result of Equation (A.8), using similar reasoning,

$$
\begin{aligned}
&|W_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n) - \bar{V}_n((t-1)/n))S'\beta(t/n)| \\
&\quad = |\bar{W}_n(1)'\Omega^{1/2}S'\beta(1) - \bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1 \bar{V}_n(r)S'd\beta(r)| \\
&\quad \leq 2\sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|k(|\beta(0)| + |\beta(1)|).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\bar{A}_n(n^{-1/2}\beta) \geq\ & |\beta(0)|^2\lambda_{min}(\bar{V}_n([\eta n]/n)) + |\beta(1)|^2\lambda_{min}(\bar{V}_n(1) - \bar{V}_n(n(1-\eta)]/n)) \\
& - 3\sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|k(|\beta(1)| + |\beta(0)|) \\
& - \sup_{r,j}|\bar{W}_n(r)'\Omega^{1/2}S's_j|k(|\beta(1)| + |\beta(0)|)
\end{aligned}
$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

implying that $\inf_{\beta\in B}\bar{A}_n(\beta) = O_p(1)$ if $\lambda_{min}(\bar{V}_n([\eta n]/n)^{-1} = O_p(1)$, $\lambda_{min}(\bar{V}_n(1) - \bar{V}_n(n(1-\eta))]/n))^{-1} = O_p(1)$, $\sup_{r,j}|\bar{W}_n(r)'\Omega^{1/2}S's_j| = O_p(1)$, and $\sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j| = O_p(1)$. These results follow from Assumption 1. Also, it now follows that under those conditions,

$$\limsup_{K\to\infty}\limsup_{n\to\infty} P(\inf_{\beta\in B}\bar{A}_n(n^{-1/2}\beta) \neq \inf_{\beta\in B\cap\{\beta:\|\beta\|\leq K\}}\bar{A}_n(n^{-1/2}\beta))$$
$$\leq \limsup_{K\to\infty}\limsup_{n\to\infty} P(\inf_{\beta\in B\cap\{\beta:\|\beta\|>K\}}\bar{A}_n(n^{-1/2}\beta) \leq \bar{A}_n(0)) = 0.$$

Finally, note that by partial integration, because $SQ_{xx}S'$ is positive definite, $\beta(0)=\beta(r)$ for $r\in[0,\eta]$ and $\beta(1)=\beta(r)$ for $r\in[1-\eta,1]$ by assumption,

$$\begin{aligned}
\bar{A}^1(\beta) &= \int_0^1 \beta(r)'SQ_{xx}S'\beta(r)dr - 2\int_0^1 \beta(r)'S\Omega^{1/2}d\bar{W}(r) + 2\bar{W}(1)'\Omega^{1/2}S'\bar{\beta}\\
&\geq \int_0^\eta \beta(r)'S'Q_{xx}S\beta(r)dr + \int_{1-\eta}^1 \beta(r)'S'Q_{xx}S\beta(r)dr - 2[\beta(r)'S\Omega^{1/2}\bar{W}(r)]_0^1\\
&\quad + 2\int_0^1 \bar{W}(r)'\Omega^{1/2}S'd\beta(r) + 2\bar{W}(1)'\Omega^{1/2}S'\int_0^1\beta(r)dr\\
&\geq \eta\beta(0)'S'Q_{xx}S\beta(0) + \eta\beta(1)'S'Q_{xx}S\beta(1) - 2|\beta(1)||S\Omega^{1/2}\bar{W}(1)|\\
&\quad - 4k\sup_{j,r\in[0,1]}|\bar{W}(r)'\Omega^{1/2}S's_j|\int_0^1 d\beta_j(r)\\
&\geq \eta(|\beta(0)|^2+|\beta(1)|^2)\lambda_{min}(Q_{xx}) - 2|S\Omega^{1/2}\bar{W}(1)|(|\beta(0)|+|\beta(1)|)\\
&\quad - 4k\sup_{j,r\in[0,1]}|\bar{W}(r)'\Omega^{1/2}S's_j|(|\beta(0)|+|\beta(1)|),
\end{aligned}$$

implying that $\inf_{\beta\in B}\bar{A}^1(\beta)$ is a proper random variable because $\sup_{r\in[0,1]}|W(r)|$ is proper and $\lambda_{min}(Q_{xx})>0$ by assumption. □

LEMMA A.9. *Under Assumption 1, if $\beta_0(\cdot)=0$,*

$$\sup_{\beta\in B_K}|\bar{A}_n(n^{-1/2}\beta) - \bar{A}^1(\beta)| \xrightarrow{a.s.} 0.$$

Proof of Lemma A.9. Defining

$$\begin{aligned}
\delta_n = &\sup_{j,l\in\{1,\dots,k\}}|s_j'S(\bar{V}_n(1)-\bar{V}(1))S's_l|\\
&+\sup_{j,l\in\{1,\dots,k\},r\in[1/n,1]}|s_j'S(\bar{V}_n(1)-\bar{V}_n(r-1/n)-\bar{V}(1)+\bar{V}(r))S's_l|\\
&+\sup_{j,l\in\{1,\dots,k\},r,s\in[0,1]}|s_j'S(\bar{V}_n(1)-\bar{V}_n(\max(r,s)-1/n)-\bar{V}(1)+\bar{V}(\max(r,s)))S's_l|\\
&+\sup_{j\in\{1,\dots,m\},r\in[0,1]}|(\bar{W}_n(r)-\bar{W}(r))'\Omega^{1/2}S's_j|\\
&+\sup_{j\in\{1,\dots,m\},r\in[0,1]}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}(\bar{V}_n(r)-\bar{V}(r))S's_j|
\end{aligned}$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

and noting that $\delta_n \xrightarrow{a.s.} 0$ by the Skorokhod construction and the assumptions of the theorem, and noting that $\sup_{r\in[0,1]}|\beta_j(r)| \leq \| \beta \|$, we have, by Lemmas A.3–Lemma A.7,

$$
\begin{aligned}
&\sup_{\beta\in B_K} |\bar{A}_n(n^{-1/2}\beta) - \bar{A}^1(\beta)| \\
&\quad \leq \sup_{\beta\in B_K} |\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_l(0)\beta_j(0)s_j' S(\bar{V}_n(1) - \bar{V}(1))S' s_l| \\
&\qquad + \sup_{\beta\in B_K} |\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_l(0)\int_0^1 s_j' S(\bar{V}_n(1) - \bar{V}_n(r-1/n) - \bar{V}(1) + \bar{V}(r))S' s_l d\beta_j(r)| \\
&\qquad + \sup_{\beta\in B_K} |\sum_{j=1}^{k}\sum_{l=1}^{k} \beta_j(0)\int_0^1 s_j' S(\bar{V}_n(1) - \bar{V}_n(r-1/n) - \bar{V}(1) + \bar{V}(r))S' s_l d\beta_l(r)| \\
&\qquad + \sup_{\beta\in B_K} |\sum_{j=1}^{k}\sum_{l=1}^{k} \int_0^1\int_0^1 s_j' S(\bar{V}_n(1) - \bar{V}_n(\max(r,s)-1/n) - \bar{V}(1) \\
&\qquad + \bar{V}(\max(r,s)))S' s_l d\beta_j(r) d\beta_l(s)| \\
&\qquad + 2\sup_{\beta\in B_K} |\int_0^1 (\bar{W}_n(r) - \bar{W}(r))'\Omega^{1/2} S' d\beta(r)| \\
&\qquad + 2\sup_{\beta\in B_K} |\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\int_0^1 (\bar{V}_n(r) - \bar{V}(r))S' d\beta(r)| \\
&\quad \leq \delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_l(0)||\beta_j(0)| \\
&\qquad + \delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_l(0)||\int_0^1 d\beta_j(r)| \\
&\qquad + \delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_j(0)||\int_0^1 d\beta_l(r)| \\
&\qquad + \delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k}\sum_{l=1}^{k} |\int_0^1\int_0^1 d\beta_j(r) d\beta_l(s)| \\
&\qquad + 2\delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k} |\int_0^1 d\beta_j(r)| \\
&\qquad + 2\delta_n \sup_{\beta\in B_K} \sum_{j=1}^{k} |\int_0^1 d\beta_j(r)| \\
&\quad \leq \delta_n K(k^2 + 2k^2 + 2k^2 + 4k^2 + 2k + 2k) \xrightarrow{a.s.} 0. \qquad \square
\end{aligned}
$$

We are now able to prove the main theorem.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Proof of Theorem 1. We will verify the conditions of Lemma A.1 for $\bar{A}_n(n^{-1/2}\beta)$ and $\bar{A}^1(\beta)$. Because the distributions of $\bar{A}_n(n^{-1/2}\beta)$ and $\bar{A}^1(\beta)$ are identical to those of $A_n(n^{-1/2}\beta)$ and $A^1(\beta)$, this suffices for proving the theorem. To verify Condition 1 of Lemma A.1, note that $\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta)$ is Borel measurable because $\bar{A}_n(\beta)$ is a function of $(\beta(1/n),\beta(2/n),\ldots,\beta(1))$, and therefore the measurability of the infimum over B_K follows from standard results, such as those of Jennrich (1969). In addition, $\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta)$ is proper because of the result of Lemma A.8. Since

$$\inf_{\beta\in B}\bar{A}_n(\beta)=\lim_{K\to\infty}\inf_{\beta\in B_K}\bar{A}_n(n^{-1/2}\beta),$$

it follows that $\inf_{\beta\in B}\bar{A}_n^1(\beta)$ is also Borel measurable, because it is the a.s. limit of a sequence of Borel measurable random variables and also is proper because of Lemma A.8. By assumption, $\inf_{\beta\in B_K}\bar{A}^1(\beta)$ is the a.s. limit of $\inf_{\beta\in B_K}\bar{A}_n^1(n^{-1/2}\beta)$, and therefore Borel measurable, and it is proper because $\inf_{\beta\in B}\bar{A}^1(\beta)$ is proper by Lemma A.8. Finally, because $\inf_{\beta\in B_K}\bar{A}^1(\beta)$ is the a.s. limit of $\inf_{\beta\in B}\bar{A}^1(\beta)$, Borel measurability follows. Condition 2 of Lemma A.1 follows from Lemma A.9. Noting that Condition 3 of Lemma A.1 follows from Lemma A.8, the proof of convergence in distribution is now complete.

To show that both expressions for the limit distribution of $\inf_{\beta\in B}A^1(\beta)$ are identical, note that because if $\beta\in B$, for any k-vector c also $\beta+c\in B$, implying that

$$\begin{aligned}
&\inf_{\beta\in B}\Big(\int_0^1\beta(r)'S'Q_{xx}S\beta(r)dr-2\int_0^1\beta(r)'S\Omega^{1/2}dW(r)+2W(1)'\Omega^{1/2}S'\int_0^1\beta(r)dr\Big)\\
&=\inf_{\beta\in B}\inf_{c\in\mathbb{R}^k}\Big(\int_0^1(\beta(r)+c)'S'Q_{xx}S(\beta(r)+c)dr-2\int_0^1(\beta(r)+c)'S\Omega^{1/2}dW(r)\\
&\quad+2W(1)'\Omega^{1/2}S'\int_0^1(\beta(r)+c)dr\Big). \qquad \textbf{(A.11)}
\end{aligned}$$

We will now concentrate out c from the above expression. Differentiating with respect to c implies that

$$2\int_0^1 S'Q_{xx}S(\beta(r)+c)dr-2\int_0^1 S\Omega^{1/2}dW(r)+2S\Omega^{1/2}W(1)=0$$

which in turn implies, because $Q_{zx}=SQ_{xx}$, that

$$S'Q_{xx}S\Big(\int_0^1\beta(r)dr+c\Big)=0,$$

and therefore, as long as Q_{xx} is nonsingular,

$$c=-\int_0^1\beta(r)dr=-\bar{\beta}.$$

Plugging this value for c into the expression of Equation (A.11) now shows that both expressions are identical. □

Our result under the alternative requires two more lemmas.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

LEMMA A.10. *Under Assumption 1,*

$$\limsup_{K\to\infty}\limsup_{n\to\infty} P(\inf_{\beta\in B}\bar{A}_n(\beta)\neq \inf_{\beta\in B_K}\bar{A}_n(\beta))=0,$$

$\inf_{\beta\in B} n^{-1}\bar{A}_n(\beta)=O_p(1)$, *and* $\inf_{\beta\in B}\bar{A}^1(\beta)$ *are proper random variables.*

Proof of Lemma A.10. For the first term on $n^{-1}\bar{A}_n(\beta)$, we have by Lemma A.8,

$$\sum_{t=1}^{n}\beta(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$
$$\geq |\beta(0)|^2\lambda_{min}(\bar{V}_n([\eta n]/n))+|\beta(1)|^2\lambda_{min}(\bar{V}_n(1)-\bar{V}_n(n(1-\eta)]/n)).$$

For the second term, we found in Lemma A.8

$$|\sum_{t=1}^{n}(\bar{W}_n(t/n)-\bar{W}_n((t-1)/n))'\Omega^{1/2}S'\beta(t/n)|$$
$$\leq 2\sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|k(|\beta(1)|+|\beta(0)|),$$

while for the fifth term, Lemma A.8 gave

$$|W_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)|$$
$$\leq 2\sup_{r,j}|\bar{W}_n(1)'\Omega^{1/2}\bar{V}_n(1)^{-1}\bar{V}_n(r)S's_j|k(|\beta(0)|+|\beta(1)|).$$

For the third term, we have, because $\beta_j(r)\leq|\beta(r)|\leq|\beta(0)|+|\beta(1)|$ as was noted in the proof of Lemma A.8,

$$|\sum_{t=1}^{n}\beta_0(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)|$$
$$\leq |\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_{0l}(0)\beta_j(0)s_j'S\bar{V}_n(1)S's_l|$$
$$+|\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_{0l}(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_j(r)|$$
$$+|\sum_{j=1}^{k}\sum_{l=1}^{k}\beta_j(0)s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(r-1/n))S's_l d\beta_{l0}(r)|$$
$$+|\sum_{j=1}^{k}\sum_{l=1}^{k}\int_0^1 s_j'S\int_0^1(\bar{V}_n(1)-\bar{V}_n(\max(r,s)-1/n))S's_l d\beta_j(r)d\beta_{l0}(s)|,$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

$$\le \sup_{j,l,r} |s_j' S\bar{V}_n(r)S's_l| \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_{0l}(0)||\beta_j(0)|$$
$$+ \sup_{j,l,r} |s_j' S\bar{V}_n(r)S's_l| \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_{0l}(0)|(|\beta_j(0)|+|\beta_j(1)|)$$
$$+ \sup_{j,l,r} |s_j' S\bar{V}_n(r)S's_l| \sum_{j=1}^{k}\sum_{l=1}^{k} |\beta_j(0)|(|\beta_{0l}(0)|+|\beta_{0l}(1)|)$$
$$+ \sup_{j,l,r} |s_j' S\bar{V}_n(r)S's_l| \sum_{j=1}^{k}\sum_{l=1}^{k} (|\beta_j(0)|+|\beta_j(1)|)(|\beta_{0l}(0)|+|\beta_{0l}(1)|)$$
$$\le \sup_{j,l,r} |s_j' S\bar{V}_n(r)S's_l| k^2(|\beta_0(0)|+|\beta_0(1)|)(|\beta(0)|+|\beta(1)|),$$

and for the fourth term, we have by the reasoning of Lemma A.6,

$$\sum_{t=1}^{n} \beta_0(t/n)'S(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))V_n(1)^{-1}\sum_{t=1}^{n}(\bar{V}_n(t/n)-\bar{V}_n((t-1)/n))S'\beta(t/n)$$
$$=(\bar{V}_n(1)'S'\beta(1)-\int_0^1 \bar{V}_n(r)S'd\beta(r))V_n(1)^{-1}(\bar{V}_n(1)'S'\beta_0(1)-\int_0^1 \bar{V}_n(r)S'd\beta_0(r)).$$

□

LEMMA A.11. *Under Assumption 1,*

$$\sup_{\beta\in B_K} |n^{-1}\bar{A}_n(\beta)-\bar{A}^2(\beta)| \xrightarrow{a.s.} 0.$$

Proof of Lemma A.11. This proof is now analogous to the proof of Lemma A.9 and hence is omitted for space. □

Proof of Theorem 2. The proof of Theorem 2 is completely analogous to the proof of Theorem 1, except that the references to Lemma A.8 need to be replaced by references to Lemma A.10, and the references to Lemma A.9 by Lemma A.11. □

A.2. Additional Simulation

A.2.1. *Additional Simulation with Different η.*

A.2.2. *Additional Simulation with DGP P'1.* We also consider a scenario in which one parameter (the intercept) is monotonically decreasing and one parameter (the slope) is monotonically increasing. The DGP is

DGP P'1

$$y_t = \begin{cases} 0.9+0.5x_t+\varepsilon_t, & \text{if } t\le 0.3n,\\ 0.7+0.7x_t+\varepsilon_t, & \text{if } 0.3n\le t\le 0.7n,\\ 0.5+0.9x_t+\varepsilon_t, & \text{otherwise.}\end{cases}$$

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Table A.1. Empirical size and power with different η.

	$\eta = 0.1$			$\eta = 0.15$			$\eta = 0.2$		
n	100	250	500	100	250	500	100	250	500
DGP S.1	.037	.048	.049	.047	.052	.048	.029	.0344	.042
DGP S.2	.039	.054	.056	.032	.049	.047	.032	.037	.044
DGP P.1	.626	.948	1.00	.646	.958	1.00	.666	.963	1.00
DGP P.2	.446	.742	.965	.458	.770	.972	.481	.786	.978
DGP P.3	.426	.778	.986	.439	.821	.987	.466	.840	.988
DGP P.4	.684	.956	1.00	.681	.949	1.00	.668	.954	1.00
DGP P.5	.482	.825	.990	.487	.837	.990	.492	.831	.986
DGP P.6	.137	.344	.700	.149	.378	.744	.175	.407	.764

Note: 5% significance level.

Table A.2. Empirical power of test under DGP P'1.

n	100	250	500
$\inf_{\beta \in B} A_n(\beta)$	.431	.735	.944
$\hat{H}$	.187	.578	.924
SupLM	.184	.498	.830
UDMax	.207	.549	.874
qLL	.232	.566	.898

Note: (1) 5% significance level; (2) $\inf_{\beta \in B} A_n(\beta)$ is our test based on isotonic regression; SupLM is Andrews' (1993) supremum LM test; UDMax is Bai and Perron's (1998) double maximum test; qLL is Elliott and Müller's (2006) efficient test based on a quasilocal level model.

The empirical power is reported in Table A.2. The proposed $\inf_{\beta \in B} A_n(\beta)$ outperforms other tests for all sample sizes, which shows the new test maintains good power under the case with one parameter decreasing and one increasing.

REFERENCES

Alogoskoufis, S., & Smith, R. (1991). The Phillips curve, the persistence of inflation, and the Lucas critique: Evidence from exchange rate regimes. *American Economic Review*, 81, 1254–1275.

Andrews, D. W. K. (1993). Tests for parameter instability and structural change with unknown change point. *Econometrica*, 61, 821–856.

Bai, J., & Perron, P. (1998). Estimating and testing linear models with multiple structural changes. *Econometrica*, 66, 47–78.

Bai, J., & Perron, P. (2003), Computation and analysis of multiple structural change models. *Journal of Applied Econometrics* 18, 1–22.

https://doi.org/10.1017/S0266466625100066 Published online by Cambridge University Press

Bernanke, B., & Blanchard, O. (2024). What caused the U.S. pandemic-era inflation?. *American Economic Journal: Macroeconomics*, forthcoming.

Chen, B., & Hong, Y. (2012). Testing for smooth structural changes in time series models via nonparametric regression. *Econometrica*, 80, 1157–1183.

Chow, G. C. (1960). Tests of equality between subsets of coefficients in two linear regressions. *Econometrica*, 28, 591–605.

Delworth, T. L., & Knutson, T. R. (2000). Simulation of early 20th century global warming. *Science*, 287, 2246–2250.

Elliott, G., & Müller, U. (2006). Efficient tests for general persistent time variation in regression coefficients. *Review of Economic Studies*, 73, 907–940.

Hansen, B. (2001). The new econometrics of structural change: Dating breaks in U.S. labor productivity. *Journal of Economic Perspectives*, 15, 117–128.

Jennrich, R. I. (1969). Asymptotic properties of non-linear least squares estimators. *The Annals of Mathematical Statistics*, 40, 633–643.

Kristensen, D. (2012). Nonparametric detection and estimation of structural change. *Econometrics Journal*, 15, 420–461.

Li, D., Phillips, P. C. B., & Gao, J. (2020). Kernel-based inference in time-varying coefficient cointegrating regression. *Journal of Econometrics*, 215, 607–632.

Lin, C. J., & Teräsvirta, T. (1994). Testing the constancy of regression parameters against continuous structural change. *Journal of Econometrics*, 62, 211–228.

Melillo, J. M. (1999). Climate change: Warm, warm on the range. *Science*, 283, 183–184.

Nordhaus, W. (2019). Climate change: The ultimate challenge for economics. *American Economic Review*, 109(6), 1991–2014.

Robertson, T., Wright, F. T., & Dykstra, R. L. (1988). *Order restricted statistical inference*. John Wiley.

Su, L., & Wang, X. (2017). On time-varying factor models: Estimation and testing. *Journal of Econometrics*, 198, 84–101.

Whittington, L. A., Alm, J., & Peters, H. E. (1990). Fertility and the personal exemption: Implicit pronatalist policy in the United States. *American Economic Review*, 80, 545–556.

Wooldridge, J. (2008). *Introductory econometrics: A modern approach*. Cengage Learning.

Wu, W. B., Woodroofe, M., & Mentz, G. (2001). Isotonic regression: Another look at the changepoint problem. *Biometrika*, 88, 793–804.

Zhang, T., & Wu, W. B. (2012). Inference of time-varying regression models. *Annals of Statistics*, 40, 1376–1402.