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ON THE TOTAL LENGTH OF EXTERNAL BRANCHES FOR BETA-COALESCENTS

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Abstract

In this paper we consider the beta$(2-\alpha,\alpha)$-coalescents with $1<\alpha<2$ and study the moments of external branches, in particular, the total external branch length $L_{\mathrm{ext}}^{(n)}$ of an initial sample of n individuals. For this class of coalescents, it has been proved that $n^{\alpha-1}T^{(n)} \xrightarrow{\mathrm{D}} T$, where $T^{(n)}$ is the length of an external branch chosen at random and T is a known nonnegative random variable. For beta$(2-\alpha,\alpha)$-coalescents with $1<\alpha<2$, we obtain $\lim_{n\to+\infty} n^{3\alpha-5}\mathbb{E}\{(L_{\mathrm{ext}}^{(n)}-n^{2-\alpha}\mathbb{E}\{T\})^2\} = ((\alpha-1)\Gamma(\alpha+1))^2\Gamma(4-\alpha)/((3-\alpha)\Gamma(4-2\alpha))$.

Keywords: Coalescent process; beta-coalescent; total external branch length; Fu and Li's statistical test

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1. Introduction

1.1. Motivation

In a Wright–Fisher haploid population model with size N, we sample n individuals at present from the total population, and look backward to see the ancestral tree until we find the most recent common ancestor (MRCA). If time is well rescaled and the population size N becomes large, then the genealogy of the sample of size n converges weakly to the Kingman n-coalescent (see [33] and [34]). During the evolution of the population, mutations may occur. We consider the infinite sites model introduced by Kimura [32]. In this model, each mutation is produced at a new site which has never been seen before and will never be seen in the future. The neutrality of mutations means that all mutants are equally privileged by the environment. Under the infinite sites model, to detect or reject the neutrality when the genealogy is given by the Kingman coalescent, Fu and Li [22] proposed a statistical test based on the total mutation numbers on the external branches and internal branches. Mutations on external branches affect only single individuals, so in practice they can be picked out according to the model setting. In this test, the ratio $L_{\mathrm{ext}}^{(n)}/L^{(n)}$ between the total external branch length $L_{\mathrm{ext}}^{(n)}$ and the total length $L^{(n)}$ measures in some sense the weight of mutations occurring on external branches among all mutations. It then makes the study of these quantities relevant.

For many populations, Kingman's coalescent describes the genealogy quite well. But for some others, when descendants of one individual can occupy a large ratio of the next generation with nonnegligible probability, it is no longer relevant, for example, in the case of some marine

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species (see [1], [9], [19], [23], and [26]). In this case, if time is well rescaled and the population size becomes large, the ancestral tree converges weakly to the Λ-coalescent which is associated with a finite measure Λ on $[0, 1]$. This coalescent allows multiple collisions and was introduced by Pitman [38] and Sagitov [39]. Among Λ-coalescents, a special and important subclass is called beta(a, b)-coalescents characterized by Λ being a beta distribution beta(a, b). The most popular distributions are those with parameters $2 - \alpha$ and α where $\alpha \in (0, 2)$.

beta-coalescents arise not only in the context of biology; they also have connections with supercritical Galton–Watson processes (see [40]), continuous-state branching processes (see [2], [6], and [20]), and continuous random trees (see [4]). If $\alpha = 1$, we recover the Bolthausen–Sznitman coalescent which appears in the field of spin glasses (see [8] and [10]) and is also connected to random recursive trees (see [25]). The Kingman coalescent is also obtained from the beta$(2 - \alpha, \alpha)$-coalescent by letting α tend to 2.

For beta$(2 - \alpha, \alpha)$-coalescents with $1 < \alpha < 2$, a central limit theorem (CLT) of the total external branch length $L_{\mathrm{ext}}^{(n)}$ is known (see [31]). The aim of this paper is to study its moments. The results obtained can be extended to more general coalescent processes (see [15]). We should say that in this case, using the moment method we are not able to obtain the right convergence speed in the CLT, which illustrates some limitations of moment calculations.

1.2. Introduction and main results

Let $\mathcal{E}$ be the set of partitions of $\mathbb{N} := \{1, 2, 3, \dots\}$ and, for $n \in \mathbb{N}$, $\mathcal{E}_n$ be the set of partitions of $\mathbb{N}_n := \{1, 2, \dots, n\}$. We denote by $\rho^{(n)}$ the natural restriction on $\mathcal{E}_n$. If $1 \le n \le m \le +\infty$ and $\pi = \{A_i\}_{i \in I}$ is a partition of $\mathbb{N}_m$, then $\rho^{(n)}\pi$ is the partition of $\mathbb{N}_n$ defined by $\rho^{(n)}\pi = \{A_i \cap \mathbb{N}_n\}_{i \in I}$. For a finite measure Λ on $[0, 1]$, we denote by $\Pi = (\Pi_t)_{t \ge 0}$ the Λ-coalescent process, introduced independently in [38] and [39]. The process $(\Pi_t)_{t \ge 0}$ is a càdlàg continuous-time Markovian process taking values in $\mathcal{E}$ with $\Pi_0 = \{\{1\}, \{2\}, \{3\}, \dots\}$. It is characterized by the càdlàg Λ n-coalescent processes $(\Pi_t^{(n)})_{t \ge 0} := (\rho^{(n)}\Pi_t)_{t \ge 0}$, $n \in \mathbb{N}$. For $n \le m \le +\infty$, we have $(\Pi_t^{(n)})_{t \ge 0} = (\rho^{(n)}\Pi_t^{(m)})_{t \ge 0}$ (where $\Pi^{(+\infty)} = \Pi$).

Let $\nu(\mathrm{d}x) = x^{-2}\Lambda(\mathrm{d}x)$. For $2 \le a \le b$, we set

$$\lambda_{b,a} = \int_0^1 x^{a-2}(1-x)^{b-a}\Lambda(\mathrm{d}x) = \int_0^1 x^a(1-x)^{b-a}\nu(\mathrm{d}x).$$

It holds that $\Pi^{(n)}$ is a Markovian process with values in $\mathcal{E}_n$, and its transition rates are given by, for $\xi, \eta \in \mathcal{E}_n$, $q_{\xi,\eta} = \lambda_{b,a}$ if η is obtained by merging a of the $b = |\xi|$ blocks of ξ and letting the $b - a$ others remain unchanged, and $q_{\xi,\eta} = 0$ otherwise. We say that a individuals (or blocks) of ξ have been coalesced in one single individual of η. We remark that the process $\Pi^{(n)}$ is an exchangeable process, which means that, for any permutation τ of $\mathbb{N}_n$, $\tau \circ \Pi^{(n)} \stackrel{\mathrm{D}}{=} \Pi^{(n)}$.

The process $\Pi^{(n)}$ finally reaches one block. This final individual is the MRCA. We denote by $\tau^{(n)}$ the number of collisions it takes for the n individuals to be coalesced to the MRCA.

We define by $R^{(n)} = (R_t^{(n)})_{t \ge 0}$ the block counting process of $(\Pi_t^{(n)})_{t \ge 0}$: $R_t^{(n)} = |\Pi_t^{(n)}|$, which equals the number of blocks/individuals at time t. Then $R^{(n)}$ is a continuous-time Markovian process taking values in $\mathbb{N}_n$, decreasing from n to 1. At state b, for $a = 2, \dots, b$, each of the $\binom{b}{a}$ groups with a individuals coalesces independently at rate $\lambda_{b,a}$. Hence, the time the process $(R_t^{(n)})_{t\geq 0}$ stays at state b is exponential with parameter

$$\begin{aligned} g_b &= \sum_{a=2}^{b} \binom{b}{a} \lambda_{b,a} \\ &= \int_0^1 (1-(1-x)^b - bx(1-x)^{b-1})\nu(\mathrm{d}x) \\ &= b(b-1)\int_0^1 t(1-t)^{b-2}\rho(t)\,\mathrm{d}t, \end{aligned} \tag{1.1}$$

where $\rho(t) = \int_t^1 \nu(\mathrm{d}x)$. We denote by $Y^{(n)} = (Y_k^{(n)})_{k\geq 0}$ the discrete-time Markov chain associated with $R^{(n)}$. This is a decreasing process from $Y_0^{(n)} = n$ which reaches 1 at the $\tau^{(n)}$th jump. The probability transitions of the Markov chain $Y^{(n)}$ are given by, for $b \geq 2$, $k \geq 1$, and $1 \leq l \leq b-1$,

$$p_{b,b-l} := \mathbb{P}\{Y_k^{(n)} = b-l \mid Y_{k-1}^{(n)} = b\} = \frac{\binom{b}{l+1}\lambda_{b,l+1}}{g_b}, \tag{1.2}$$

and 1 is an absorbing state.

We introduce the discrete-time process $X_k^{(n)} := Y_{k-1}^{(n)} - Y_k^{(n)}$, $k \geq 1$ with $X_0^{(n)} = 0$. This process counts the number of blocks we lose at the kth jump. For $i \in \{1, \dots, n\}$, we define

$$T_i^{(n)} := \inf\{t \mid \{i\} \notin \Pi_t^{(n)}\}$$

as the length of the ith external branch and $T^{(n)}$ the length of a randomly chosen external branch. By exchangeability, $T_i^{(n)} \overset{\mathrm{D}}{=} T^{(n)}$. We denote by $L_{\mathrm{ext}}^{(n)} := \sum_{i=1}^n T_i^{(n)}$ the total external branch length of $\Pi^{(n)}$, and by $L^{(n)}$ the total branch length.

For several measures Λ, many asymptotic results on the external branches and their total external lengths of the Λ n-coalescent are already known.

1. If $\Lambda = \delta_0$, Dirac measure on 0, $\Pi^{(n)}$ is Kingman's n-coalescent. Then
   (a) $nT^{(n)}$ converges in distribution to T, which is a random variable with density $f_T(x) = 8/(2+x)^3 \mathbf{1}_{\{x\geq 0\}}$ (see [7], [12], and [27]);
   (b) $L_{\mathrm{ext}}^{(n)}$ converges in L^2 to 2 (see [18] and [22]). A CLT is also proved in [27].
2. If Λ is the uniform probability measure on [0, 1], $\Pi^{(n)}$ is the Bolthausen–Sznitman n-coalescent. Then $(\log n)T^{(n)}$ converges in distribution to an exponential variable with parameter 1 (see [21] and [41]). For moment results of $L_{\mathrm{ext}}^{(n)}$, see [14], and for the CLT, see [30].
3. If $\nu_{-1} = \int_0^1 x^{-1}\Lambda(\mathrm{d}x) < +\infty$, which includes the case of the beta$(2-\alpha, \alpha)$-coalescent with $0 < \alpha < 1$, then
   (a) $T^{(n)}$ converges in distribution to an exponential variable with parameter ν_{-1} (see [24] and [37]);
   (b) $L^{(n)}/n$ converges in distribution to a random variable L whose distribution coincides with that of $\int_0^{+\infty} \mathrm{e}^{-X_t}\,\mathrm{d}t$, where X_t is a certain subordinator (see [17, p. 1405] and [36]), and $L_{\mathrm{ext}}^{(n)}/L^{(n)}$ converges in probability to 1 (see [37]).

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4. If Λ is the beta$(2-\alpha, \alpha)$ measure with $1 < \alpha < 2$, then we obtain the beta$(2-\alpha, \alpha)$-coalescents. Note that $n^{\alpha-1}T^{(n)}$ converges in distribution to T which is a random variable with density function (see [16])

$$f_T(x) = \frac{1}{(\alpha-1)\Gamma(\alpha)}\left(1 + \frac{x}{\alpha\Gamma(\alpha)}\right)^{-\alpha/(\alpha-1)-1}\mathbf{1}_{\{x\geq 0\}}. \qquad (1.3)$$

For CLTs of $L_{\text{ext}}^{(n)}$ and $L^{(n)}$, see [31] and [29], respectively.

In the rest of the paper, we only consider the beta$(2-\alpha, \alpha)$-coalescents, $1 < \alpha < 2$. In that case, we have

$$\nu(\mathrm{d}x) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}x^{-1-\alpha}(1-x)^{\alpha-1}\,\mathrm{d}x,$$

where T denotes a random variable with density (1.3). If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two real sequences, we define $a_n \sim b_n$ when $\lim_{n\to+\infty} a_n/b_n = 1$ is true.

Theorem 1.1. (i) *The total external branch length* $L_{\text{ext}}^{(n)}$ *satisfies*

$$\lim_{n\to+\infty} n^{3\alpha-5}\mathbb{E}\{(L_{\text{ext}}^{(n)} - n^{2-\alpha}\mathbb{E}\{T\})^2\} = \Delta(\alpha),$$

where $\mathbb{E}\{T\} = \alpha(\alpha-1)\Gamma(\alpha)$ *and*

$$\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^2\Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}.$$

(ii) *As a consequence,* $n^{\alpha-2}L_{\text{ext}}^{(n)} \to \mathbb{E}\{T\}$ *in the* L^2 *(second-order moment) distance.*

Remark 1.1. For the second part of the theorem, the almost sure convergence in probability can be found from Berestycki *et al.* [3]–[5].

From the first part of the theorem, we obtain $n^{(5-3\alpha)/2}$ as the convergence speed for $L_{\text{ext}}^{(n)}$ tending to $n^{2-\alpha}\mathbb{E}\{T\}$ in L^2 distance. But, as shown in [31],

$$\frac{L_{\text{ext}}^{(n)} - n^{2-\alpha}\mathbb{E}\{T\}}{n^{1/\alpha+1-\alpha}} \xrightarrow{\text{D}} \frac{\alpha(2-\alpha)(\alpha-1)^{1/\alpha+1}\Gamma(\alpha)}{\Gamma(2-\alpha)^{1/\alpha}}\zeta,$$

where ζ is a stable random variable with parameter α. Our moment method fails to obtain the correct speed of convergence in distribution.

To prove Theorem 1.1, the first idea is to write

$$\mathbb{E}\{(L_{\text{ext}}^{(n)} - n^{2-\alpha}\mathbb{E}\{T\})^2\} = n\operatorname{var}(T_1^{(n)}) + n(n-1)\operatorname{cov}(T_1^{(n)}, T_2^{(n)}) + (n\mathbb{E}\{T_1^{(n)}\} - n^{2-\alpha}\mathbb{E}\{T\})^2. \qquad (1.4)$$

Hence, we have to obtain results on the moments of the external branches. This is the subject of the next theorems. The first theorem provides the asymptotic behaviour for the covariance of two external branch lengths.

Theorem 1.2. *The asymptotic covariance of two external branch lengths is given by*

$$\lim_{n\to+\infty} n^{3(\alpha-1)}\operatorname{cov}(T_1^{(n)}, T_2^{(n)}) = \frac{\int_0^1((1-x)^{2-\alpha}-1)^2\nu(\mathrm{d}x)}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3 = \Delta(\alpha).$$

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Remark 1.2. Here $\Delta(\alpha)$ is the limit only in the case of beta$(2-\alpha, \alpha)$-coalecents, but the result can be extended to more general Λ-coalescent (see [15]).

Note that $\Delta(\alpha)$ is strictly positive implies that $\mathrm{cov}(T_1^{(n)}, T_2^{(n)})$ is of order $n^{3-3\alpha}$ and $T_1^{(n)}$, $T_2^{(n)}$ are positively correlated in the limit which is similar to the Boltausen–Sznitman coalescent and the opposite of Kingman's coalescent (negatively correlated) (see [14]). To prove this theorem, we need the asymptotic behaviours of $\mathbb{E}\{T_1^{(n)}T_2^{(n)}\}$ and $\mathbb{E}\{T_1^{(n)}\}$ (Theorem 2.1). From Theorem 2.1, it follows that the third term in (1.4) satisfies

$$(n\mathbb{E}\{T_1^{(n)}\} - n^{2-\alpha}\mathbb{E}\{T\})^2 = O(n^{6-4\alpha}).$$

The next theorem gives the asymptotic behaviour of moments of one external branch length, hence, we can estimate $n\,\mathrm{var}(T_1^{(n)})$. We then see that $n(n-1)\,\mathrm{cov}\,(T_1^{(n)}, T_2^{(n)})$ is dominant in $\mathbb{E}\{(L_{\mathrm{ext}}^{(n)} - n^{2-\alpha}\mathbb{E}\{T\})^2\}$ (see (1.4)). Then, we obtain Theorem 1.1.

Theorem 1.3. *For* beta$(2-\alpha, \alpha)$*-coalescent, we have*

(i) *if* $0 \le \beta < \alpha/(\alpha-1)$ *then* $\lim_{n\to+\infty} \mathbb{E}\{(n^{\alpha-1}T_1^{(n)})^\beta\} = \mathbb{E}\{T^\beta\}$*;*

(ii) *if* $\beta \ge \alpha/(\alpha-1)$ *then* $\lim_{n\to+\infty} \mathbb{E}\{(n^{\alpha-1}T_1^{(n)})^\beta\} = +\infty$.

1.3. Organization of this paper

In Sections 2 and 3, we provide estimates of $\mathbb{E}\{T_1^{(n)}\}$ and $\mathbb{E}\{T_1^{(n)}T_2^{(n)}\}$, respectively. Both $\mathbb{E}\{T_1^{(n)}\}$ and $\mathbb{E}\{T_1^{(n)}T_2^{(n)}\}$ satisfy the same type of recurrence which allows us to obtain their estimates and leads to an estimate of $\mathrm{cov}(T_1^{(n)}, T_2^{(n)})$ in Section 3. The main tool is Lemma A.1, the proof of which is in Appendix A. In Section 4 we deal with Theorem 1.3. In Appendix A we provide proofs omitted in the main text.

2. First moment of $T_1^{(n)}$ by recursive methods

2.1. Preliminaries

For $s > -\alpha$, we define the measure

$$\nu^{(s)}(\mathrm{d}x) := (1-x)^s\nu(\mathrm{d}x) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}x^{-1-\alpha}(1-x)^{\alpha-1+s}\,\mathrm{d}x.$$

The collision rates of the Λ-coalescent associated with the measure $\nu^{(s)}$ is

$$g_n^{(s)} := \int_0^1 (1-(1-x)^n - nx(1-x)^{n-1})\nu^{(s)}(\mathrm{d}x) \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$$

when $n \to \infty$.

We introduce the quantity $\rho^{(s)}(t) := \int_t^1 \nu^{(s)}(\mathrm{d}x)$.

Lemma 2.1. *For* $s > -\alpha$*, when* $t \to 0$*, we have*

$$\rho^{(s)}(t) = \frac{t^{-\alpha}}{\Gamma(\alpha+1)\Gamma(2-\alpha)} - \frac{(\alpha-1+s)t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + o(t^{1-\alpha}),$$

$$\int_t^1 \rho^{(s)}(x)\,\mathrm{d}x = \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha}).$$

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Then $\lim_{t\to 0+}(\int_t^1 \rho^{(s)}(x)\,\mathrm{d}x - t^{1-\alpha}/(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha))$ *exists, and its value is*

$$C^{(s)} = \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)}.$$

In particular, if $s \geq 1-\alpha$, $C^{(s)} = \Gamma(\alpha+s)/\Gamma(s+1)\Gamma(\alpha)(1-\alpha)$.

Proof. The result for $\rho^{(s)}(t)$ is straightforward since

$$\rho^{(s)}(t) = \int_t^1 \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha}(1-x)^{\alpha-1+s}\,\mathrm{d}x.$$

For $\int_t^1 \rho^{(s)}(x)\,\mathrm{d}x$, using integration by parts, we have

$$\begin{aligned}\int_t^1 \rho^{(s)}(x)\,\mathrm{d}x &= -t\rho^{(s)}(t) + \frac{\int_t^1 x^{-\alpha}(1-x)^{\alpha-1+s}\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} \\ &= -\frac{t^{1-\alpha}}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} + \frac{\int_t^1 (x^{-\alpha}(1-x)^{\alpha-1+s} - x^{-\alpha})\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} \\ &\quad + \frac{\int_t^1 x^{-\alpha}\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha}) \\ &= \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)} + \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} \\ &\quad - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} + O(t^{2-\alpha}),\end{aligned}$$

which also provides the existence and the first definition of $C^{(s)}$.

If $s = 1-\alpha$, $C^{(s)} = 1/(1-\alpha)\Gamma(\alpha)\Gamma(2-\alpha)$. If $s > 1-\alpha$, again using integration by parts, we obtain

$$C^{(s)} = \frac{\int_0^1 x^{-\alpha}((1-x)^{\alpha-1+s}-1)\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)} - \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} = \frac{\Gamma(\alpha+s)}{\Gamma(s+1)\Gamma(\alpha)(1-\alpha)}.$$

We define two values

$$\begin{aligned}A &:= \int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x)\nu^{(1)}(\mathrm{d}x), \\ B &:= \int_0^1 ((1-x)^{2(1-\alpha)} - 1 - 2(\alpha-1)x)\nu^{(2)}(\mathrm{d}x),\end{aligned}$$

which we will use on numerous occasions.

Lemma 2.2. *If* A, B *are defined as above, then*

$$A = \frac{\alpha^2-\alpha-1}{\alpha-1}, \qquad B = \frac{1}{\Gamma(\alpha+1)(\alpha-1)}\left(\frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)} + (\alpha^2-\alpha-1)\Gamma(\alpha+2)\right).$$

Proof. Note that

$$A = \frac{\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x)x^{-1-\alpha}(1-x)^{\alpha}\,\mathrm{d}x}{\Gamma(\alpha)\Gamma(2-\alpha)}.$$

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Using integration by parts: $x^{-1-\alpha}\,\mathrm{d}x = -\mathrm{d}x^{-\alpha}/\alpha$,

$$A = \frac{\int_0^1 (\alpha^2 x(1-x)^{\alpha-1} + (1-x)^{\alpha} - 1)x^{-\alpha}\,\mathrm{d}x}{\Gamma(\alpha+1)\Gamma(2-\alpha)}.$$

Again, using integration by parts: $x^{-\alpha}\,\mathrm{d}x = -\mathrm{d}x^{1-\alpha}/(\alpha-1)$,

$$A = \frac{\int_0^1 (\alpha(\alpha-1)(1-x)^{\alpha-1} - \alpha^2(\alpha-1)x(1-x)^{\alpha-2})x^{1-\alpha}\,\mathrm{d}x}{(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha)} = \frac{\alpha^2-\alpha-1}{\alpha-1}.$$

In the same way, we obtain

$$B = \frac{1}{\Gamma(\alpha+1)(\alpha-1)}\biggl(\frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)} + (\alpha^2-\alpha-1)\Gamma(\alpha+2)\biggr).$$

2.2. The main result

Theorem 2.1. *We have*

$$\begin{aligned}\mathbb{E}\{T_1^{(n)}\} &= (\alpha-1)\Gamma(\alpha+1)n^{1-\alpha} \\ &\quad + \frac{(\alpha-1)^2(\Gamma(\alpha+1))^2}{2-\alpha}(A + (\alpha-1)C^{(1)} - C^{(0)})n^{2(1-\alpha)} + o(n^{2(1-\alpha)}).\end{aligned}$$

The idea is to use the recurrence satisfied by $\mathbb{E}\{T_1^{(n)}\}$ (see [14]):

$$\mathbb{E}\{T_1^{(n)}\} = \frac{1}{g_n} + \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\mathbb{E}\{T_1^{(k)}\}. \tag{2.1}$$

Let $L = (\alpha-1)\Gamma(\alpha+1)$ and

$$Q = \frac{(\alpha-1)^2(\Gamma(\alpha+1))^2}{2-\alpha}(A + (\alpha-1)C^{(1)} - C^{(0)}).$$

We transform the recurrence (2.1) to

$$\begin{aligned}&(\mathbb{E}\{n^{\alpha-1}T_1^{(n)}\} - L)n^{\alpha-1} - Q \\ &= \biggl(\frac{n^{\alpha-1}}{g_n} - \biggl(1 - \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\Bigl(\frac{n}{k}\Bigr)^{\alpha-1}\biggr)L\biggr)n^{\alpha-1} - Q\biggl(1 - \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\Bigl(\frac{n}{k}\Bigr)^{2(\alpha-1)}\biggr) \\ &\quad + \sum_{k=2}^{n-1}\Bigl(\frac{n}{k}\Bigr)^{2(\alpha-1)} p_{n,k}\frac{k-1}{n}(k^{\alpha-1}(\mathbb{E}\{k^{\alpha-1}T_1^{(k)}\} - L) - Q).\end{aligned} \tag{2.2}$$

Hence, we obtain a recurrence

$$a_n = b_n + \sum_{k=2}^{n-1} q_{n,k}a_k \tag{2.3}$$

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with

$$a_n = (\mathbb{E}\{n^{\alpha-1}T_1^{(n)}\} - L)n^{\alpha-1} - Q,$$

$$b_n = \bigg(\frac{n^{\alpha-1}}{g_n} - \bigg(1 - \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\bigg(\frac{n}{k}\bigg)^{\alpha-1}\bigg)L\bigg)n^{\alpha-1} - Q\bigg(1 - \sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\bigg(\frac{n}{k}\bigg)^{2(\alpha-1)}\bigg),$$

$$q_{n,k} = \bigg(\frac{n}{k}\bigg)^{2(\alpha-1)} p_{n,k}\frac{k-1}{n}.$$

With this notation, the theorem can be written as $\lim_{n\to+\infty} a_n = 0$. It is then natural to study the behaviour of b_n when n tends to ∞. To this aim, we obtain asymptotics of $1/g_n$, and

$$\sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_l}{(n)_l}\bigg(\frac{n}{k}\bigg)^r, \qquad r \geq 0,\ l \in \mathbb{N}, \tag{2.4}$$

where $(n)_l$ is defined as (the same for $(k-1)_l$),

$$(n)_l = \begin{cases} n(n-1)(n-2)\cdots(n-l+1) & \text{if } n \geq l \geq 1, \\ 0 & \text{if } l > n \geq 1. \end{cases}$$

2.2.1. *Asymptotics of* $1/g_n$. For any $c, d \in \mathbb{R}$, we have

$$\frac{\Gamma(n+c)}{\Gamma(n+d)} = n^{c-d}\bigg(1 + (c-d)\frac{c+d-1}{2}n^{-1} + O(n^{-2})\bigg). \tag{2.5}$$

This is a straightforward consequence of Stirling's formula:

$$\Gamma(z) = \sqrt{2\pi}z^{z-1/2}\mathrm{e}^{-z}\bigg(1 + \frac{1}{12z} + O\bigg(\frac{1}{z^2}\bigg)\bigg), \qquad z > 0.$$

Then, for any real numbers a and $b > -1$,

$$\begin{aligned}\int_0^1 (1-t)^{n+a}t^b\,\mathrm{d}x &= \frac{\Gamma(n+a+1)\Gamma(b+1)}{\Gamma(n+a+b+2)} \\ &= \Gamma(b+1)n^{-1-b}\bigg(1 + (-1-b)\frac{b+2a+2}{2}n^{-1} + O(n^{-2})\bigg). \end{aligned}\tag{2.6}$$

Using (2.6), we obtain the following lemma.

Lemma 2.3. *For* beta$(2-\alpha,\alpha)$*-coalescents, we have*

$$g_n = \frac{n^\alpha}{\Gamma(\alpha+1)} - \bigg(\frac{\alpha(\alpha-1)}{2\Gamma(\alpha+1)} + \frac{2-\alpha}{\Gamma(\alpha)}\bigg)n^{\alpha-1} + o(n^{\alpha-1})$$

and

$$\frac{1}{g_n} = \Gamma(\alpha+1)\bigg(1 + \bigg(-\frac{\alpha^2}{2} + \frac{3\alpha}{2}\bigg)n^{-1} + o(n^{-1})\bigg)n^{-\alpha}. \tag{2.7}$$

Proof. The proof is straightforward using Lemma 2.1 and $g_n = n(n-1)\int_0^1 t(1-t)^{n-2}\rho(t)\,\mathrm{d}t$.

2.2.2. *Calculus of (2.4).*

Lemma 2.4. *Consider any Λ-coalescent process associated with measure ν. Let $l \in \{1, 2, \dots, n-2\}$ be fixed. Then, for any real function f,*

$$\sum_{k=2}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} f(k) = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \mathbb{E}^{\nu^{(l)}}\{f(n - X_1^{(n-l)})\},$$

where $\mathbb{E}^{\nu^{(l)}}\{\cdot\}$ indicates that the Λ-coalescent is associated with the measure $\nu^{(l)}$.

Proof. Recall the definitions of g_n and $p_{n,k}$ (see (1.1) and (1.2)). We have

$$\begin{aligned}
\sum_{k=2}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} &= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1} \nu(\mathrm{d}x)}{g_n} \\
&= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1-l} \nu^{(l)}(\mathrm{d}x)}{g_n} \\
&= \sum_{k=1}^{n-1-l} \frac{\int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1} (1-x)^{k-1} \nu^{(l)}(\mathrm{d}x)}{g_n} \\
&= \frac{g_{n-l}^{(l)}}{g_n}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{k=2}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} f(k) \\
&\quad = \bigg(\sum_{k=2}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \bigg) \frac{\sum_{k=2}^{n-1} p_{n,k}((k-1)_l/(n)_l) f(k)}{\sum_{k=2}^{n-1} p_{n,k}(k-1)_l/(n)_l} \\
&\quad = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \frac{\sum_{k=l+1}^{n-1} \int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1-l} f(k) \nu^{(l)}(\mathrm{d}x)}{g_{n-l}^{(l)}} \\
&\quad = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \frac{\sum_{k=1}^{n-1-l} \int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1} (1-x)^{k-1} f(k+l) \nu^{(l)}(\mathrm{d}x)}{g_{n-l}^{(l)}} \\
&\quad = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \mathbb{E}^{\nu^{(l)}}\{f(Y_1^{(n-l)} + l)\} \\
&\quad = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \mathbb{E}^{\nu^{(l)}}\{f(n - X_1^{(n-l)})\}.
\end{aligned}$$

This completes the proof of the lemma.

In consequence,

$$\sum_{k=2}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r = \mathbb{E}\bigg\{ \frac{(n-1-X_1^{(n)})_l}{(n)_l} \bigg\} \mathbb{E}^{\nu^{(l)}}\bigg\{ \bigg(\frac{n}{n - X_1^{(n-l)}} \bigg)^r \bigg\}. \tag{2.8}$$

We have

$$\mathbb{E}\bigg\{\frac{(n-1-X_1^{(n)})_l}{(n)_l}\bigg\}, \qquad \mathbb{E}^{\nu^{(l)}}\bigg\{\bigg(\frac{n}{n-X_1^{(n-l)}}\bigg)^r\bigg\}.$$

The latter is obtained from Proposition B.1 in Appendix A. The former is studied in the following lemma.

Lemma 2.5. *Consider a* beta$(2-\alpha,\alpha)$*n-coalescent. Let* $l\in\{1,2,\dots,n-2\}$ *be fixed. We have*

$$\mathbb{E}\bigg\{\frac{(n-1-X_1^{(n)})_l}{(n)_l}\bigg\} = 1-\frac{l\alpha}{n(\alpha-1)}+\Gamma(\alpha+1)\bigg(\sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\,\nu(\mathrm{d}x)-C^{(0)}l\bigg)n^{-\alpha}+o(n^{-\alpha}).$$

Proof. We have

$$\mathbb{E}\bigg\{\frac{(n-1-X_1^{(n)})_l}{(n)_l}\bigg\} = \mathbb{E}\bigg\{1-\sum_{i=0}^{l-1}\frac{X_1^{(n)}+1}{n-i}+\sum_{j=2}^{l}\sum_{\substack{\{i_1,\dots,i_j \text{ all different};\\ 0\le i_1,\dots,i_j\le l-1\}}}(-1)^j\frac{(X_1^{(n)}+1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}\bigg\}.$$

For $\mathbb{E}\{\sum_{i=0}^{l-1}(X_1^{(n)}+1)/(n-i)\}$, we use Lemma B.1 in Appendix B. Using Lemma B.2, we obtain

$$\mathbb{E}\bigg\{\sum_{j=2}^{l}\sum_{\substack{\{i_1,\dots,i_j \text{ all different};\\ 0\le i_1,\dots,i_j\le l-1\}}}(-1)^j\frac{(X_1^{(n)}+1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}\bigg\} = n^{-\alpha}\Gamma(\alpha+1)\sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\,\nu(\mathrm{d}x)+O(n^{-2}).$$

Now we can provide the estimate of (2.4) using (2.8), Lemma 2.5, and Proposition B.1.

Proposition 2.1. *Consider a* beta$(2-\alpha,\alpha)$*n-coalescent. Let* $l\in\{1,2,\dots,n-2\}$ *and* $r\in[0,\alpha+l)$ *be fixed. We have*

$$\sum_{k=2}^{n-1}p_{n,k}\frac{(k-1)_l}{(n)_l}\bigg(\frac{n}{k}\bigg)^r = 1+\frac{(r-l\alpha)}{n(\alpha-1)} + \Gamma(\alpha+1)\bigg(\int_0^1((1-x)^{-r}-1-rx)\nu^{(l)}(\mathrm{d}x) + \sum_{j=2}^{l}\binom{l}{j}(-1)^j\int_0^1 x^j\,\nu(\mathrm{d}x)+rC^{(l)}-lC^{(0)}\bigg)n^{-\alpha} + o(n^{-\alpha}).$$

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2.3. Proof of Theorem 2.1

Recall the transformation (2.2) and the associated recurrence (2.3). The aim is to prove that $\lim_{n\to+\infty} a_n = 0$ for a_n in (2.3). Using Proposition 2.1, we obtain

$$1-\sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\left(\frac{n}{k}\right)^{\alpha-1} = \frac{1}{n(\alpha-1)} - \Gamma(\alpha+1)(A+(\alpha-1)C^{(1)}-C^{(0)})n^{-\alpha}+o(n^{-\alpha}),$$

and

$$1-\sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\left(\frac{n}{k}\right)^{2(\alpha-1)} = \frac{2-\alpha}{n(\alpha-1)} + O(n^{-\alpha}).$$

Hence, we obtain $b_n = o(n^{-1})$.

Let $\varepsilon > 0$ such that $2(\alpha-1)+\epsilon < \alpha$. We have

$$1-\sum_{k=2}^{n-1} p_{n,k}\frac{k-1}{n}\left(\frac{n}{k}\right)^{2(\alpha-1)+\varepsilon} = O(n^{-1}) > 0.$$

The recurrence (2.3) satisfies the assumptions of Lemma A.1, which leads to $\lim_{n\to+\infty} a_n = 0$.

3. Estimate of $\mathbb{E}\{T_1^{(n)}T_2^{(n)}\}$ and proof of Theorem 1.2

Using Theorem 1.1 in [14], we have

$$\mathbb{E}\{T_1^{(n)}T_2^{(n)}\} = \frac{2\mathbb{E}\{T_1^{(n)}\}}{g_n} + \sum_{k=2}^{n-1} p_{n,k}\frac{(k-1)_2}{(n)_2}\mathbb{E}\{T_1^{(k)}T_2^{(k)}\}.$$

As a consequence of (2.7) and Theorem 2.1,

$$\begin{aligned}\frac{2\mathbb{E}\{T_1^{(n)}\}}{g_n} &= 2(\Gamma(\alpha+1))^2 n^{1-2\alpha}\\ &\quad\times\left(\alpha-1+\frac{(\alpha-1)^2\Gamma(\alpha+1)}{2-\alpha}(A+(\alpha-1)C^{(1)}-C^{(0)})n^{1-\alpha}\right)+o(n^{2-3\alpha}).\end{aligned}$$

Thanks to the recurrence method described in the previous section, from a direct calculation it follows that

$$\begin{aligned}\mathbb{E}\{T_1^{(n)}T_2^{(n)}\} &= ((\alpha-1)\Gamma(\alpha+1))^2 n^{2(1-\alpha)}\\ &\quad+\frac{\alpha-1}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3\\ &\quad\times\left(B+2(\alpha-1)C^{(2)}+1-2C^{(0)}+\frac{2}{2-\alpha}(A+(\alpha-1)C^{(1)}-C^{(0)})\right)\\ &\quad\times n^{3(1-\alpha)}+o(n^{3(1-\alpha)}).\end{aligned}$$

Now together with Theorem 2.1, we can obtain the estimate of

$$\begin{aligned}\mathrm{cov}(T_1^{(n)},T_2^{(n)}) &= \frac{((\alpha-1)\Gamma(\alpha+1))^3}{3-\alpha}\\ &\quad\times(B-2A+2(\alpha-1)(C^{(2)}-C^{(1)})+1)n^{3(1-\alpha)}+o(n^{3(1-\alpha)}).\end{aligned}$$

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Then

$$\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^3}{3-\alpha}(B - 2A + 2(\alpha-1)(C^{(2)} - C^{(1)}) + 1).$$

It is straightforward to see that

$$\Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^2\Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}$$

by recalling the values of A, B, $C^{(1)}$, and $C^{(2)}$. Then, we prove

$$\Delta(\alpha) = \frac{\int_0^1 ((1-x)^{2-\alpha} - 1)^2 \nu(\mathrm{d}x)}{3-\alpha}((\alpha-1)\Gamma(\alpha+1))^3.$$

Note that

$$B - 2A = \int_0^1 ((1-x)^{2(2-\alpha)} - 2(1-x)^{2-\alpha} + 1 - x^2 + 2(\alpha-1)x^2(1-x))\nu(\mathrm{d}x).$$

By definition,

$$\begin{aligned} C^{(2)} - C^{(1)} &= \lim_{t\to+\infty}\int_t^1 (\rho^{(2)}(x) - \rho^{(1)}(x))\,\mathrm{d}x \\ &= \lim_{t\to 0}\int_t^1 x(\nu^{(2)}(\mathrm{d}x) - \nu^{(1)}(\mathrm{d}x)) \\ &= \int_0^1 -x^2(1-x)\nu(\mathrm{d}x), \end{aligned}$$

and $\int_0^1 x^2\nu(\mathrm{d}x) = 1$. This concludes the proof.

4. Proof of Theorem 1.3

Note that $n^{\alpha-1}T_1^{(n)} \xrightarrow{\mathrm{D}} T$ and if $\beta \ge \alpha/(\alpha-1)$, we obtain $\mathbb{E}\{T^\beta\} = +\infty$, hence; $\mathbb{E}\{(n^{\alpha-1}T_1^{(n)})^\beta\}$ converges to $+\infty$ (see [28, Lemma 4.11]). If $0 \le \beta_1 < \beta_2 < \alpha/(\alpha-1)$ and $(\mathbb{E}\{(n^{\alpha-1}T_1^{(n)})^{\beta_2}\}, n \ge 2)$ is bounded, then $((n^{\alpha-1}T_1^{(n)})^{\beta_1}, n \ge 2)$ is uniformly integrable (see [28, Lemma 4.11] and [11, Section 8.3, Problem 14]). Then, we need only to prove that for $\beta \in [2, \alpha/(\alpha-1))$, $(\mathbb{E}\{(n^{\alpha-1}T_1^{(n)})^\beta\}, n \ge 2)$ is bounded.

We will prove by induction on n that there exists a constant $C > 0$ such that, for all $n \ge 2$, $(\mathbb{E}\{n^{\alpha-1}T_1^{(n)}\})^\beta \le C$. We first assume that, for all $2 \le k \le n-1$, $(\mathbb{E}\{k^{\alpha-1}T_1^{(k)}\})^\beta \le C$ and then we will prove that (if C is large enough) $(\mathbb{E}\{n^{\alpha-1}T_1^{(n)}\})^\beta \le C$.

Writing the decomposition of $T_1^{(n)}$ at the first coalescence, we have

$$T_1^{(n)} = \frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}}\bar{T}_1^{(k)},$$

where

- $H_{n,k}$ is the event: {From n individuals, we have k individuals after the first coalescence, and individual 1 is not involved in this collision}, $2 \le k \le n-1$;
- e_0 is a unit exponential random variable, $\bar{T}_1^{(k)} \stackrel{\mathrm{D}}{=} T_1^{(k)}$, and all these random variables e_0, $\bar{T}_1^{(k)}$, $\mathbf{1}_{\{H_{n,k}\}}$ are independent. Note that $\mathbb{P}\{H_{n,k}\} = p_{n,k}(k-1)/n$ (see (2.1)).

Using Lemma D.1 in Appendix D, we have the following inequality:

$$\mathbb{E}\{(T_1^{(n)})^\beta\} = \mathbb{E}\bigg\{\bigg(\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\bigg)^\beta\bigg\} \le I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}, \tag{4.1}$$

where

$$I_{n,1} = \mathbb{E}\bigg\{\bigg(\frac{e_0}{g_n}\bigg)^\beta\bigg\}, \qquad I_{n,2} = \mathbb{E}\bigg\{\bigg(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\bigg)^\beta\bigg\},$$

$$I_{n,3} = \mathbb{E}\bigg\{\beta 2^{\beta-1} \frac{e_0}{g_n} \bigg(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\bigg)^{\beta-1}\bigg\}, \qquad I_{n,4} = \mathbb{E}\bigg\{\beta 2^{\beta-1} \bigg(\frac{e_0}{g_n}\bigg)^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\bigg\}.$$

We first bound $I_{n,1}$. Recall that $g_n \sim n^\alpha/\Gamma(\alpha+1)$. Hence, there exists a constant $K_1 > 0$ (which depends on β) such that, for any $n \ge 2$,

$$n^{(\alpha-1)\beta} I_{n,1} \le \frac{K_1}{n}. \tag{4.2}$$

We now consider $I_{n,2}$. Note that $(\alpha-1)\beta < \alpha+1$. Hence, using Proposition 2.1, we have

$$\begin{aligned} n^{(\alpha-1)\beta} I_{n,2} &= n^{-(\alpha-1)\beta} \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \bigg(\frac{n}{k}\bigg)^{(\alpha-1)\beta} \mathbb{E}\{(k^{\alpha-1} T_1^{(k)})^\beta\} \\ &\le C \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \bigg(\frac{n}{k}\bigg)^{(\alpha-1)\beta} \\ &= C\bigg(1 - \frac{\alpha - (\alpha-1)\beta}{n(\alpha-1)} + o(n^{-1})\bigg) \\ &\le C\bigg(1 - \frac{\alpha - (\alpha-1)\beta}{2n(\alpha-1)}\bigg) \quad \text{for } n \ge N, \end{aligned} \tag{4.3}$$

where N is a fixed positive integer.

We now proceed to $I_{n,3}$. Note that, for $2 \le k \le n-1$,

$$\mathbb{E}\{(k^{\alpha-1} T_1^{(k)})^{\beta-1}\} \le (\mathbb{E}\{(k^{\alpha-1} T_1^{(k)})^\beta\})^{(\beta-1)/\beta} \le C^{(\beta-1)/\beta}.$$

Hence, we have

$$\begin{aligned} n^{(\alpha-1)\beta} I_{n,3} &= n^{(\alpha-1)\beta} \mathbb{E}\bigg\{\beta 2^{\beta-1} \frac{e_0}{g_n} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} (\bar{T}_1^{(k)})^{\beta-1}\bigg\} \\ &\le C^{(\beta-1)/\beta} \beta 2^{\beta-1} n^{\alpha-1} g_n^{-1} \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \bigg(\frac{n}{k}\bigg)^{(\alpha-1)(\beta-1)} \\ &= C^{(\beta-1)/\beta} n^{\alpha-1} \beta 2^{\beta-1} g_n^{-1} \bigg(1 - \frac{\alpha - (\alpha-1)(\beta-1)}{n(\alpha-1)} + o(n^{-1})\bigg) \\ &\le \frac{C^{(\beta-1)/\beta} K_2}{n}, \end{aligned} \tag{4.4}$$

where K_2 is a positive constant. In the second equality, we have used Proposition 2.1.

While, for any $n \geq 2$,

$$\begin{aligned} n^{(\alpha-1)\beta} I_{n,4} &= n^{(\alpha-1)\beta} \mathbb{E}\bigg\{\beta 2^{\beta-1}\bigg(\frac{e_0}{g_n}\bigg)^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\bigg\} \\ &\leq \beta 2^{\beta-1} \mathbb{E}\{e_0^{\beta-1}\}(g_n)^{1-\beta} n^{(\alpha-1)(\beta-1)} \mathbb{E}\{n^{\alpha-1} T_1^{(n)}\} \\ &\leq \frac{K_3}{n^{\beta-1}} \\ &\leq \frac{K_3}{n}, \end{aligned} \tag{4.5}$$

where K_3 is a positive constant. We have used Lemma 2.1 to bound $\mathbb{E}\{n^{\alpha-1} T_1^{(n)}\}$.

Using (4.1)–(4.5), we have proved that for any $n, n \geq N$, if there exists $C > 0$ such that, for all $2 \leq k \leq n-1$, $\mathbb{E}\{(k^{\alpha-1} T_1^{(k)})^\beta\} \leq C$, then

$$\mathbb{E}\{(n^{\alpha-1} T_1^{(n)})^\beta\} \leq \bigg[C + \bigg(K_1 - C\frac{\alpha-(\alpha-1)\beta}{2(\alpha-1)} + C^{(\beta-1)/\beta} K_2 + K_3\bigg)\bigg] n^{-1}.$$

Let C be large enough such that

$$K_1 - C\frac{\alpha-(\alpha-1)\beta}{2(\alpha-1)} + C^{(\beta-1)/\beta} K_2 + K_3 < 0.$$

Then $\mathbb{E}\{(n^{\alpha-1} T_1^{(n)})^\beta\} \leq C$, which concludes the proof.

Appendix A. The main recurrence tool

Lemma A.1. *We consider the recurrence* $a_n = b_n + \sum_{k=1}^{n-1} q_{n,k} a_k, n \geq 1$. *We assume that all parameters* $a_1, b_n, q_{n,k}$ *with* $n \geq 1$, $1 \leq k \leq n-1$ *are nonnegative,* $b_n = o(n^{-1})$, *and that there exist* $\varepsilon > 0$ *and* $C > 0$ *such that* $1 - \sum_{k=1}^{n-1} q_{n,k}(n/k)^\varepsilon \geq Cn^{-1}$ *for large enough* n. *Then* $\lim_{n\to+\infty} a_n = 0$.

Proof. Let $(\bar{c}_n)_{n\geq 1}$ be a positive increasing sequence such that

$$\lim_{n\to+\infty} \bar{c}_n = +\infty, \qquad \lim_{n\to+\infty} n b_n \bar{c}_n = 0.$$

Define another sequence $(c_n)_{n\geq 1}$ by $c_1 = \bar{c}_1$. For $n \geq 1$,

$$c_{n+1} = \min\bigg\{c_n\bigg(\frac{n+1}{n}\bigg)^\varepsilon, \bar{c}_{n+1}\bigg\}.$$

Then, we have $\lim_{n\to+\infty} c_n = +\infty$, $c_n b_n = o(n^{-1})$ and, for any $1 \leq k \leq n-1$, $c_n/c_k \leq (n/k)^\varepsilon$. In consequence,

$$1 - \sum_{k=1}^{n-1} q_{n,k} \frac{c_n}{c_k} \geq \frac{C}{n}$$

for large enough n. Let $n_1 > 0$ such that, for $n > n_1$, we have

$$1 - \sum_{k=1}^{n-1} q_{n,k} \frac{c_n}{c_k} > \frac{C}{n}, \qquad c_n b_n < \frac{C}{2n}$$

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and pick a number C' such that $C' > \max\{1, c_k a_k;\ 1 \le k \le n_1\}$. We transform the original recurrence to

$$c_n a_n = c_n b_n + \sum_{k=1}^{n-1} \left(q_{n,k} \frac{c_n}{c_k} \right) c_k a_k.$$

Then

$$c_{n_1+1} a_{n_1+1} \le \frac{C}{2(n_1+1)} + \left(1 - \frac{C}{n_1+1}\right) C' \le C'.$$

By induction, we prove that the sequence $(c_n a_n)_{n\ge 1}$ is bounded by C'. Since c_n tends to ∞, we obtain $\lim_{n\to+\infty} a_n = 0$.

Remark A.1. We refer the reader to [35] for a rather detailed survey on this kind of recurrence relationship.

Appendix B. Asymptotic behaviours of $X_1^{(n)}$

Lemma B.1. *Consider the coalescent process with related measure $\nu^{(s)}$, where $s > -\alpha$. Then*

$$\mathbb{E}^{\nu^{(s)}}\{X_1^{(n)}\} = \frac{1}{\alpha - 1} + \Gamma(\alpha+1) C^{(s)} n^{1-\alpha} + o(n^{1-\alpha}),$$

Proof. We have (see [13])

$$\mathbb{E}^{\nu^{(s)}}\{X_1^{(n)}\} = \frac{\int_0^1 (1-t)^{n-2} (\int_t^1 \rho^{(s)}(r)\,\mathrm{d}r)\,\mathrm{d}t}{\int_0^1 (1-t)^{n-2} t \rho^{(s)}(t)\,\mathrm{d}t}.$$

From Lemma 2.1, we obtain the expansions of $\rho^{(s)}(t)$ and $\int_t^1 \rho^{(s)}(r)\,\mathrm{d}r$. Using (2.5), we obtain

$$\int_0^1 (1-t)^{n-2} \left(\int_t^1 \rho^{(s)}(r)\,\mathrm{d}r \right) \mathrm{d}t = \frac{n^{\alpha-2}}{(\alpha-1)\Gamma(\alpha+1)} + C^{(s)} n^{-1} + o(n^{-1})$$

and $\int_0^1 (1-t)^{n-2} t \rho^{(s)}(t)\,\mathrm{d}t = n^{\alpha-2}/\Gamma(\alpha+1) + O(n^{\alpha-3})$.

Lemma B.2. *If $s > -\alpha$ and $k \ge 2$,*

$$\mathbb{E}^{\nu^{(s)}}\left\{ \left(\frac{X_1^{(n)}}{n} \right)^k \right\} = \Gamma(\alpha+1) \int_0^1 x^k \nu^{(s)}(\mathrm{d}x) n^{-\alpha} + O(n^{-\min\{1+\alpha, k\}}).$$

Proof. Let $B_{n,x}$ denote a binomial random variable with parameter (n, x), $n \ge 2$, $0 \le x \le 1$. Recall that, for $2 \le i \le n$,

$$\mathbb{P}^{\nu^{(s)}}\{X_1^{(n)} = i - 1\} = \int_0^1 \frac{\binom{n}{i} x^i (1-x)^{n-i} \nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}} = \int_0^1 \frac{\mathbb{P}\{B_{n,x} = i\} \nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}}.$$

Here, $\mathbb{P}^{\nu^{(s)}}$ means that $X_1^{(n)}$ is related to the coalescent process with measure $\nu^{(s)}$. Then

$$\begin{aligned}
\mathbb{E}^{\nu^{(s)}}\left\{ \left(\frac{X_1^{(n)}}{n} \right)^k \right\} &= \frac{\int_0^1 \mathbb{E}\{((B_{n,x}-1)/n)^k \mathbf{1}_{\{B_{n,x}\ge 1\}}\}) \nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}} \\
&= \int_0^1 n^{-k} \mathbb{E}\Bigg\{ (B_{n,x}^k - B_{n,x}) + \sum_{i=1}^{k-1} \binom{k}{i} (-1)^i (B_{n,x}^{k-i} - B_{n,x}) \\
&\qquad + (-1)^k (1 - B_{n,x}) \mathbf{1}_{\{B_{n,x}\ge 1\}}) \Bigg\} \nu^{(s)}(\mathrm{d}x) (g_n^{(s)})^{-1}.
\end{aligned}$$

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Using Lemma C.1 in Appendix C, we obtain $\mathbb{E}\{(B_{n,x}^k - B_{n,x})\} = (nx)^k + O(n^{k-1})x^2$. Then

$$\begin{aligned}\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{X_1^{(n)}}{n}\right)^k\right\} &= \frac{\int_0^1 n^{-k}((nx)^k + O(n^{k-1})x^2)\nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}} \\ &\quad + \frac{n^{-k}\int_0^1(-1)^k(1-nx-(1-x)^n)\nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}} \\ &= \Gamma(\alpha+1)\int_0^1 x^k\nu^{(s)}(\mathrm{d}x)n^{-\alpha} + O(n^{-\min\{1+\alpha,k\}}).\end{aligned}$$

In the second equality, we have used $g_n^{(s)} \sim n^\alpha/\Gamma(\alpha+1)$ and also the fact that

$$\int_0^1(1-nx-(1-x)^n)\nu^{(s)}(\mathrm{d}x) \le g_n^{(s)} = \int_0^1(1-nx(1-x)^{n-1}-(1-x)^n)\nu^{(s)}(\mathrm{d}x).$$

This completes the proof.

Proposition B.1. *For $s \in \mathbb{N}\cup\{0\}$ and $0 \le r < \alpha + s$, we have*

$$\begin{aligned}\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n-s)}}\right)^r\right\} &= 1 + \frac{r}{n(\alpha-1)} \\ &\quad + \Gamma(\alpha+1)\left(\int_0^1((1-x)^{-r}-1-rx)\nu^{(s)}(\mathrm{d}x) + rC^{(s)}\right)n^{-\alpha} \\ &\quad + o(n^{-\alpha}).\end{aligned}$$

Proof. Using a Taylor expansion, for $m \ge 2$ and $n \ge s+2$, we have

$$\begin{aligned}&\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n-s)}}\right)^r\right\} \\ &\quad = \mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{1}{1-X_1^{(n-s)}/n}\right)^r\right\} \\ &\quad = \mathbb{E}^{\nu^{(s)}}\left\{1 + r\frac{X_1^{(n-s)}}{n} + \sum_{k=2}^m\frac{\Gamma(k+r)}{\Gamma(r)k!}\left(\frac{X_1^{(n-s)}}{n}\right)^k\right. \\ &\qquad\left. + \frac{\Gamma(m+1+r)}{\Gamma(r)m!}\int_0^{X_1^{(n-s)}/n}(1-t)^{-r-m-1}\left(\frac{X_1^{(n-s)}}{n}-t\right)^m \mathrm{d}t\right\}.\end{aligned}$$

Using Lemma B.1 and Lemma B.2, we have, for $m \ge 2$,

$$\lim_{n\to+\infty} n^\alpha\mathbb{E}^{\nu^{(s)}}\left\{\sum_{k=2}^m\frac{\Gamma(k+r)}{\Gamma(r)k!}\left(\frac{X_1^{(n-s)}}{n}\right)^k\right\} = \Gamma(\alpha+1)\sum_{k=2}^m\frac{\Gamma(k+r)}{\Gamma(r)k!}\int_0^1 x^k\nu^{(s)}(\mathrm{d}x).$$

In consequence,

$$\lim_{m\to+\infty}\lim_{n\to+\infty} n^\alpha\mathbb{E}^{\nu^{(s)}}\left\{\sum_{k=2}^m\frac{\Gamma(k+r)}{\Gamma(r)k!}\left(\frac{X_1^{(n-s)}}{n}\right)^k\right\} = \Gamma(\alpha+1)\int_0^1((1-x)^{-r}-1-rx)\nu^{(s)}(\mathrm{d}x).$$

It remains to estimate

$$\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}\left\{\int_0^{X_1^{(n-s)}/n}(1-t)^{-r-m-1}\left(\frac{X_1^{(n-s)}}{n}-t\right)^m \mathrm{d}t\right\},$$

which is the sum of two terms $P_1(m,n,s,y)$ and $P_2(m,n,s,y)$, with $0<y<1$, defined by

$$\begin{aligned}
&P_1(m,n,s,y)\\
&\quad=\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}\left\{\int_0^{X_1^{(n-s)}/n}(1-t)^{-r-m-1}\left(\frac{X_1^{(n-s)}}{n}-t\right)^m \mathrm{d}t\,\mathbf{1}_{\{X_1^{(n-s)}\geq ny\}}\right\},\\
&P_2(m,n,s,y)\\
&\quad=\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}\left\{\int_0^{X_1^{(n-s)}/n}(1-t)^{-r-m-1}\left(\frac{X_1^{(n-s)}}{n}-t\right)^m \mathrm{d}t\,\mathbf{1}_{\{X_1^{(n-s)}<ny\}}\right\}.
\end{aligned}$$

We first focus on $P_1(m,n,s,y)$. By Proposition C.1 in Appendix C, we have

$$\begin{aligned}
P_1(m,n,s,y)&\leq\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n-s)}}\right)^r\mathbf{1}_{\{X_1^{(n-s)}\geq ny\}}\right\}\\
&\leq\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n-s}{n-s-X_1^{(n-s)}}\right)^r\mathbf{1}_{\{X_1^{(n-s)}\geq(n-s)y\}}\right\}\\
&\leq n^{-\alpha}K_4y^{-\alpha}(1-y)^{\bar{r}-r},
\end{aligned}\tag{B.1}$$

where $\bar{r}\in(r,\alpha+s)$ and K_4 is a number depending only on $\bar{r}$ and $\nu^{(s)}$ (it is important that it does not depend on y).

We now provide an upper bound for $P_2(m,n,s,y)$,

$$\begin{aligned}
&n^\alpha P_2(m,n,s,y)\\
&\quad=n^\alpha\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\mathbb{E}^{\nu^{(s)}}\left\{\int_0^{X_1^{(n-s)}/n}(1-t)^{-r-1}\left(\frac{X_1^{(n-s)}/n-t}{1-t}\right)^m \mathrm{d}t\,\mathbf{1}_{\{X_1^{(n-s)}<ny\}}\right\}.
\end{aligned}$$

For $t\in[0,x)$ with $0<x\leq1$, we have $(x-t)/(1-t)\leq x$. Then

$$\int_0^{X_1^{(n-s)}/n}\left(\frac{X_1^{(n-s)}/n-t}{1-t}\right)^m\mathrm{d}t\leq\left(\frac{X_1^{(n-s)}}{n}\right)^{m+1}.$$

Hence, using Lemma B.2, for $m>2$,

$$\begin{aligned}
n^\alpha P_2(m,n,s,y)&\leq n^\alpha\frac{\Gamma(m+1+r)}{\Gamma(r)m!}(1-y)^{-r-1}\mathbb{E}\left\{\left(\frac{X_1^{(n-s)}}{n}\right)^{m+1}\right\}\\
&=(1-y)^{-r-1}\frac{\Gamma(m+1+r)}{\Gamma(r)m!}\left(\Gamma(\alpha+1)\int_0^1x^{m+1}\nu^{(s)}(\mathrm{d}x)+O(n^{-1})\right).
\end{aligned}$$

By Lemma C.2 in Appendix C, we have

$$\int_0^1x^{m+1}\nu^{(s)}(\mathrm{d}x)=\int_0^1x^{m+1}(1-x)^{\bar{r}}\nu^{(-\bar{r}+s)}(\mathrm{d}x)\leq K_5m^{-\bar{r}},$$

where K_5 is a positive real number depending only on $\bar{r}$ and $\nu^{(s)}$.

Note that $\Gamma(m+r+1)/\Gamma(r)m! \sim m^r/\Gamma(r)$. Hence,

$$P_2(m,n,s,y) \leq n^{-\alpha}(1-s)^{-r-1}m^r(O(m^{-\bar{r}})+o(n^{-1})). \tag{B.2}$$

Combining (B.1) and (B.2), we obtain

$$\lim_{m\to+\infty}\limsup_{n\to+\infty} n^{\alpha}(P_1(m,n,s,y)+P_2(m,n,s,y))=0.$$

This convergence together with Lemmas B.1 and B.2 yield this proposition.

Appendix C. Some necessary results for Appendix B

Lemma C.1. *Let $B_{n,x}$ be a binomial random variable with parameter (n,x), $n\geq 2$, $0\leq x\leq 1$. Let k be an integer such that $2\leq k\leq n$. Then*

$$nx + n(n-1)\cdots(n-k+1)x^k \leq \mathbb{E}\{B_{n,x}^k\} \leq (nx)^k + \binom{k}{2}n^{k-1}x^2.$$

Proof. Write $B_{n,x} = Y_1+\cdots+Y_n$, where $Y_1,\ldots,Y_n$ are independent Bernoulli random variables. Let $S := \{\{i_1,\ldots,i_k\}; 1\leq i_1,\ldots,i_k\leq n\}$. Then

$$\begin{aligned}\mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_1} Y_{i_1}\cdots Y_{i_k}\bigg\} &+ \mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_3} Y_{i_1}\cdots Y_{i_k}\bigg\}\\ &\leq \mathbb{E}\{(B_{n,x})^k\}\\ &\leq \mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_2} Y_{i_1}\cdots Y_{i_k}\bigg\} + \mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_3} Y_{i_1}\cdots Y_{i_k}\bigg\},\end{aligned}$$

where

(i) if $S_1 := \{\{i_1,\ldots,i_n\}\in A; i_1=\cdots=i_k\}$ then

$$\mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_1} Y_{i_1}\cdots Y_{i_k}\bigg\} = nx;$$

(ii) if $S_2 := \{\{i_1,\ldots,i_n\}\in A;$ there exists $1\leq p<q\leq k, i_p=i_q\}$ then

$$\mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_2} Y_{i_1}\cdots Y_{i_k}\bigg\} \leq \binom{k}{2}n^{k-1}x^2;$$

(iii) if $S_3 := \{\{i_1,\ldots,i_n\}\in A;$ for all $1\leq p<q\leq k, i_p\neq i_q\}$ then

$$\mathbb{E}\bigg\{\sum_{\{i_1,\ldots,i_k\}\in S_3} Y_{i_1}\cdots Y_{i_k}\bigg\} = n(n-1)\cdots(n-k+1)x^k.$$

The lemma is then immediate from the above calculations.

Lemma C.2. *Consider any Λ-coalescent such that $\rho(t) = Ct^{-\alpha}+o(t^{-\alpha})$. Then, for every $s\geq 0$, $n\geq 2$, $\int_0^1 x^n(1-x)^s\nu(\mathrm{d}x)\leq K_6 n^{-s}$, where K_6 is a positive constant which depends only on s and ν.*

Proof. It is clear that there exists $K_7 > 0$ such that $\rho(t) \le K_7 t^{-\alpha}$ for all $0 < t \le 1$. Then

$$\begin{aligned}\int_0^1 x^n(1-x)^s \nu(\mathrm{d}x) &= \int_0^1 \rho(t)(n-(n+s)t)t^{n-1}(1-t)^{s-1}\,\mathrm{d}t \\ &\le \int_0^1 \rho(t)(n-nt)t^{n-1}(1-t)^{s-1}\,\mathrm{d}t \\ &\le nK_7\int_0^1 t^{n-1-\alpha}(1-t)^s\,\mathrm{d}t \\ &= nK_7\frac{\Gamma(n-\alpha)\Gamma(s+1)}{\Gamma(n-\alpha+s+1)} \\ &\le K_6 n^{-s}\end{aligned}$$

for some K_6 which only depends on K_7 and s. This completes the proof of the lemma.

Proposition C.1. *Let $s > -\alpha$ and $0 \le r < \alpha + s$, $\bar{r} \in (r, \alpha + s)$. Then there exists a constant K_{11} depending only on $\bar{r}$ and s such that, for all $y \in (0, 1)$, $n \ge 2$,*

$$\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n)}}\right)^r \mathbf{1}_{\{X_1^{(n)}\ge ny\}}\right\} \le n^{-\alpha}K_{11}y^{-\alpha}(1-y)^{\bar{r}-r}.$$

Proof. Define $\lceil x\rceil = \min\{m \in \mathbb{Z}; m \ge x\}$. We have

$$\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n)}}\right)^r \mathbf{1}_{\{X_1^{(n)}\ge ny\}}\right\} = \sum_{k=\lceil ny\rceil}^{n-1}\frac{\int_0^1\binom{n}{k+1}x^{k+1}(1-x)^{n-k-1}(n/(n-k))^r\nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}}.$$

Using (2.5), there exist two positive constants K_8, K_9 such that, for all $k \in \{1, 2, \dots, n-1\}$,

$$K_8\frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)} \le \binom{n}{k+1}\left(\frac{n}{n-k}\right)^r \le K_9\frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)}.$$

Moreover, using integration by parts, for $1 \le l \le n-1$ and $0 \le x \le 1$, we have

$$\begin{aligned}&\sum_{k=l}^{n-1}\frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)}x^{k+1}(1-x)^{n-k-1+r} \\ &\quad= \frac{\Gamma(n+1+r)}{\Gamma(l+1)\Gamma(n-l+r)}\int_0^x t^l(1-t)^{n-l+r-1}\,\mathrm{d}t + \frac{\Gamma(n+1+r)}{\Gamma(n+1)\Gamma(1+r)}x^n(1-x)^r \\ &\qquad - \frac{\Gamma(n+1+r)}{\Gamma(n)\Gamma(1+r)}\int_0^x t^{n-1}(1-t)^r\,\mathrm{d}t.\end{aligned} \tag{C.1}$$

From Lemma 2.1, we have $\rho^{(-\bar{r}+s)}(t) = t^{-\alpha}/\Gamma(2-\alpha)\Gamma(\alpha+1) + o(t^{-\alpha})$. Then there exists $K_{10} > 0$, such that $\rho^{(-\bar{r}+s)}(t) \le K_{10}t^{-\alpha}$ for all $t \in (0, 1]$. We have

$$\begin{aligned}&\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n)}}\right)^{\bar{r}} \mathbf{1}_{\{X_1^{(n)}\ge ny\}}\right\} \\ &\quad= \sum_{k=\lceil ny\rceil}^{n-1}\frac{\int_0^1\binom{n}{k+1}(n/(n-k))^{\bar{r}}x^{k+1}(1-x)^{n-k-1}\nu^{(s)}(\mathrm{d}x)}{g_n^{(s)}} \\ &\quad= \sum_{k=\lceil ny\rceil}^{n-1}\frac{\int_0^1\binom{n}{k+1}(n/(n-k))^{\bar{r}}x^{k+1}(1-x)^{n-k-1+\bar{r}}\nu^{(-\bar{r}+s)}(\mathrm{d}x)}{g_n^{(s)}}\end{aligned}$$

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$$
\begin{aligned}
&\le K_9 \frac{\int_0^1 (\Gamma(n+1+\bar r)/\Gamma(\lceil ny\rceil+1)\Gamma(n-\lceil ny\rceil+\bar r)) \int_0^x t^{\lceil ny\rceil}(1-t)^{n-\lceil ny\rceil+\bar r-1}\,\mathrm{d}t\,\nu^{(-\bar r+s)}(\mathrm{d}x)}{g_n^{(s)}}\\
&\quad + K_9 \frac{\int_0^1 (\Gamma(n+1+\bar r)/\Gamma(n+1)\Gamma(1+\bar r)) x^n (1-x)^{\bar r}\nu^{(-\bar r+s)}(\mathrm{d}x)}{g_n^{(s)}}\\
&\le K_9 \frac{\int_0^1 (\Gamma(n+1+\bar r)/\Gamma(\lceil ny\rceil+1)\Gamma(n-\lceil ny\rceil+\bar r))\rho^{(-\bar r+s)}(t) t^{\lceil ny\rceil}(1-t)^{n-\lceil ny\rceil+\bar r-1}\,\mathrm{d}t}{g_n^{(s)}}\\
&\quad + K_9 \frac{\int_0^1 (\Gamma(n+1+\bar r)/\Gamma(n+1)\Gamma(1+\bar r)) x^n (1-x)^{\bar r}\nu^{(-\bar r+s)}(\mathrm{d}x)}{g_n^{(s)}}\\
&\le K_9 K_{10} \frac{\Gamma(n+1+\bar r)\Gamma(\lceil ny\rceil+1-\alpha)/\Gamma(\lceil ny\rceil+1)\Gamma(n+1+\bar r-\alpha)}{g_n^{(s)}}\\
&\quad + K_6 K_9 \frac{(\Gamma(n+1+\bar r)/\Gamma(n+1)\Gamma(1+\bar r))n^{-\bar r}}{g_n^{(s)}}\\
&\le K_{11} s^{-\alpha} n^{-\alpha},
\end{aligned}
$$

where for the first inequality we use (C.1) with $l = \lceil ny\rceil$, in the second inequality, we use integration by parts and for the third inequality we bound $\rho^{(-\bar r+s)}(x)$ by $K_{10}x^{-\alpha}$ and also use Lemma C.2. For the last inequality, we use (2.5). Here, K_{11} is a constant which depends only on $\bar r$ and $\nu^{(s)}$. Then, we obtain

$$
\begin{aligned}
\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n)}}\right)^r \mathbf{1}_{\{X_1^{(n)}\ge ny\}}\right\} &\le (1-y)^{\bar r - r}\mathbb{E}^{\nu^{(s)}}\left\{\left(\frac{n}{n-X_1^{(n)}}\right)^{\bar r} \mathbf{1}_{\{X_1^{(n)}\ge ny\}}\right\}\\
&\le K_{11} y^{-\alpha}(1-y)^{\bar r - r} n^{-\alpha},
\end{aligned}
$$

which completes the proof.

Remark C.1. If $r \ge \alpha + s$, Proposition C.1 is false. Assume that $s = 0$, $r \ge \alpha$, and for any fixed $0 < y < 1$, $n \ge 1/(1-y)$, we have $ny \le n-1$ it then follows that

$$
\begin{aligned}
P_1(m,n,s,y) &\ge \mathbb{E}\left\{\left(\left(\frac{n}{n-X_1^{(n)}}\right)^r - 1 - r\frac{X_1^{(n)}}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!}\left(\frac{X_1^{(n)}}{n}\right)^k\right)\mathbf{1}_{\{X_1^{(n)}=n-1\}}\right\}\\
&= \mathbb{P}\{X_1^{(n)} = n-1\}\left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!}\left(\frac{n-1}{n}\right)^k\right)\\
&= \frac{\int_0^1 x^n\nu(\mathrm{d}x)}{g_n}\left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!}\left(\frac{n-1}{n}\right)^k\right)\\
&\sim Cn^{-2\alpha}\left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!}\left(\frac{n-1}{n}\right)^k\right),
\end{aligned}
$$

where C is a positive number. Then

$$
\liminf_{n\to+\infty} n^\alpha P_1(m,n,s,y) \ge C \quad \text{for all } 0 < y < 1.
$$

Hence, this remark justifies the constraint $0 \le r < \alpha + s$.

Appendix D. Results that are used to prove Theorem 1.3

Lemma D.1. *Let* $a > 0, b > 0, \beta \geq 1$. *Then*

$$0 < (a+b)^\beta \leq a^\beta + b^\beta + \beta 2^{\beta-1} a b^{\beta-1} + \beta 2^{\beta-1} b a^{\beta-1}.$$

Proof. If $0 \leq m \leq 1$ then

$$(1+m)^\beta \leq 1 + \beta 2^{\beta-1} m \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

We use the fact that the function $m \mapsto (1+m)^\beta$ is convex and that $\beta 2^{\beta-1}$ is the derivative of $(1+m)^\beta$ at $m = 1$.

If $1 < m$ then

$$(1+m)^\beta = m^\beta \left(1 + \frac{1}{m}\right)^\beta \leq (m)^\beta \left(1 + \beta 2^{\beta-1} \frac{1}{m}\right) \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Hence, for all $m \geq 0$,

$$(1+m)^\beta \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Then, for all $a > 0, b > 0$,

$$\begin{aligned}(a+b)^\beta &= a^\beta \left(1 + \frac{b}{a}\right)^\beta \\ &\leq a^\beta \left(1 + \left(\frac{b}{a}\right)^\beta + \beta 2^{\beta-1} \frac{b}{a} + \beta 2^{\beta-1} \left(\frac{b}{a}\right)^{\beta-1}\right) \\ &= a^\beta + b^\beta + \beta 2^{\beta-1} a b^{\beta-1} + \beta 2^{\beta-1} b a^{\beta-1}.\end{aligned}$$

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