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Randers Metrics of Constant Scalar Curvature

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Abstract. Randers metrics are a special class of Finsler metrics. Every Randers metric can be expressed in terms of a Riemannian metric and a vector field via Zermelo navigation. In this paper, we show that a Randers metric has constant scalar curvature if the Riemannian metric has constant scalar curvature and the vector field is homothetic.

1 Introduction

For a Finsler manifold (M, F), the flag curvature $\mathbf{K} = \mathbf{K}(x, y, P)$ at a point x is a function of tangent plane $P \subseteq T_xM$ and nonzero vector $y \in P$. This quantity tells us how curved the space is. The Ricci curvature $\mathbf{Ric} = \mathbf{Ric}(x, y)$ is the average value of the flag curvature over the "flags" P containing a vector $y \in T_xM$. Further averaging on the Ricci curvature gives the so-called scalar curvature

$$r(x) := \frac{n+2}{\omega_n} \int_{B_x} \mathbf{Ric}(x, y) dV_x,$$

where B_x the unit ball of F_x in T_xM, dV_x is the Busemann volume form on T_xM, and ω_n is the volume of the unit ball in R^n ([5]). It is a natural problem in Finsler geometry to understand the geometric properties of Finsler metrics of constant flag curvature, constant Ricci curvature or constant scalar curvature. In this paper, we shall focus on the scalar curvature of a special class of Finsler metrics in the form

$$F = \alpha + \beta,$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold. This class of Finsler metrics was introduced by G. Randers in 1941 in his study on general relativity, and hence they are named after him. Randers metrics also arise naturally from the navigation problem on a manifold M with a Riemannian metric $h = \sqrt{h_{ij}(x)y^iy^j}$ under the influence of an external force field $V = V^i(x)\frac{\partial}{\partial x^i}$ on M. The least time path from one point to another is a geodesic of the Randers metric F

$$F = \frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda} - \frac{V_0}{\lambda}, \tag{1.1}$$

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where $V_0 := V_i y^i$, $V_i := h_{ij} V^j$ and $\lambda := 1 - V_i V^i$. It is easy to see that every Randers metric can be expressed in the form (1.1); see [2, 4]. The pair (h, V) is called the navigation data of F.

The classification of Randers metrics of constant flag curvature has been completed [2], and Randers metrics with constant Ricci curvature can be characterized by Riemannian Einstein metrics and homothetic vector field via Zermelo navigation [1]. The main purpose of this paper is to bring attention to the scalar curvature and to show how to construct Randers metrics of constant scalar curvature.

Theorem 1.1 *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M that is expressed by navigation data (h, V) in (1.1). Let $r(x)$ and $\bar{r}(x)$ denote the scalar curvature of F and h respectively. Suppose that V is homothetic with respect to h, namely,*

$$V_{i;j} + V_{j;i} = -4ch_{ij}, \tag{1.2}$$

where $V_i := h_{ij} V^j$ and $c = constant$. Then $r(x) = \bar{r}(x) - n(n-1)c^2$. Hence, if h has constant scalar curvature, $\bar{r} = n(n-1)\mu$, then F has constant scalar curvature $r(x) = n(n-1)(\mu - c^2)$.

Note that if V is a Killing vector field with respect to h, *i.e.,* V satisfies (1.2) with $c = 0$, then the scalar curvature of F is equal to that of h.

There are many non-trivial examples of Randers metrics with constant scalar curvature but not constant Ricci curvature.

Example 1.2 Let $n \geq 3$ and $\varepsilon = \sqrt{(n-2)/n}$. The product Riemannian metric h on $M = S^{n-1}(\varepsilon) \times R$ has constant scalar curvature

$$\bar{r}(x) = (n-1)(n-2)\varepsilon^{-2} = n(n-1).$$

Let $V = \partial/\partial t$ be the vector field tangent to R in $M = S^{n-1}(\varepsilon) \times R$. Then V is a Killing vector field on (M, h). Let F be the Randers metric defined by (1.1). Then it has constant scalar curvature $r(x) = n(n-1)$.

Example 1.3 Let $n \geq 4$ and H^{n-2} be the Riemannian hyperbolic space of constant curvature -1. Let $\varepsilon = 1/\sqrt{n^2 - 3n + 3}$ and let $S^2(\varepsilon)$ be the standard sphere of radius ε in R^n. The product Riemannian metric h on $M = S^2(\varepsilon) \times H^{n-2}$ has constant scalar curvature

$$\bar{r}(x) = 2\varepsilon^{-2} - (n-2)(n-3) = n(n-1).$$

Let $\overline{V}$ be a Killing vector field on $S^2(\varepsilon)$. We can extend $\overline{V}$ to a Killing vector field $V = \overline{V} \oplus \{0\}$ on $M = S^2(\varepsilon) \times H^{n-2}$. Let F be the Randers metric defined by (1.1). Then it has constant scalar curvature $r(x) = n(n-1)$.

2 Preliminaries

Consider a Randers metric $F = \alpha + \beta$ on a manifold M. We can express it in the form (1.1), namely,

$$F = \frac{\sqrt{\lambda h^2 + V_0^2}}{\lambda} - \frac{V_0}{\lambda}, \tag{2.1}$$

where $h = \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric, $V = V^i(x)\frac{\partial}{\partial x^i}$ is a vector field, and $V_0 = V_i(x)y^i$. Let $\mathbf{Ric} = \mathbf{Ric}(x, y)$ and $\overline{\mathbf{Ric}} = \overline{\mathbf{Ric}}(x, y)$ denote the Ricci curvature of F and h respectively. Since h is Riemannian, $\overline{\mathbf{Ric}}(x, y) = \overline{Ric}_{ij}(x)y^i y^j$ is quadratic in $y \in T_xM$. For the vector $\xi = \xi^i \frac{\partial}{\partial x^i} \in T_xM$ with

$$\xi^i := y^i - F(x, y)V^i(x), \tag{2.2}$$

let

$$\widetilde{h} := \sqrt{h_{ij}(x)\xi^i\xi^j}, \quad \widetilde{\mathbf{Ric}} := \overline{Ric}_{ij}(x)\xi^i\xi^j.$$

Note that the transformation $\psi\colon y \in T_xM \setminus \{0\} \to \xi \in T_xM \setminus \{0\}$ defined by (2.2) is a diffeomorphism. It is easy to verify that $\widetilde{h}(x, \xi) = F(x, y)$.

We have the following lemma.

Lemma 2.1 ([3]) *Let $F = \alpha + \beta$ be a Randers metric on an n-manifold M given by (1.1) with navigation data (h, V). Suppose that V is homothetic with respect to h, namely,*

$$V_{i;j} + V_{j;i} = -4ch_{ij},$$

where $V_i := h_{ij}V^j$ and $c = c(x)$ is a scalar function on M. Then for any scalar function $\mu = \mu(x)$ on M

$$\mathbf{Ric} - (n-1)\Big\{\frac{3c_{x^m}y^m}{F} + \sigma\Big\}F^2 = \widetilde{\mathbf{Ric}} - (n-1)\mu\widetilde{h}^2, \tag{2.3}$$

where $\sigma := \mu(x) - c^2(x) - 2c_{x^m}(x)V^m(x)$.

For a Finsler metric $F = F(x, y)$ on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is given by

$$\sigma_F(x) := \frac{\omega_n}{\mathrm{Vol}\{(y^i) \in R^n \mid F(x, y) < 1\}},$$

where $\omega_n := \mathrm{Vol}(B^n(1))$. On the tangent space T_xM, there is a natural coordinate system (y^i) determined by the natural basis $\{\partial/\partial x^i\}$ for T_xM. Then the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ induces a volume form dV_x on T_xM:

$$dV_x = \sigma_F(x)dy^1 \cdots dy^n.$$

We use this volume form dV_x to average the Ricci curvature over the unit ball $B_x := \{y \in T_xM \mid F(x, y) < 1\}$ and define the scalar curvature of F by

$$r(x) := \frac{n+2}{\omega_n}\int_{B_x} \mathbf{Ric}\, dV_x.$$

This definition is given in [5]. We will show that if F is Einstein, *i.e.*,

$$\mathbf{Ric} = (n-1)\sigma F^2,$$

then it is easy to see that $r(x) = n(n-1)\sigma$. Moreover, if F is weakly Einstein, *i.e.*,

$$\mathbf{Ric} = (n-1)\Big\{\frac{3c_{x^m}y^m}{F} + \sigma F\Big\},$$

then $r(x) = 3(n^2-1)c_{x^m}V^m + n(n-1)\sigma$.

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3 Proof of Theorem 1.1

Let $F = \alpha+\beta$ be a Randers metric given by (2.1) with navigation data (h, V). Suppose that V satisfies $V_{i;j} + V_{j;i} = -4ch_{ij}$, where $c = c(x)$ is a scalar function. This is equivalent to the S-curvature being isotropic, $\mathbf{S} = (n+1)cF$. Theorem 1.1 follows from the next theorem.

Theorem 3.1 *The scalar curvatures of F and h are related by*

$$r(x) - \bar{r}(x) = -n(n-1)(c^2 + 2c_{x^m}V^m) + 3(n^2-1)c_{x^m}V^m. \tag{3.1}$$

Proof To prove Theorem 3.1, we need (2.3) in Lemma 2.1. We can rewrite equation (2.3) as

$$\mathbf{Ric} - \widetilde{\mathbf{Ric}} = 3(n-1)c_{x^m}y^mF + (n-1)\sigma F^2, \tag{3.2}$$

where $\sigma = -c^2 - 2c_{x^m}V^m$. By evaluating equation (3.2) on the unit tangent ball with respect to Busemann–Hausdorff volume form dV_x, we obtain (3.1). ■

Fix a coordinate system (x^i) at x. There are two coordinate systems (y^i) and (ξ^i) in T_xM, which are related by the following coordinate transformation, $\psi\colon (y^i) \to (\xi^i)$ given by

$$\xi^i = y^i - F(x,y)V^i(x).$$

Let $B_x := \{(y^i) \mid F(x,y) < 1\}$ and $\widetilde{B}_x := \{(\xi^i) \mid \widetilde{h}(x,\xi) < 1)\}$. Since $F(x,y) = \widetilde{h}(x,\xi)$ for $\xi^i = y^i - F(x,y)V^i(x)$, one can see that the image of B_x under the map ψ is $\widetilde{B}_x$. Note that the Riemannian volume of $\widetilde{h} = \sqrt{h_{ij}\xi^i\xi^j}$ on T_xM in (ξ^i) is given by

$$dV_{\widetilde{h}} = \sqrt{\det(h_{ij})}d\xi^1\cdots d\xi^n.$$

The unit balls B_x and $\widetilde{B}_x$ are related by $\widetilde{B}_x = B_x - (V^i)$ in R^n. Thus,

$$\begin{aligned}\sigma_x &= \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\,\{(y^i)|\, F(x,y)<1\}} = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\,(B_x)}\\ &= \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\,(B_x - V)} = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\,\{(\xi^i)|\,\widetilde{h}(x,\xi)<1\}} = \sqrt{\det(h_{ij})}.\end{aligned}$$

Furthermore, the coordinate transformation ψ has Jacobian $1/(1+\widetilde{h}_{\xi^k}V^k)$. Thus, the Busemann–Hausdorff volume form $d\widetilde{V}_x = (\psi^{-1})^*dV_x$ can be expressed in (ξ^i)

$$d\widetilde{V}_x = (1+\widetilde{h}_{\xi^k}V^k)dV_{\widetilde{h}} = \sqrt{\det h_{ij}}(1+\widetilde{h}_{\xi^k}V^k)d\xi^1\cdots d\xi^n.$$

Integrating (3.2) on B_x or $\widetilde{B}_x$, we obtain

$$\int_{B_x}\mathbf{Ric}\,dV_x - \int_{\widetilde{B}_x}\widetilde{\mathbf{Ric}}\,d\widetilde{V}_x = (n-1)\sigma\int_{B_x}F^2\,dV_x + 3(n-1)c_{x^m}\int_{B_x}y^mF\,dV_x. \tag{3.3}$$

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Using the definition of the coordinate transformation ψ, we have $\widetilde{h}(x,\xi) = F(x,y)$. We have

$$\int_{\widetilde{B}_x} \widetilde{\mathbf{Ric}}\, d\widetilde{V}_x = \int_{\widetilde{B}_x} \widetilde{\mathbf{Ric}}\, dV_{\widetilde{h}} + \int_{\widetilde{B}_x} \widetilde{\mathbf{Ric}}\, \widetilde{h}_{\xi^k} V^k\, dV_{\widetilde{h}},$$

$$\int_{B_x} F^2 dV_x = \int_{\widetilde{B}_x} \widetilde{h}^2 d\widetilde{V}_x = \int_{\widetilde{B}_x} \widetilde{h}^2 (1 + \widetilde{h}_{\xi^k} V^k) dV_{\widetilde{h}},$$

$$c_{x^m} \int_{B_x} y^m F dV_x = c_{x^m} \int_{\widetilde{B}_x} (\xi^m + \widetilde{h} V^m) \widetilde{h} (1 + \widetilde{h}_{\xi^k} V^k) dV_{\widetilde{h}}.$$

By definition,

$$\int_{B_x} \mathbf{Ric}\, dV_x = \frac{\omega_n}{n+2} r(x), \qquad \int_{\widetilde{B}_x} \widetilde{\mathbf{Ric}}\, dV_{\widetilde{h}} = \frac{\omega_n}{n+2} \overline{r}(x). \tag{3.4}$$

Since $\widetilde{\mathbf{Ric}}\, \widetilde{h}_{\xi^k} V^k$ is an odd function in ξ on $\widetilde{B}_x$, we have

$$\int_{\widetilde{B}_x} \widetilde{\mathbf{Ric}}\, \widetilde{h}_{\xi^k} V^k\, dV_{\widetilde{h}} = 0. \tag{3.5}$$

It is easy to verify that

$$\int_{\widetilde{B}_x} \widetilde{h}^2\, dV_{\widetilde{h}} = \frac{n}{n+2} \omega_n. \tag{3.6}$$

Note that $\widetilde{h}^2 \widetilde{h}_{\xi^k} V^k(x) = \widetilde{h} h_{jk}(x) \xi^j V^k(x)$ is an odd function on $\widetilde{B}_x$. Thus

$$\int_{\widetilde{B}_x} \widetilde{h}^2 \widetilde{h}_{\xi^k} V^k(x) dV_{\widetilde{h}} = 0.$$

Then

$$\int_{\widetilde{B}_x} \widetilde{h}^2\, d\widetilde{V}_x = \int_{\widetilde{B}_x} \widetilde{h}^2 (1 + \widetilde{h}_{\xi^k} V^k) dV_{\widetilde{h}} = \frac{n}{n+2} \omega_n. \tag{3.7}$$

Since both $\widetilde{h}\xi^m$ and $\widetilde{h}^2 \widetilde{h}_{\xi^k} V^m V^k$ are odd functions on $\widetilde{B}_x$, we have

$$\int_{\widetilde{B}_x} \widetilde{h} \xi^m dV_{\widetilde{h}} = \int_{\widetilde{B}_x} \widetilde{h}^2 \widetilde{h}_{\xi^k} V^m V^k dV_{\widetilde{h}} = 0.$$

Then

$$\begin{aligned} c_{x^m} \int_{B_x} y^m F dV_x &= c_{x^m} \int_{\widetilde{B}_x} (\xi^m + \widetilde{h} V^m) \widetilde{h} (1 + \widetilde{h}_{\xi^k} V^k) dV_{\widetilde{h}} \\ &= c_{x^m} V^m \int_{\widetilde{B}_x} \widetilde{h}^2 dV_{\widetilde{h}} + c_{x^m} \int_{\widetilde{B}_x} \xi^m \widetilde{h} \widetilde{h}_{\xi^k} V^k dV_{\widetilde{h}} \\ &= \frac{n}{n+2} \omega_n c_{x^m} V^m + c_{x^m} \int_{\widetilde{B}_x} \xi^m \widetilde{h} \widetilde{h}_{\xi^k} V^k dV_{\widetilde{h}}. \end{aligned}$$

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We claim that

$$c_{x^m} \int_{\widetilde{B}_x} \xi^m \widetilde{\widetilde{h}}h_{\xi^k} V^k dV_{\widetilde{h}} = \frac{1}{n+2} \omega_n c_{x^m} V^m. \tag{3.8}$$

To prove (3.8), we choose a special coordinate system at x such that $h_{ij} = \delta_{ij}$. Then $\widetilde{B}_x = \{ (\xi^i) \mid |(\xi^i)| < 1 \}$ and

$$c_{x^m} \int_{\widetilde{B}_x} \xi^m \widetilde{\widetilde{h}}h_{\xi^k} V^k dV_{\widetilde{h}} = c_{x^m} V^k \int_{\widetilde{B}_x} \xi^k \xi^m \, d\xi^1 \cdots d\xi^n.$$

Note that if $k \neq m$,

$$\int_{\widetilde{B}_x} \xi^k \xi^m \, d\xi^1 \cdots d\xi^n = 0.$$

For each fixed $k = m$,

$$\int_{\widetilde{B}_x} \xi^m \xi^k \, d\xi^1 \cdots d\xi^n = \frac{1}{n} \int_{\widetilde{B}_x} \sum_{i=1}^{n} (\xi^i)^2 \, d\xi^1 \cdots d\xi^n = \frac{1}{n+2} \omega_n.$$

Thus

$$c_{x^m} V^k \int_{\widetilde{B}_x} \xi^k \xi^m \, d\xi^1 \cdots d\xi^n = \frac{1}{n+2} \omega_n c_{x^m} V^m.$$

This proves (3.8). Therefore,

$$c_{x^m} \int_{B_x} y^m F dV_x = \frac{n+1}{n+2} \omega_n c_{x^m} V^m. \tag{3.9}$$

Plugging (3.4)–(3.9) into (3.3), we obtain (3.1). This completes the proof of Theorem 3.1. ∎

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