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UNIFORM DISTRIBUTION AND LATTICE POINT COUNTING

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Abstract

A well-known theorem of Hardy and Littlewood gives a three-term asymptotic formula, counting the lattice points inside an expanding, right triangle. In this paper a generalisation of their theorem is presented. Also an analytic method is developed which enables one to interpret the coefficients in the formula. These methods are combined to give a generalisation of a "height-counting" formula of Győry and Pethő which itself was a generalisation of a theorem of Lang.

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In their paper [5] Hardy and Littlewood consider the lattice points inside a right triangle, two of whose sides lie on the x-y axis, the third lying on the line $x + \theta y = Q$. Here Q is a real parameter and $\theta > 0$ is a real, irrational number. They obtained a three-term asymptotic formula counting the number of lattice points when Q is large. If T_Q denotes this triangle they showed that

(1) $$\#(\mathbb{Z}^2 \cap T_Q) = T_1 Q^2 = T_2 Q + o(Q).$$

In fact their interest in this problem lay in the relationship between the precise order of the error in (1) and the continued fraction expansion for θ. For example, they showed that if θ has bounded convergents then the error is

(2) $$O(\log Q).$$

Several authors (see [2], [10], [11]) have considered generalisations of (1). In [2] the author applied such a generalisation to the study of the values taken

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by a general sum of S-units. Since it appears that these techniques will find other applications it seems worthwhile to present a general formulation of a lattice point counting theorem which may be of independent interest.

For applications it is often desirable to have explicit formulations of the coefficients of an asymptotic formula in terms of invariants of the underlying data. The class-number formula is a classical example (see [7]). In Theorem 1 below the constant A is easy to interpret, being the volume of the region $\mathfrak{p}$. In Section 2 an analytic method will be presented which in principle gives an explicit interpretation of the constant B. In Section 3 a concrete example is presented, one which has been applied to count solutions of the norm-form equation (see [3]). It is a generalisation and refinement of a Theorem of Gyröy and Pethö [4], who in turn had generalised a Theorem of Lang in [9].

1. Lattice points

Suppose $r > 1$ is an integer and $\mathfrak{p}$ is a *bounded* region of $\mathbb{R}^r$ defined by a system of linear inequalities

$$L_i(x) \leq 1, \qquad 1 \leq i \leq k \tag{3}$$

and

$$\varphi_j(x) \leq 0, \qquad 1 \leq j \leq l, \tag{4}$$

where it is assumed that

(i) $L_1, \dots, L_k$ and $\varphi_1, \dots, \varphi_l$ have real coefficients,

(ii) $L_1, \dots, L_k$ have rank r,

(iii) for every $1 \leq i \leq k$ the coefficients of $L_i(x)$ span a $\mathbb{Q}$-vector space of dimension greater than or equal to 2. Write

$$M(x) = \max_{1 \leq i \leq k} \{L_i(x)\}. \tag{5}$$

For real $Q > 0$, define

$$\mathfrak{p}_Q = \{x \in \mathbb{R}^r : M(x) \leq Q, \ \varphi_j(x) \leq 0, \ j = 1, \dots, l\}. \tag{6}$$

THEOREM 1. *There exist constants A and B with*

$$\#(\mathbb{Z}^r \cap \mathfrak{p}_Q) = AQ^r + BQ^{r-1} + o(Q^{r-1}), \quad \textit{as} \quad Q \to \infty. \tag{7}$$

Obviously Theorem 1 is a generalisation of the result of Hardy and Littlewood because one may take

$$L_1(x) = x_1 + \theta x_2, \qquad L_2(x) = x_1 - \theta x_2, \qquad \varphi_1(x) = -x_1, \qquad \varphi_2(x) = -x_2,$$

(the form $L_2(x)$ playing no role at all).

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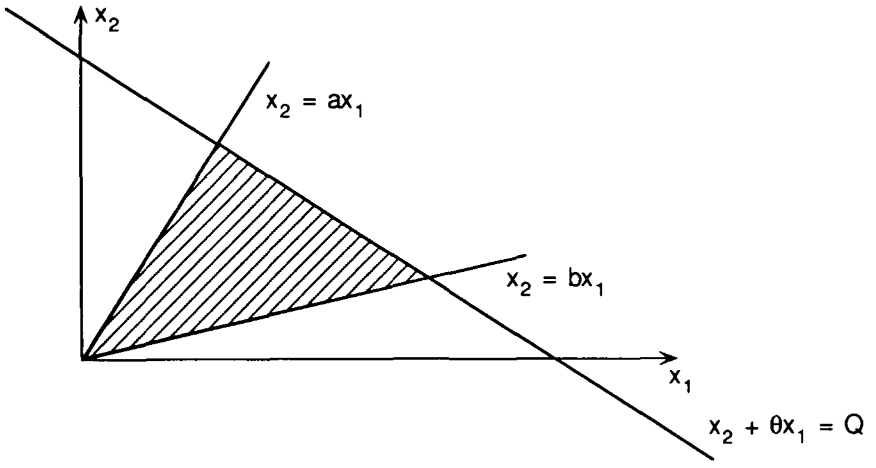
(8)



A simple case of (3) and (4) is show in Diagram (8) and this turns out to be the induction step required later.
Here a, $b \in \mathbb{R} \cup \{\infty\}$ (≥ 0) and $0 < \theta < 1$ is irrational. Let W_Q denote the shaded region.

LEMMA 1. *There are constants* ω_1, ω_2 *such that*

(9) $$\#(\mathbb{Z}^2 \cap W_Q) = \omega_1 Q^2 + \omega_2 Q + o(Q), \quad \textit{as} \quad Q \to \infty.$$

COROLLARY. *Suppose* T_Q *is a triangle bounded by the lines*

$$a_i x_1 + b_i x_2 = Q, \qquad i = 1, 2, 3.$$

Provided $a_i/b_i \notin \mathbb{Q}$ *for* $i = 1, 2, 3$, *there is a formula*

(10) $$\#(\mathbb{Z}^2 \cap T_Q) = t_1 Q^2 + t_2 Q + o(Q), \quad \textit{as} \quad Q \to \infty.$$

The proof of the corollary is trivial; T_Q is a union of differences of regions like W_Q.

Consider the lattice points $x \in \mathbb{Z}^2$ together with the unit square C_x centered at x. Write C_Q for the C_x with $x \in \mathbb{Z}^2$, with positive coordinates which intersect the line $x_2 + \theta x_1 = Q$ non-trivially. Write $d(x)$ for the distance from x to the line along the x_2-axis, and $d = (\theta + 1)/2$ so that

(11) $$-d \leq d(x) \leq d \quad \text{for} \quad x \in C_Q.$$

LEMMA 2. *The sequence* $d(x)$ *for* $x \in C_Q$ *is dense and uniformly distributed* (u.d.) *in the interval* $[-d, d]$ *if and only if* (9) *holds.*

PROOF. We will use freely the equivalent formulations of u.d. as given in [1]. Suppose (9) holds. Replace Q by $Q + \varepsilon$. The number of lattice points

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caught in the strip is $2A\varepsilon + o(Q)$. Thus the number with $d(x)$ lying in the region

$$-d \le \beta \le d(x) \le \alpha \le d$$

is $2A(\alpha - \beta)Q + o(Q)$, compared with $4AdQ + o(Q)$ in total. The ratio tends to $(\alpha - \beta)/2d$ as expected.

Conversely the area of W_Q is clearly AQ^2. This can be written as

$$AQ^2 = \sum_{x\,:\,C_x \subset W} 1 + \sum_{x \in C_Q} \text{area}\,(C_x \cap W_Q). \tag{12}$$

The area of $C_X \cap W_Q$ is a continuous function on $d(x)$ (piecewise polynomial in fact) and since d_x is u.d. in $[-d, d]$ it follows that the sum in (12) can be replaced by an integral (see [1]) to yield

$$A_1 Q + o(Q). \tag{13}$$

Counting those x with $x \in W_Q \cap C_Q$ is the same as counting 1 every time $d(x) \ge 0$. Again the u.d. of $d(x)$ allows the sum to be replaced by an integral and it follows that

$$\sum_{x \in W_Q \cap C_Q} 1 = A_2 Q + o(Q). \tag{14}$$

The counting techniques for those x with C_x intersecting either of the lines $x_2 = ax_1$ or $x_2 = bx_1$ is similar. The only difference is that a and b are not necessarily irrational. If one is rational then close to that line a sequence of distances is obtained which is discrete in the interval. However the u.d. property is still valid. Notice that in this case there could be lattice points sitting on the lines $x_2 = ax_1$, $x_2 = bx_1$.

To summarize, we have

$$\sum_{x \in C_Q} \text{area}\,(C_x \cap W_Q) = A_3 Q + o(Q). \tag{15}$$

$$\sum_{x \in C_Q \cap W_Q} 1 = A_4 Q + o(Q). \tag{16}$$

Putting (12), (15), (16) together yields

$$A_4 Q + AQ^2 = \sum_{x \in \mathbb{Z}^2 \cap W_Q} 1 + A_3 Q + o(Q)$$

and this is the form of (9).

Lemma 3. *The sequence $d(x)$ is u.d. in the interval $[-d, d]$.*

Proof. Treat first the case $a = \infty$, $b = 0$. Consider the values of

$$Q - x_2 - \theta x_1 \tag{17}$$

in the range $[-d, d]$, that is,

$$0 < Q - \theta x_1 + d - x_2 < 2d = 1 + \theta. \tag{18}$$

Notice that $1 < 2d < 2$. Let $\{t\} = t - [t]$, $[t]$ denoting the integer part of t. As x_1 runs from 1 to $[Q/\theta] = N$ the expression $\{Q - \theta x_1 + d\}$ is u.d. in $[0, 1]$ and there are N values. Asymptotically θN lie in the range $[0, \theta]$. We are counting all values $\{Q - \theta x_1 + d\}$ for $1 \le x_1 \le N$ together with the values $1 + \{Q - \theta x_1 + d\}$ for $\{Q - \theta x_1 + d\} \le \theta$. This gives a sequence of $(1+\theta)N$ points (asymptotically) in the interval $[0, 1+\theta]$ and this sequence is u.d.

For $b > 0$ it suffices to count as before but with x_1 running from 1 to $[cN]$ for some $c > 0$. Now the result for W_Q is obtained as the difference between two regions with $b > 0$, $a = \infty$.

PROOF OF THEOREM 1. The faces F_i of $\mathfrak{p}_Q$ correspond to sections of the hyperplanes $L_i(x) = Q$. Fix i and consider those $x \in \mathbb{Z}^r$ for which $C_x \cap F_i \neq \varnothing$. There must be a pair of coefficients in $L_i(x)$ say a_1 and a_2 with $\theta = a_1/a_2 \notin \mathbb{Q}$ (by condition (iii)). Then Diagram (8) applies. Here though $Q = Q(x_3, \dots, x_r)$ depends on the other variables. Let (as in Lemma 3) $d(x)$ denote the distance along the x_2-axis. We claim that these distances are dense and uniformly distributed within an interval

$$-d \le d(x) = \frac{1}{a_2}(Q - L_i(x)) \le d.$$

Use Weyl's criterion and set

$$S = \sum_{x\,:\,C_x \cap F_i \neq \varnothing} e^{[\pi i k\, d(x)/d]} \tag{19}$$

where k is a nonzero integer. Every $(r-2)$ tuple $(x_3, \dots, x_r)$ which occurs in $x = (x_1, \dots, x_r)$ in the sum in (19) gives rise to a value $Q(x_3, \dots, x_r)$ and then to a diagram (8). The sum over the resulting pairs x_1, x_2 is therefore $o(\#S)$, where $\#S$ denotes the number of terms in the sum, because of Lemma 3. Thus the sum in (19) must be $o(\#T)$ where $\#T$ denotes the total number of terms and this shows Weyl's criterion is satisfied.

Now Theorem 1 follows from exactly the same counting argument as in the proof of Lemma 2. The key point is that discrete sums may be replaced by integrals, using the uniform distribution property just established.

2. Analytic theory

In this section the analytic theory of counting lattice points in regions defined by (3) and (4) will be developed. It turns out that when condition

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(4) is vacuous a very simple interpretation of the constant B in (7) can be given. This is sufficient for the applications in [2] and [3].

Given the notation in Section 1, define

$$M(z) = \sum_{x \in \mathbf{Z}^r \cap D} \frac{1}{M(x)^z}, \qquad z \in \mathbb{C}, \tag{20}$$

where D denotes that region of $\mathbb{R}^r$ defined by (4). Let $\| \ \|$ denote the 'max-norm' on $\mathbb{R}^r$,

$$\|x = (x_1, \dots, x_r)\| = \max\{|x_i|\}.$$

LEMMA 4. *The boundedness of* $\mathfrak{p}$ *implies*

$$M(x) > C\|x\| \quad \textit{for all } x \in D, \tag{21}$$

where $C > 0$ *is constant.*

PROOF. Define $F(\mathfrak{p})$ to be those $x \in \mathfrak{p}$ with $M(x) = 1$. Notice that $M(x) \geq 0$ always since if $M(x) < 0$ and $\lambda > 1$ then

$$M(\lambda x) = \lambda M(x) < M(x).$$

Thus $\mathfrak{p}$ contains λx for arbitrarily large λ, contradicting the boundedness of $\mathfrak{p}$. Notice also that $\mathfrak{p}$ contains the origin. Thus, given $x \in D$, project x centrally onto $F(\mathfrak{p})$ by dividing by $\mu > 0$ say. Thus $M(x/\mu) = 1$ and x/μ lies on $F(\mathfrak{p})$. This means that the coefficients of x/μ are uniformly bounded so

$$\|x\| < d\mu = dM(x) \quad \text{for } d > 0,$$

are required.

LEMMA 5. *$M(z)$ converges absolutely in the half-plane* $\mathrm{Re}(z) > r$ *and uniformly on compact subsets of* $\mathrm{Re}(z) > r$.

PROOF. This is trivial since we can use (21) to compare $M(z)$ with the series $\sum_{0 \notin x \in \mathbf{Z}^r} \|x\|^{-z}$.

THEOREM 2. *The series $M(z)$ has meromorhpic continuation to the half-plane* $\mathrm{Re}(z) \geq r - 1$ *where it is analytic apart from simple poles at* $z = r$ *and* $z = r - 1$.

NOTES 1. In fact Theorem 2 follows directly from Theorem 1. The residues of the poles at $z = r$ (respectively $r - 1$) are given by rA (respectively $((r-1)B)$. We will now prove Theorem 2 in a way which allows simple

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formulae for the residues to be given, at least in the case where condition (4) is vacuous. See Note 1 after Theorem 4 for an example of the dichotomy.

2. If condition (4) is vacuous then the series $M(z)$ has a continuation to $\mathrm{Re}(z) > r-2$ with simple poles only as stated.

PROOF OF THEOREM 2. We will treat only the case where the condition (4) is vacuous. Write

$$I(z) = \int_{\mathbf{R}^r - F} M(y)^{-z}\,dy \tag{22}$$

where F is a compact ball about the origin. Notice that we are free to change F because this only adds to $I(z)$ an entire function and does not affect the form of the results we seek.

The domain of integration can be decomposed into a finite union of regions of the form

$$\begin{gathered}\eta_1(x_2, \dots, x_r) \geq x_1 \geq \nu_1(x_2, \dots, x_r), \\ \eta_2(x_3, \dots, x_r) \geq x_2 \geq \nu_2(x_3, \dots, x_r), \\ \vdots \\ \eta_{r-1}(x_r) \geq x_{r-1} \geq \nu_{r-1}(x_r), \\ \infty \geq x_r \geq \gamma > 0,\end{gathered} \tag{23}$$

where ν_i, ν_j are linear forms in the variables shown and we can take formally (if necessary) $\eta_i = \infty$ (respectively 0) or $\nu_j = 0$ (respectively $-\infty$). Then the integral can be evaluated directly as a multiple integral, and it is analytic in the complex plane apart for simple poles at $z = 1, \dots, r$.

Compare the sum with the integral

$$M(z) - I(z) = \sum_{x\in\mathbf{Z}^r}\left(M(x)^{-z} - \int_{C_x} M(y)^{-z}\,dy\right), \tag{24}$$

where C_x denotes the unit cube centered at x and where it is assumed that those x have been omitted for which $C_x \cap F \neq \varnothing$, obviously a finite number. Write this as

$$\sum_{x\in\mathbf{Z}^r}\left(M(x)^{-z} - \int_{C_o} M(x+y)^{-z}\,dy\right), \tag{25}$$

where C_o denotes the unit cube about the origin. It is certainly true that

$$M(x+y) = M(x) + O(1) \tag{26}$$

for $x \in \mathbb{Z}^r$ and $y \in C_o$, the constant implied by $O(1)$ being uniform. In fact it is true that

$$M(x+y) = M(x) + L_i(y) \quad \text{for some } 1 \leq i \leq k \tag{27}$$

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for all $x \in \mathbb{Z}^r$ and all $y \in C_0$ apart from those x which cluster around a finite number of hyperplanes in $\mathbb{R}^r$. let C denote those x which satisfy (27), C' denoting the rest. Then

$$M(z) - I(z) = \sum_{x \in C} \left(M(x)^{-z} - M(x)^{-z} \int_{C_o} (1 + \sum L_i(y)/M(x))^{-z}\, dy \right)$$
$$+ \sum_{x \in C'} \left(M(x)^{-z} + M(x)^{-z} \int_{C_o} (1 + O(1)/M(x))^{-z}\, dy \right).$$

Expand the inner brackets by the binomial theorem. This gives the meromorphic continuation to $\mathrm{Re}(z) > r - 1$. Apart from smaller order terms

$$\text{(28)} \qquad M(z) - I(z) = \sum_{x \in C} \frac{1}{M(x)^{z+1}} \int_{C_0} \sum L_i(y)\, dy + \sum_{x \in C'} \frac{O(1)}{M(x)^{z+1}}.$$

The second series converges for $\mathrm{Re}(z) > r - 2$ because it is a sum over lattice points clustered around a finite collection of hyperplanes. It follows that apart from the addition of a function which is analytic in $\mathrm{Re}(z) > r - 2$ the following formula holds:

$$\text{(29)} \qquad M(z) = I(z) + I(z+1) \int_{C_0} \sum L_i(y)\, dy.$$

The case where there are auxiliary inequalities runs along similar lines. We do not present the details because the resulting formulae do not allow explicit computations of the residues.

NOTE. For a connection with the classical literature consider the case (1) $L_1(x) = \theta x_1 + x_2$, (2) $\varphi_1(x) = -x_1$, $\varphi_2(x) = -x_2$, $\varphi_3(x) = ax_1 - x_2$ ($a > 1$ say). Then we see that the proof of Theorem 2 proceeds in a similar fashion, taking account of the inequalities in (4). The only difference is that in equation (29) the following sum is introduced.

$$\text{(30)} \qquad \frac{1}{(a+\theta)^s} \sum_{x_1=1}^{\infty} \frac{(\frac{1}{2} - \{ax_1\})}{x_1^s}.$$

This series has been studied in special cases by Hecke (see [6] or [8]).

Next we go on to establish a technical result which shows that the bulk of the lattice lie inside easily described regions. These regions are important for applications where general problems about counting solutions of equations are reduced to counting lattice points inside regions described below (see [2] and [3]).

Recall the definition (5) and define $M^*(x)$ to be the second largest of the linear forms, $M^{**}(x)$ to be the third largest. Repetitions are allowed so we might have $M(x) = M^*(x)$, etc. Given constants θ_1, θ_2 define

(31)
$$T_0 = T_0(\theta_1, \theta_2) = \{x \in \mathbb{Z}^r : M^*(x) + \theta_1 \log M(x) + \theta_2 < M(x)\}$$
$$\tilde{T}_0 = \tilde{T}_0(\theta_1, \theta_2) = \{x \in \mathbb{Z}^r : M^{**}(x) + \theta_1 \log M(x) + \theta_2 < M(x)\}.$$

THEOREM 3. *For* $1 > \varepsilon > 0$,

(32)
$$\text{(i) } \#(T_0 \cap \mathfrak{p}_Q) = AQ^r + O(Q^{r-1+\varepsilon}),$$
$$\text{(ii) } \#(\tilde{T}_0 \cap \mathfrak{p}_Q) = AQ^r + BQ^{r-1} + O(Q^{r-2+\varepsilon}).$$

PROOF. (i) Count those x with

$$M^*(x) \le M(x) \le M^*(x) + \theta_1 \log M(x) + \theta_2, \qquad M(x) \le Q.$$

The number of those points is bounded by the greater of the value (in r-space) of this region and the volume (in $(r-1)$-space) of the boundary. Clearly this is $O(Q^{r-1}(\log Q)^\alpha)$ for some $\alpha > 0$.

(ii) This is similar.

3. Application

Let K denote an algebraic number field of finite degree n over $\mathbb{Q}$, $n = [K : \mathbb{Q}]$. Let O_K denote the ring of algebraic integers in K and O_K^*, the group of units of O_K. The structure of O_K^* is known to be

(33)
$$O_K^* \cong T \times \mathbb{Z}^r, \qquad \text{for some } r \in \mathbb{N},$$

where T denotes the group of roots of unity in K. (See [7] for definitions and results in algebraic number theory.) Let $\sigma_1, \dots, \sigma_n$ denote the embeddings of K into $\mathbb{C}$. We will count complex embeddings in conjugate pairs so we distinguish only $r+1$ distinct embeddings. Define, for $\alpha \in K$,

(34)
$$H(\alpha) = \max_{1 \le i \le r+1} \{|\sigma_i(\alpha)|\}.$$

Similarly $H^*(\alpha)$ (respectively $H^{**}(\alpha)$) denotes the second respectively third) largest member of the set. Also let $\omega_K = |T|$, the number of roots of unity in K.

Choose a basis for O_K^* modulo T. Then the collection of $\log|\sigma_i(u)|$ becomes a set of $r+1$ linear forms $\zeta_1(x), \dots, \zeta_{r+1}(x)$ on $\mathbb{R}^r$. Define the regulator R_K by

(35)
$$R_k = |\det(\zeta_i(e_j))|, \qquad i = 1, \dots, r, \; e_j = (0, \dots, \underset{\substack{\uparrow \\ j}}{1}, \dots, 0).$$

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Notice that on $\mathbb{R}^r$,

$$\sum_{i=1}^{r+1} \zeta_i(x) = 0 \tag{36}$$

so the definition of R_K is independent of the choice of labelling for the ζ_i, also of the choice of basis for O_K^*.

For ease of presentation we are going to assume that K is totally real.

Theorem 4. *Take any totally real number field K.*

(i) *Let $U(q) = \#\{u \in O_K^* : H(u) < q\}$. Then*

$$U(q) = \frac{\omega_K (r+1)^r}{R_K r!}(\log q)^r + \frac{\omega_K (r+1)^{r-1}}{R_K (r-1)!}(\log q)^{r-1} + o((\log q)^{r-1}). \tag{37}$$

(ii) *Let $U_i(q) = \#\{u \in O_K^* : H(u) = |\sigma_i(u)| < q\}$. Then*

$$U_i(q) = \frac{\omega_K (r+1)^{r-1}}{R_K r!}(\log q)^r + u_i (\log q)^{r-1} + o((\log q)^{r-1}). \tag{38}$$

(iii) *Let*

$$V(q) = \#\left\{u \in O_K^* : \frac{H^{**}(u)}{H(u)} < \frac{\theta_3}{(\log H(u))^{\theta_4}}\right\}, \quad \theta_3 > 0, \quad \theta_4 > 1.$$

Then

$$U(q) = V(q) + o((\log q)^{r-1}). \tag{39}$$

Notes. (1) It is a curious fact that we are unable to give u_i explicitly in formula (38). This is a feature of the phenomenon observed earlier. The derivation of this formula involves nontrivial data in the definition (4). In application we need to apply (38) as it stands so we consent ourselves with the observation

$$\sum_{i=1}^{r+1} u_i = \frac{\omega_K (r+1)^{r-1}}{R_K (r-1)!}.$$

(2) These formulae are generalisations and refinements of the theorems of Györy and Pethö in [4] and Lang in [9] (see page 58).

(3) We could have counted instead those $u \in O_K^*$ with $N_{K|\mathbb{Q}}(u) = 1$. The only difference would be to replace ω_K by ω_K / i_K, where i_K is the index of O_K^* of the norm-1 subgroup.

Proof. This is simply a matter of identification. The forms $\zeta_1, \dots, \zeta_{r+1}$ correspond to $L_1, \dots, L_k$, so that $\log H(u)$ identifies with $M(x)$, the vector

x corresponding to the vector of exponents for u with respect to the chosen basis for O_K^* (mod T). For example, the condition

$$\frac{H^{**}(u)}{H(u)} < \frac{\theta_3}{(\log H(u)^{\theta_4}}$$

is identified with

$$M^{**}(u) + \theta_1 \log M(x) + \theta_2 < M(x)$$

so that $\theta_1 = \theta_4$ and $\theta_2 = -\log \theta_3$.

Thus it is that formulae (i) and (ii) follow from Theorem 1 while formula (iii) follows from Theorem 3. Theorem 2 can be applied in the following way to find the explicit nature of the coefficients in the formulae. From equation (29) we wish to find the residues of the function

$$I(z) = \int_{\mathbf{R}^r - F} Z(y)^{-z}\, dy, \qquad Z(y) = \max_{1 \le i \le r+1} \{\zeta_i(y)\}$$

at its poles $z = r$, $z = r - 1$. Clearly

$$I(z) = (r+1)! \int_{\mathbf{R}^r - F} \zeta_1^{-z}\, dy$$

where we assume that $\zeta_1 \ge \zeta_2 \ge \cdots \ge \zeta_{r+1} = -\zeta_1 - \cdots - \zeta_r$. Change the variables so that $y_i' = \zeta_i(y)$. The Jacobian is R_K^{-1}. Now change again so that

$$y_i'' = y_i' + \cdots + 2y_i' + \cdots + y_r'.$$

The Jacobian this time is $(r+1)^{-1}$. To ensure that a compact ball is removed, integrate over the region $y_r'' > 1$. Now the integral is

$$(r+1)^z \int_{y_1'' \ge \cdots \ge y_r'' > 1} (ry_1'' - y_2'' - \cdots - y_2'')^{-z}\, dy'',$$

and this is an easy multiple integral. The residues of the poles of $I(z)$ are

$$\frac{(r+1)^r}{R_K(r-1)!} \quad \text{for} \quad z = r, \qquad \text{and} \qquad \frac{(r+1)^{r-1}}{R_K(r-2)!} \quad \text{for} \quad z = r-1.$$

Finally, observe that a factor of ω_K must be inserted in the formulae in order to apply Theorem 2, because we worked modulo T and used the fact that H is invariant under T is the sense that $H(\alpha t) = H(\alpha)$ for all $\alpha \in K$, $t \in T$.

References

[1] J. W. S. Cassels, *An Introduction to Diophantine Approximation* (CUP, Cambridge, 1957).

[2] G. R. Everest, 'Counting the values taken by sums of S-units', *J. Number Theory* **35** (1990), 269–286.

[3] G. R. Everest, 'On the solution of the norm-form equation', to appear in Amer. J. Math. (1992)

[4] K. Győry, A Pethö, 'Uber die verteilung der lösungen von Normforem Gleichungen III', *Acta. Arith.* **37** (1980), 143–165.

[5] G. H. Hardy and J. E. Littlewood, 'Some problems of diophantine approximation: the lattice points of a right angled triangle', *Proc. London Math. Soc.* (2) **20** (1922), 15–36.

[6] E. Hecke, 'Über analytische Funktionen und die Verteilung von Zahlen mod eins', *Abh. Math. Sem. Hamburg* **1** (1921), 54–76.

[7] S. Lang, *Algebraic Number Theory*, (Addison-Wesley, Massachusetts, 1970).

[8] S. Lang, *Introduction to Diophantine Approximations*, (Addison-Wesley, Massachusetts, 1966).

[9] S. Lang, *Diophantine Geometry* (Wiley, New York, 1962).

[10] D. H. Lehmer, 'Lattice points of an n-dimensional tetrahedron', *Duke Math. Journal* **7** (1940), 341–353.

[11] D. C. Spencer, 'The lattice points of tetrahedra', *J. Math. Phys.* **21** (1942), 189–197.

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