Canad. J. Math. Vol. 77 (3), 2025 pp. 916–939
http://dx.doi.org/10.4153/S0008414X2400018X

A classification of anomalous actions through model action absorption

Sergio Girón Pacheco

Abstract. We discuss a strategy for classifying anomalous actions through model action absorption. We use this to upgrade existing classification results for Rokhlin actions of finite groups on C*-algebras, with further assuming a UHF-absorption condition, to a classification of anomalous actions on these C*-algebras.

1 Introduction

Connes' classification of automorphisms on the hyperfinite II_1 factor $\mathcal{R}$ [7, 8] paved the way toward a classification of symmetries of simple operator algebras. Over the next decade, this was followed by V. F. R. Jones' [29] classification of finite group actions on $\mathcal{R}$ and Ocneanu's [36] classification of actions of countable amenable groups on $\mathcal{R}$. To achieve these classification results, an important role is played by adaptations of Connes' noncommutative Rokhlin lemma, which yields that outer group actions on $\mathcal{R}$ satisfy a condition often called the Rokhlin property that is analogous to properties of ergodic measure preserving actions of amenable groups on probability spaces [38, 42]. In the C*-setting, the analogous property is not automatic. However, there has been substantial progress in the classification of those group actions on C*-algebras that satisfy the Rokhlin property [12, 16, 18–20, 22, 23, 35]. Very recently, groundbreaking results toward a classification of group actions without the need for the Rokhlin property have appeared [15, 25, 26].

Connes, V. F. R. Jones, and Ocneanu also classify group homomorphisms $G \to \mathrm{Out}(\mathcal{R})$ up to outer conjugacy [8, 29, 36]. Such a homomorphism is called a *G-kernel* on $\mathcal{R}$. The classification of G-kernels on injective factors was completed by Katayama and Takesaki [31]. These can be understood as the first classification results for quantum symmetries of $\mathcal{R}$ which do not arise as group actions. *Quantum symmetry* is a broad term that encapsulates generalized notions of symmetry that appear in topological and conformal field theories. These symmetries are often encoded through the action of a higher category equipped with a product operation such that the category weakly resembles a group. In the case of G-kernels, these can be understood

Received by the editors November 9, 2023; revised February 1, 2024; accepted February 12, 2024.
Published online on Cambridge Core March 7, 2024.
The author was supported by the Ioan and Rosemary James Scholarship awarded by St John's College and the Mathematical Institute, University of Oxford, as well as by project G085020N funded by the Research Foundation Flanders (FWO).
AMS subject classification: **46L35, 46L55, 46L40.**
Keywords: G-kernels, anomalous actions, classification, Rokhlin property.

as actions of 2-groups or tensor categories [13, 27]. The study of quantum symmetries of $\mathcal{R}$ was developed through the subfactor theory of Jones [30] culminating in Popa's classification of subfactors $\mathcal{N} \subset \mathcal{R}$ with amenable standard invariant [40].

In comparison to the success in understanding the existence and classification of G-kernels on von Neumann algebras, the study of G-kernels on C*-algebras has up to recently been underdeveloped. In [27], C. Jones studies the closely related notion of *ω-anomalous action*.[1] In his paper, C. Jones provides a C*-adaptation of V. F. R. Jones' work [28], laying out a systematic way to construct anomalous actions on C*-crossed products. C. Jones also establishes existence and no-go theorems for anomalous actions on abelian C*-algebras. In [13], Evington and the author lay out an algebraic K-theory obstruction to the existence of anomalous actions on tracial C*-algebras. Recently, Izumi [24] has developed a cohomological invariant for G-kernels. This invariant introduces new obstructions to the existence of G-kernels which also apply in the non-tracial setting. Further, Izumi uses this invariant to classify G-kernels of some poly-$\mathbb{Z}$ groups on strongly self-absorbing Kirchberg algebras satisfying the universal coefficient theorem (abbreviated as UCT).

This paper provides a classification of anomalous actions with the Rokhlin property on C*-algebras where K-theoretic obstructions vanish. The Rokhlin property for finite group actions was first systematically studied by Izumi [22, 23]. In his work, Izumi uses the Rokhlin property to boost existing classification results of Kirchberg algebras in the UCT class [32, 39] and unital, simple, separable, nuclear, tracially approximate finite-dimensional (TAF) algebras in the UCT class [34] by their K-theory, to a classification of finite group actions with the Rokhlin property on these classes of C*-algebras by the induced module structure on K-theory [23, Theorems 4.2 and 4.3].[2]

The strategy of this paper is to bootstrap Izumi's classification of G actions with the Rokhlin property, for finite groups G, to achieve analogous classification results for anomalous actions. To do this, we will assume that our C*-algebra A satisfies a UHF absorbing condition. To be precise, that the A is stable under tensoring with the UHF algebra $M_{|G|^\infty} \cong \bigotimes_{i\in\mathbb{N}} M_{|G|}$. This property is considered, for example, in [2, 16] and in some cases follows immediately from the existence of Rokhlin G actions on A ([23, Theorems 3.4 and 3.5], [16, Theorem 5.2]). Further assuming the Rokhlin property, we will establish a model action absorption result (Proposition 4.5). Second, we will use the model action absorption combined with a trick, that builds on ideas of Connes in the cyclic group case [8, Section 6]. This trick lets us use the existence of anomalous action on the UHF-algebra $M_{|G|^\infty}$ to reduce the classification of anomalous actions to the classification of cocycle actions. We may not apply this method by replacing $M_{|G|^\infty}$ by $\mathcal{Z}$ or $\mathcal{O}_\infty$ due to the obstruction results of [13, Theorem A] and [24, Theorem 3.6]. This argument allows us to prove the following.

Theorem A *(cf. Theorems 5.2 and 5.3) Let G be a finite group, and let $A \cong A \otimes M_{|G|^\infty}$ be either a Kirchberg algebra in the UCT class or a unital, simple, separable, nuclear*

[1] In the case that a C*-algebra A has trivial center, the study of ω-anomalous actions on A is equivalent to the study of G-kernels on A [27, Section 2.3].

[2] TAF algebras are C*-algebras that may be locally approximated by finite-dimensional C*-algebras in trace (see [33, Definitions 1 and 2]).

TAF algebra in the UCT class. If $(\alpha, u), (\beta, v)$ are anomalous G actions on A with the Rokhlin property, then (α, u) is cocycle conjugate to (β, v) through an automorphism that is trivial on K-theory if and only if $K_i(\alpha_g) = K_i(\beta_g)$ for all $g \in G$ and the anomalies of (α, u) and (β, v) coincide.

Similarly, we can boost Nawata's classification of Rokhlin G actions on $\mathcal{W}$ (see [35]) to a classification of anomalous actions on $\mathcal{W}$.

Theorem B *(cf. Theorem 5.4) Let G be a finite group, and let $(\alpha, u), (\beta, v)$ be anomalous G actions on $\mathcal{W}$ with the Rokhlin property, then (α, u) is cocycle conjugate to (β, v) if and only if the anomalies of (α, u) and (β, v) coincide.*

As a consequence of the results of [16], we may also apply this strategy to classify anomalous actions with the Rokhlin property on C*-algebras that arise as inductive limits of one-dimensional non commutative CW complexes (see Theorem 5.6).

The procedure utilized for the proof of Theorem A can be expected to work in more generality. The reason we restrict to unital, simple, nuclear TAF algebras in the tracial setting is due to the need to apply classification results for (cocycle) group actions. With more novel stably finite classification results in hand [5], and using similar techniques to [22, 23], a classification of finite group actions with the Rokhlin property on simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebra satisfying the UCT through the induced module structure on the Elliott invariant is plausible. A strategy to approach this classification problem has been proposed by Szabó in private communications. With such a result in hand, one could apply the abstract Lemma 5.1 to yield the equivalent to Theorem A in the generality of simple, separable, nuclear, $M_{|G|^\infty}$-stable C*-algebras satisfying the UCT.

Recent advances in the classification of more general symmetries on C*-algebras pave the way toward a classification of quantum symmetries. Significant results in this direction are the classification of AF-actions of fusion categories on AF-algebras [6], as well as Yuki Arano's announcement of an adaptation of Izumi's techniques in [22] to actions of fusion categories with the Rokhlin property. In the final section of this paper, we connect our results to the work in [6]. We demonstrate the existence of an AF ω-anomalous G-action with the Rokhlin property on $M_{|G|^\infty}$ which we denote by θ^ω_G. This has structural implications for anomalous actions with the Rokhlin property on any AF-algebra A. Indeed, combined with Theorem A, the existence of θ^ω_G implies that every anomalous action on A with the Rokhlin property, that consists of automorphisms that act trivially on K-theory, is automatically AF (see Corollary 6.3). Under some assumptions on the anomaly, an application of the classification results of [6] establishes the converse (see Corollary 6.3). This partial converse exhibits a difference in behavior between anomalous actions and group actions (see the discussion following Corollary 6.3).

The paper is organized as follows. In Section 2, we recall some necessary background on anomalous actions. Section 3 recalls the construction of model anomalous actions on UHF algebras. In Section 4, we prove a model action absorbing result for finite group anomalous actions. In Section 5, we set out an abstract lemma for the classification of anomalous actions (Lemma 5.1) which we use to prove our main results. Finally, in Section 6, we discuss an application of the classification result to AF-actions.

2 Preliminaries

Throughout, A and B will be used to denote C*-algebras and G, Γ, K will be used to denote countable discrete groups. We let $\mathbb{T} \subset \mathbb{C}$ be the circle group. We denote the multiplier algebra of A by $M(A)$. Any automorphism $\alpha \in \mathrm{Aut}(A)$ extends uniquely to an automorphism of $M(A)$, we denote this extension also by α. For a unitary $u \in M(A)$, we write $\mathrm{Ad}(u)$ for the automorphism $a \mapsto uau^*$ of A and the group of inner automorphisms on A by $\mathrm{Inn}(A)$. Recall that a G-kernel of A is a group homomorphism $G \to \mathrm{Aut}(A)/\mathrm{Inn}(A) = \mathrm{Out}(A)$. We now recall the definition of an anomalous action from [27, Definition 1.1]. In the case that A has trivial center, this notion coincides with a lift of a G-kernel into $\mathrm{Aut}(A)$.

Definition 2.1 An *anomalous action* of a countable discrete group G on a C*-algebra A consists of a pair (α, u) where

$$\alpha : G \to \mathrm{Aut}(A),$$
$$u : G \times G \to U(M(A))$$

are a pair of maps such that

$$\alpha_g \alpha_h = \mathrm{Ad}(u_{g,h})\alpha_{gh}, \text{ for all } g, h \in G, \tag{2.1}$$

$$\alpha_g(u_{h,k})u_{g,hk}u^*_{gh,k}u^*_{g,h} \in \mathbb{T} \cdot 1_{M(A)}, \text{ for all } g, h, k \in G. \tag{2.2}$$

First, note that in (2.1) and (2.2), we have used the subscript notation α_g and $u_{g,h}$ instead of $\alpha(g)$ and $u(g,h)$ for $g, h \in G$. We will use this throughout when notationally convenient.

As shown in [10, Lemma 7.1], the formula in (2.2) defines a circle valued 3-cocycle, i.e., an element of $Z^3(G, \mathbb{T})$. We will call this the *anomaly* of the action and denote it by $o(\alpha, u)$. For $\omega \in Z^3(G, \mathbb{T})$, we say (α, u) is a (G, ω) action on A to mean that (α, u) is an anomalous action of G on A with anomaly ω.[3] If $\omega = 1$, then we call (α, u) a *cocycle action*. Note that any anomalous action (α, u) induces a G-kernel when passing to the quotient group $\mathrm{Out}(A)$, we denote its associated G-kernel by $\overline{\alpha}$. For any G-kernel $\overline{\alpha}$ on A, we denote by $\mathrm{ob}(\overline{\alpha}) \in H^3(G, Z(U(M(A))))$ its 3-cohomology invariant (see, e.g., [13, Section 2.1]).

The reader should be warned that there is a slight variation in Definition 2.1 to the definitions of anomalous actions in [13, 27]. Given our conventions in Definition 2.1, a (G, ω) action induces an $\overline{\omega}$ anomalous action as in [27, Definition 1.1], this is seen by taking $m_{g,h} = u^*_{g,h}$.

Throughout this paper, we will denote the algebra of bounded sequences of A quotiented by those sequences going to zero in norm by A_∞. For a *-closed subset S of A_∞, we may consider the commutant C*-algebra $A_\infty \cap S' = \{x \in A_\infty : [x, S] = 0\}$ and the annihilator $A_\infty \cap S^\perp = \{x \in A_\infty : xS = Sx = 0\}$. We may then denote Kirchberg's sequence algebra by

$$F(S, A_\infty) = (A_\infty \cap S')/(A_\infty \cap S^\perp).$$

[3] In [27], the anomaly ω is carried as part of the data. We prefer to see the anomaly as an invariant of the pair (α, u).

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

In the case that S is the C*-algebra of constant sequences in A_∞, we denote this simply by $F(A) = F(A, A_\infty)$ and $F(A)$ the *central sequence algebra* of A. Note that $F(A)$ is a unital C*-algebra whenever A is σ-unital. Indeed, the unit is given by $h = (h_n)$ for any sequential approximate unit h_n for A.

Any automorphism $\theta \in \mathrm{Aut}(A)$ induces an automorphism θ of A_∞ through $(a_n) \mapsto (\theta(a_n))$ for any $(a_n) \in A_\infty$.[4] If a subset S of A_∞ is invariant under both θ and θ^{-1}, then so are $A_\infty \cap S'$ and $A_\infty \cap S^\perp$ and θ induces an automorphism of $F(S, A_\infty)$.

Definition 2.2 For an anomalous action (α, u) of a group G on a C*-algebra A and a *-closed subset $S \subset A_\infty$, we say S is (α, u)*-invariant* if $\alpha_g(S) \subset S$ for all $g \in G$ and $u_{g,h}S + Su_{g,h} \subset S$ for all $g, h \in G$.

Note that whenever S is (α, u)-invariant, then the automorphisms $\mathrm{Ad}(u_{g,h})$ also preserve S for all $g, h \in G$ and so $\alpha_g^{-1} = \mathrm{Ad}(u_{g,g^{-1}})\alpha_g$ preserve S for all $g \in G$.

Remark 2.1 When A is equipped with a (G, ω) action (α, u), it induces a (G, ω) action on A_∞. In fact, α induces a group action on $F(A)$ as $\mathrm{Ad}(u)(x) - x \in A_\infty \cap A^\perp$ for any $x \in A_\infty \cap A'$ and $u \in U(M(A))$. Similarly, if $S = S^*$ is an (α, u) invariant subset of A_∞, then α induces a group action on $F(S, A_\infty)$ (see [45, Remarks 1.8 and 1.10]).

We will be interested in anomalous actions with the Rokhlin property. This notion was introduced in [22, Definition 3.10] for actions of finite groups on unital C*-algebras and later generalized by Nawata and Santiago for non-unital C*-algebras (see [35, 43]). Its definition in the setting of anomalous actions is ad verbatim, we will only require it for σ-unital C*-algebras.

Definition 2.3 An anomalous action (α, u) of a finite group G on a σ-unital C*-algebra A is said to have the *Rokhlin property*, if there exist projections $p_g \in F(A)$ for $g \in G$ such that:

(1) $\sum_{g\in G} p_g = 1$,
(2) $\alpha_g(p_h) = p_{gh}$.

Remark 2.2 The Rokhlin property also makes sense for G-kernels. In this case, a G-kernel $\overline{\alpha}$ of a finite group G on a σ-unital C*-algebra A satisfies the Rokhlin property if for any/some lift (α, u) of $\overline{\alpha}$ there exists a partition of unity of projections $p_g \in F(A)$ for $g \in G$ such that $\alpha_g(p_h) = p_{gh}$ for all $g, h \in G$.

Our main goal is to classify anomalous actions with the Rokhlin property. To make sense of this question, we first need to introduce equivalence relations for anomalous actions. Before we do so, we start by introducing some notation that will allow us to streamline future definitions.

Definition 2.4 Let (α, u) be an anomalous action of a group G on a C*-algebra A. If $\mathbb{v}_g \in U(M(A))$ for $g \in G$, then the pair $(\alpha^{\mathbb{v}}, u^{\mathbb{v}})$ with

$$\alpha_g^{\mathbb{v}} = \mathrm{Ad}(\mathbb{v}_g)\alpha_g, \quad g \in G,$$
$$u_{g,h}^{\mathbb{v}} = \mathbb{v}_g\alpha_g(\mathbb{v}_h)u_{g,h}\mathbb{v}_{gh}^*, \quad g, h \in G$$

is an anomalous action. We say that $(\alpha^{\mathbb{v}}, u^{\mathbb{v}})$ is a *unitary perturbation* of (α, u).

[4] Note the abuse of notation.

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

It is a straightforward that $o(\alpha, u) = o(\alpha^{\mathbb{v}}, u^{\mathbb{v}})$ for any map $\mathbb{v} : G \to U(M(A))$.

Definition 2.5 Let A, B be C*-algebras, let (α, u) be an anomalous G action on A, and let (β, v) be an anomalous action on B. Then we say that:

(i) (α, u) is *conjugate* to (β, v) if there exists an isomorphism $\theta : A \to B$ such that $\alpha_g = \theta\beta_g\theta^{-1}$ and $v_{g,h} = \theta(u_{g,h})$ for all $g, h \in G$.

(ii) (α, u) is *cocycle conjugate* to (β, v) if there exist unitaries $s_g \in U(M(A))$ for $g \in G$ such that (α^s, u^s) is conjugate to (β, v). We denote this by $(\alpha, u) \simeq (\beta, v)$.

(iii) If A and B are equal and $(\alpha, u) \simeq (\beta, v)$ with the conjugacy holding through an automorphism θ such that $K_i(\theta) = \mathrm{id}_{K_i(A)}$ for $i = 1, 2$, we say (α, u) and (β, v) are *K-trivially cocycle conjugate*. We denote this by $(\alpha, u) \simeq_K (\beta, v)$.

Finally, recall the definition of a unitary one cocycle.

Definition 2.6 Let α be a (G, ω) action on a C*-algebra A. We call a map $v : G \to U(M(A))$ such that $v_g\alpha_g(v_h) = v_{gh}$ an *α-cocyle*.

3 Model actions

Given a finite group G and $\omega \in Z^3(G, \mathbb{T})$ a normalized 3-cocycle, [13, Theorem C] constructs a (G, ω) action on $M_{|G|^\infty}$. This result is based on a construction of C. Jones in [27] which in turn is based on a construction of V. F. R. Jones in the setting of von Neumann algebras [28].

In this section, we recall this construction as we will need its specific form to deduce properties of the action. First, recall that a 3-cocycle $\omega : G^{\times 3} \to \mathbb{T}$ is called normalized if $\omega(g, h, k) = 1$ whenever either g, h or k are the identity. In [27], C. Jones shows that if ω is a normalized 3-cocycle and one has the following data:

- a group Γ and a surjection $\rho : \Gamma \twoheadrightarrow G$ such that $\rho^*(\omega)$ is a coboundary,
- a normalized 2-cochain $c : \Gamma \times \Gamma \to \mathbb{T}$ such that $\rho^*(\omega) = dc$,
- a C*-algebra B and an action $\pi : \Gamma \to \mathrm{Aut}(B)$,

one can induce a (G, ω) action on the twisted reduced crossed product $B \rtimes^r_{\pi,\bar{c}} K$, with $K = \ker(\rho)$ (see [4] for a reference on twisted crossed products).[5] The automorphic data of this (G, ω) action are given by

$$\theta_g\left(\sum_{k\in K} a_k v_k\right) = \sum_{k\in K} c_{\hat{g}k\hat{g}^{-1},\hat{g}^{-1}}\overline{c_{\hat{g},k}}\pi_{\hat{g}}(a_k)v_{\hat{g}k\hat{g}^{-1}}, \tag{3.1}$$

for $a_k \in B$, v_k the canonical unitaries in $M(B \rtimes^r_{\pi,\bar{c}} K)$, $g \in G$ and $g \mapsto \hat{g}$ a choice of set-theoretic section to $\rho : \Gamma \to G$.[6] In fact, given an arbitrary finite group G, C. Jones constructs a finite group Γ, a surjection ρ, and a 2 cochain c with the conditions needed above and additionally $c|_{\ker(\rho)} = 1$. Additionally, to Γ and c, the extra data considered in [13, Theorem C] are:

- $B = \bigotimes_{i\in\mathbb{N}} \mathcal{B}(l^2(\Gamma))$,
- $\pi = \mathrm{Ad}(\lambda_\Gamma)^{\otimes\infty}$,

[5] For $c \in C^2(G, \mathbb{T})$, we denote by $\bar{c}$ the 2-cochain given by $\bar{c}_{g,h} = \overline{c_{g,h}}$ for $g, h \in G$.

[6] Note that (3.1) is different to the formula in [27, Lemma 3.2]. This is due to our change of conventions when defining anomalous actions.

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

with λ_Γ the left regular representation and $\mathrm{Ad}(\lambda_\Gamma)_\gamma(T) = \lambda_\Gamma(\gamma)T\lambda_\Gamma(\gamma)^*$ for all $T \in \mathcal{B}(l^2(\Gamma))$ and $\gamma \in \Gamma$. In this case, the crossed product $B \rtimes_\pi^r K$ is shown to be isomorphic to the UHF algebra $M_{|G|^\infty}$. C. Jones' construction then yields a (G, ω) action on $M_{|G|^\infty}$ through (3.1) for any $\omega \in Z^3(G, \mathbb{T})$, we denote it by (s_G^ω, u_G^ω).

Proposition 3.1 *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$, then (s_G^ω, u_G^ω) has the Rokhlin property.*

Proof We use the notation set up in the previous paragraphs. Furthermore, denote by $r_i : \mathcal{B}(l^2(\Gamma)) \to B$ the unital embedding into the ith tensor factor. As $A = B \rtimes_\pi K$ is unital, $F(A)$ coincides with $A_\infty \cap A'$, so it suffices to find a partition of unity $p_g \in A_\infty \cap A'$ for $g \in G$ such that $\alpha_g(p_h) = p_{gh}$ for all $g, h \in G$.

Let e_K in $\mathcal{B}(l^2(\Gamma))$ be the projection onto $l^2(K)$, that is,

$$e_K\left(\sum_{\gamma\in\Gamma} \mu_\gamma \gamma\right) = \sum_{\gamma \in K} \mu_\gamma \gamma$$

for any complex scalars μ_γ. Let $p_n = r_n(e_K)$ for $n \in \mathbb{N}$. Note that the projection $p = (p_n) \in B_\infty$ commutes with any constant sequence of elements in B. Moreover, p commutes with the subalgebra $C^*(K) \subset (B \rtimes K)_\infty$. Indeed, e_K is invariant under $\mathrm{Ad}(\lambda_\Gamma)_k$ for any $k \in K$ and therefore for any $n \in \mathbb{N}$ and $k \in K$,

$$\begin{aligned} v_k p_n v_k^* &= \mathrm{Ad}(\lambda_\Gamma)_k^{\otimes\infty}(r_n(e_K)) \\ &= r_n(\mathrm{Ad}(\lambda_\Gamma)_k e_K)) \\ &= r_n(e_K) \\ &= p_n. \end{aligned}$$

Therefore, $p \in A_\infty \cap A'$.

We claim that the projections $p_g := s_G^\omega(g)(p) = (s_G^\omega(g)(p_n))_{n\in\mathbb{N}}$ form a set of Rokhlin projections. We start by showing that the sum $\sum_{g\in G} s_G^\omega(g)(p) = 1$. Let $n \in \mathbb{N}$ and $g \in G$, then as the cochain c is normalized, it follows from (3.1) that

$$\begin{aligned} s_G^\omega(g)(p_n) &= \pi_{\hat{g}}(p_n) \\ &= \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}^{\otimes\infty}(p_n) \\ &= \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}^{\otimes\infty}(r_n(e_K)) \\ &= r_n(\mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)). \end{aligned} \tag{3.2}$$

The maps r_n are unital, so it suffices to show that $\sum_{g\in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K) = 1_{\mathcal{B}(l^2(\Gamma))}$. To see this, let $\gamma \in \Gamma$, $g \in G$, and $\delta_\gamma \in l^2(\Gamma)$ the point mass at γ, then

$$\begin{aligned} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)(\delta_\gamma) &= \lambda_\Gamma(\hat{g})e_K\lambda_\Gamma(\hat{g}^{-1})(\delta_\gamma) \\ &= \lambda_\Gamma(\hat{g})e_K(\delta_{\hat{g}^{-1}\gamma}) \\ &= \begin{cases} \delta_\gamma, & \text{if } \gamma \in \hat{g}K, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

The left K cosets are pairwise disjoint and cover the whole group Γ. Therefore, it follows that $\sum_{g\in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)(\delta_\gamma) = \delta_\gamma$ for every $\gamma \in \Gamma$. As the operators

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

$\sum_{g \in G} \mathrm{Ad}(\lambda_\Gamma)_{\hat{g}}(e_K)$ and $\mathrm{id}_{\mathcal{B}(l^2(\Gamma))}$ coincide on a spanning set of $l^2(\Gamma)$, these operators are equal.

It remains to show that for $g, h \in G$ the projections $s_G^\omega(g)p_h = p_{gh}$. This follows as $s_G^\omega(g)p_h = s_G^\omega(g)s_G^\omega(h)p = \mathrm{Ad}(u_G^\omega(g,h))s_G^\omega(gh)p = \mathrm{Ad}(u_G^\omega(g,h))p_{gh} = p_{gh}$ where the last equality in the chain holds as p_{gh} commutes with A. ∎

4 Absorption of model actions

In this section, we show that any Rokhlin anomalous action of a finite group G, on an $M_{|G|^\infty}$-stable C*-algebra, absorbs the action

$$s_G = \bigotimes_{i=0}^{\infty} \mathrm{Ad}(\lambda_G)$$

up to cocycle conjugacy.[7] This result is similar in nature to $(i) \Rightarrow (iii)$ of [16, Theorem 5.2]. The methods utilized in this chapter are an adaptation of V. F. R. Jones' work [29] to the C*-setting.

In his work [44–46], Szabó establishes the theory of strongly self-absorbing C*-dynamical systems as an equivariant version of strongly self-absorbing C*-algebras that were introduced in [47]. We recall the main definition below.

Definition 4.1 Let G be a locally compact group. A group action γ on a unital, separable C*-algebra $\mathcal{D}$ is called *strongly self-absorbing* if there exists an equivariant isomorphism $\varphi : (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$ such that there exist unitaries $u_n \in U(\mathcal{D} \otimes \mathcal{D})$ fixed by $\gamma \otimes \gamma$ with

$$\lim_{n \to \infty} \|\varphi(a) - u_n(a \otimes 1_{\mathcal{D}})u_n^*\| = 0$$

for all $a \in \mathcal{D}$. That is, the maps φ and $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$ are *approximately unitarily equivalent* or in short $\varphi \approx_{a.u} \mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}$.

The relevant example of a strongly self-absorbing action for this paper is s_G. That s_G is strongly self-absorbing follows as a consequence of [45, Example 5.1].

In [45, Theorem 3.7], Szabó shows equivalent conditions for a cocycle action to tensorially absorb a strongly self-absorbing action. Although Szabó's theory only treats the case of cocycle actions absorbing a given strongly self-absorbing group action, many of the arguments follow in exactly the same way when replacing cocycle actions by anomalous actions that may have nontrivial anomaly. The proofs of [45, Lemma 2.1 and Theorem 2.6] and [45, Theorem 3.7 and Corollary 3.8], for example, make no use of the anomaly associated with (α, u) and (β, w) being trivial. Under this observation, we can state a specific case of [45, Corollary 3.8].

Theorem 4.1 (cf. [45, Theorem 2.8]) *Let A and $\mathcal{D}$ be separable C*-algebras, and let G be a finite group. Assume that $(\alpha, u) : G \curvearrowright A$ is an anomalous action. Let $\gamma : G \curvearrowright \mathcal{D}$ be a group action such that $(\mathcal{D}, \gamma)$ is strongly self-absorbing. If there exists an equivariant*

[7] Note that for G finite the C*-algebras $M_{|G|}$ and $B(l^2(G))$ are canonically isomorphic, we identify them throughout this paper.

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

and unital **-homomorphism*

$$(\mathcal{D},\gamma)\to(F(A),\alpha),$$

then (A,α,u) *is cocycle conjugate to* $(A\otimes\mathcal{D},\alpha\otimes\gamma,u\otimes 1_{\mathcal{D}})$ *through a map* $\varphi: A\to A\otimes\mathcal{D}$ *that is approximately unitarily equivalent to* $\mathrm{id}_A\otimes 1_{\mathcal{D}}$.

We still require a few more results before we can achieve the model action absorption. These are based on known results in the setting of finite group actions on unital C*-algebras. These generalize line by line to anomalous actions of finite groups on unital C*-algebras, we adapt the arguments also for non-unital C*-algebras.

Lemma 4.2 (cf. [21, Theorem 3.3]) *Let A be a C*-algebra, let G be a finite group, and let (α,u) be an anomalous action of G on A with the Rokhlin property. If $B=B^*$ is a separable (α,u)-invariant subset of A_∞ and there exists a unital* **-homomorphism $M\to F(B,A_\infty)$ for some separable, unital C*-algebra M, then there exists a unital* **-homomorphism $M\to F(B,A_\infty)^\alpha$.*

Proof Fix a unital homomorphism $\psi: M\to F(B,A_\infty)$ and choose a linear lift $\psi_0: M\to A_\infty\cap B'$. Then one has that:

(i) $(\psi_0(m)\psi_0(m')-\psi_0(mm'))b=0,\quad \forall m,m'\in M, b\in B,$
(ii) $(\psi_0(m^*)-\psi_0(m)^*)b=0,\quad \forall m\in M, b\in B,$
(iii) $\psi_0(1)b-b=0,\quad \forall b\in B.$

Let $S=B\cup_{g\in G}\alpha_g(\psi_0(M))\cup_{g\in G}\alpha_g(\psi_0(M))^*$, so $S=S^*$. By the Rokhlin property followed by a standard reindexing argument, there exist positive contractions $f_g\in A_\infty\cap S'$ such that:

(iv) $(\alpha_g(f_h)-f_{gh})a=0,\quad \forall g,h\in G, a\in S,$
(iiv) $(\sum_{g\in G}f_g)a-a=0\quad \forall a\in S,$
(iiiv) $f_gf_ha-\delta_{g,h}a=0\quad \forall g,h\in G, a\in S.$

Now consider the linear mapping $\varphi: M\to A_\infty\cap B'$ given by

$$\varphi(m)=\sum_{g\in G}\alpha_g(\psi_0(m))f_g.$$

First, for $m,m'\in M$ and $b\in B$, it follows from (i) and (iiiv) that

$$\begin{aligned}\varphi(m)\varphi(m')b&=\sum_{g,h\in G}\alpha_g(\psi_0(m))f_g\alpha_h(\psi_0(m'))f_hb\\&=\sum_{g,h\in G}\alpha_g(\psi_0(m))\alpha_h(\psi_0(m'))f_gf_hb\\&=\sum_{g\in G}\alpha_g(\psi_0(m))\alpha_g(\psi_0(m'))bf_g\\&=\sum_{g\in G}\alpha_g(\psi_0(mm'))bf_g\\&=\sum_{g\in G}\alpha_g(\psi_0(mm'))f_gb\\&=\varphi(mm')b.\end{aligned}$$

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

Also for $k \in G$, $m \in M$ and $b \in B$ it follows using (iv) that

$$\begin{aligned}\alpha_k(\varphi(m))b &= \sum_{g\in G} \alpha_k\left(\alpha_g(\psi_0(m))\right)\alpha_k(f_g)b \\ &= \sum_{g\in G} \mathrm{Ad}(u_{k,g})(\alpha_{kg}(\psi_0(m)))f_{kg}b \\ &= \sum_{g\in G} \mathrm{Ad}(u_{k,g})(\alpha_{kg}(\psi_0(m)))bf_{kg} \\ &= \varphi(m)b.\end{aligned}$$

Where in the last line we have used that B is u invariant and so the observation in Remark 2.1 applies. Therefore, the map

$$m \mapsto \varphi(m) + A_\infty \cap B^\perp$$

defines a homomorphism from M into $(F(B, A_\infty))^\alpha$. This homomorphism is unital through combining (iii) and (iiv) and *-preserving by (ii). ∎

In the next lemma, recall that if α is an action of a group G on a C*-algebra A, an α-cocycle is a family of unitaries $v_g \in U(M(A))$ for $g \in G$ such that $v_g\alpha_g(v_h) = v_{gh}$.

Lemma 4.3 (cf. [19, Lemma III.1]) *Let A be a separable C*-algebra, and let G be a finite group. Let (α, u) be an anomalous action of G on A with the Rokhlin property. Let $B = B^*$ be a separable (α, u)-invariant subset of A_∞. For any α-cocycle v_g for the action induced by α on $F(B, A_\infty)$, there exists a unitary $u \in F(B, A_\infty)$ with $u^*\alpha_g(u) = v_g$.*

Proof Let $v_g \in U(F(B, A_\infty))$ be an α-cocycle. Choosing lifts $v'_g \in A_\infty \cap B'$ for v_g, one has:

(i) $v'_g(v'_g)^*b - b = 0, \quad \forall g \in G, b \in B,$
(ii) $(v'_g)^*v'_gb - b = 0, \quad \forall g \in G, b \in B,$
(iii) $v'_g\alpha_g(v'_h)b - v'_{gh}b = 0, \quad \forall g, h \in G, b \in B.$

Let $S = B \cup \{\alpha_h(v'_g), \alpha_h(v'_g)^* : g, h \in G\}$. As in the previous lemma, one may apply the Rokhlin property combined with a reindexing argument to get a family of positive elements $f_g \in A_\infty \cap S'$ such that:

(iv) $(\alpha_g(f_h) - f_{gh})a = 0, \quad \forall g, h \in G, a \in S,$
(iiv) $\sum_{g\in G} f_ga - a = 0, \quad \forall a \in S,$
(iiiv) $f_gf_ha - \delta_{g,h}a = 0 \quad \forall g, h \in G, a \in S.$

Let $u = \sum_{g\in G} v'_gf_g \in A_\infty \cap B'$. Then, for any $b \in B$ by (ii), (iiv), and (iiiv), it follows that

$$\begin{aligned}u^*ub &= \sum_{g,h\in G} f_g(v'_g)^*v'_hf_hb \\ &= \sum_{g,h\in G} (v'_g)^*v'_hbf_gf_h \\ &= \sum_{g\in G}(v'_g)^*v'_gbf_g \\ &= b.\end{aligned}$$

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

Similarly, $uu^*b = b$ for any $b \in B$. Moreover, (iii), (i), (iv) and (iiv) imply that for $b \in B$ and $g \in G$,

$$\begin{aligned} u\alpha_g(u^*)b &= \sum_{h,k} v'_h f_h \alpha_g(f_k)\alpha_g(v'_k)^* b \\ &= \sum_{h,k} v'_h \alpha_g(v'_k)^* b f_h \alpha_g(f_k) \\ &= \sum_{k} v'_{gk} \alpha_g(v'_k)^* b f_{gk} \\ &= \sum_{k} v'_g b f_{gk} \\ &= v'_g b. \end{aligned}$$

Therefore, by passing to the quotient, u defines a unitary in $F(B, A_\infty)$ such that $u\alpha_g(u^*) = v_g$ for all $g \in G$. ∎

For a finite group G, we denote by $e_{g,h} \in \mathcal{B}(l^2(G))$ the canonical matrix units defined by

$$e_{g,h}(f)(k) = \begin{cases} f(h), & \text{if } k = g, \\ 0, & \text{otherwise}, \end{cases}$$

for $f \in l^2(G)$. The proof of the next lemma is based on the proof of [29, Proposition 3.4.1].

Lemma 4.4 *Let G be a finite group, and let A be a separable C^*-algebra such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. Let (α, u) be an anomalous action with the Rokhlin property of G on A. Then there exists a G-equivariant unital embedding*

$$(\mathbb{M}_{|G|^\infty}, s_G) \to (F(A), \alpha).$$

Proof To prove this, we inductively construct unital equivariant *-homomorphisms $\phi_n : (\mathcal{B}(l^2(G)), \mathrm{Ad}(\lambda_G)) \to (F(A), \alpha)$ for $n \in \mathbb{N}$ with commuting images. Then the map defined by $a_1 \otimes \cdots \otimes a_n \otimes \ldots \longmapsto \prod_{i \in \mathbb{N}} \phi_i(a_i)$ will induce an s_G to α equivariant map into $F(A)$.

Suppose $\phi_1, \phi_2, \ldots, \phi_n : (\mathcal{B}(l^2(G)), \mathrm{Ad}(\lambda_G)) \to (F(A), \alpha)$ are equivariant maps with commuting images and let $\psi_i : B(l^2(G)) \to A_\infty \cap A'$ be linear lifts of ϕ_i for $1 \le i \le n$, then:

(i) $\psi_i(m)\psi_j(m')a - \psi_j(m')\psi_i(m)a = 0, \quad \forall a \in A, m, m' \in B(l^2(G)), 1 \le i \ne j \le n,$
(ii) $\alpha_g(\psi_i(m))a - \psi_i(\lambda_G(g)(m))a = 0, \quad \forall a \in A, m \in B(l^2(G)), 1 \le i \le n, g \in G,$
(iii) $\psi_i(m)^*a - \psi_i(m^*)a = 0, \quad \forall a \in A, m \in B(l^2(G)), 1 \le i \le n,$
(iv) $\psi_i(1)a - a = 0, \quad \forall \in A, 1 \le i \le n.$

Let

$$S = \{\psi_i(m)a : m \in B(l^2(G)),\ a \in A,\ 1 \le i \le n\}.$$

Then S is separable, $S = S^*$, and S is (α, u) invariant. We check that $u_{g,h}S \subset S$ for all $g, h \in G$, the remaining conditions follow similarly. For $a \in A$, $m \in B(l^2(G))$, and $1 \le$

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

$i \leq n$, letting $m' = \mathrm{Ad}(\lambda_G)_{h^{-1}g^{-1}}(m)$, one has that

$$\begin{aligned} u_{g,h}\psi_i(m)a &= u_{g,h}\psi_i(\mathrm{Ad}(\lambda_G)_{gh}(m'))a \\ &\overset{(ii)}{=} u_{g,h}\alpha_{gh}(\psi_i(m'))a \\ &= \alpha_g\alpha_h(\psi_i(m'))u_{g,h}a \\ &\overset{(ii)}{=} \psi_i(\mathrm{Ad}(\lambda_G)_{gh}(m'))u_{g,h}a \in S. \end{aligned}$$

As $A \cong A \otimes M_{|G|^\infty}$, pick a unital embedding from $B(l^2(G))$ into $F(S, A_\infty)$. (As $A \otimes \mathbb{M}_{|G|^\infty} \cong A$, there exists a unital embedding of $B(l^2(G))$ into $F(A)$ by [47, Theorem 2.2]. Moreover, by reindexing, one can also choose a homomorphism as stated.) It follows from Lemma 4.2 that there exists a unital embedding $B(l^2(G)) \to F(S, A_\infty)^\alpha$. Let $(e'_{g,h})_{g,h\in G}$ in $F(S, A_\infty)^\alpha$ be the images of $e_{g,h}$ under this unital embedding. The permutation unitary $v_g = \sum_{h\in G} e'_{gh,h}$ gives a unitary representation of G on $F(S, A_\infty)^\alpha$ and as $\alpha_g(v_h) = v_h$ it follows that v_g is an α-cocycle. Therefore, by Lemma 4.3, there exists a unitary $u \in F(S, A_\infty)$ such that $u\alpha_g(u^*) = v_g$. Now, $f_{g,h} = u^* e'_{g,h} u$ for $g, h \in G$ is a set of matrix units such that

$$\begin{aligned} \alpha_k(f_{g,h}) &= \alpha_k(u^*)e'_{g,h}\alpha_k(u) \\ &= u^* v_k e'_{g,h} v_k^* u \\ &= u^* \left(\sum_{h',h''\in G} e'_{kh',h'} e'_{g,h} e'_{h'',kh''} \right) u \\ &= u^*(e'_{kg,kh})u \\ &= f_{kg,kh}. \end{aligned}$$

Hence, the *-homomorphism

$$\begin{aligned} \phi_{n+1} : \mathcal{B}(l^2(G)) &\to F(S, A_\infty), \\ e_{g,h} &\mapsto f_{g,h} \end{aligned}$$

defines an $\mathrm{Ad}(\lambda_G)$ to α equivariant *-homomorphisms. Moreover, the image of ϕ_{n+1} commutes with ϕ_i for all $1 \leq i \leq n$. Considering ϕ_{n+1} as a unital equivariant homomorphism into $A_\infty \cap A'/A_\infty \cap A^\perp$, the induction argument is complete. ∎

We have collected all the necessary ingredients to prove the model action absorption.

Proposition 4.5 *Let G be a finite group, and let A be a separable C^*-algebra such that $A \cong A \otimes \mathbb{M}_{|G|^\infty}$. Let (α, u) be a (G, ω) action on A with the Rokhlin property. Then (α, u) and $(\alpha \otimes s_G, u \otimes 1_{\mathbb{M}_{|G|^\infty}})$ are cocycle conjugate through an isomorphism that is approximately unitarily equivalent to $\mathrm{id}_A \otimes 1_{\mathbb{M}_{|G|^\infty}}$.*

Proof By Lemma 4.4, there exists a G-equivariant unital embedding $(\mathbb{M}_{|G|^\infty}, s_G) \to (F(A), \alpha)$. Thus, the result follows from Theorem 4.1. ∎

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

5 Classification

We now discuss the abstract approach to bootstrapping the classification of group actions on a given class of C*-algebras to a classification of anomalous actions. This method is a generalization of that used by Connes in [8, Section 6], a similar strategy was recently used in [24] to classify G-kernels of poly-$\mathbb{Z}$ groups on $\mathcal{O}_2$.

Before proceeding with the result, we set up notation. For a group G, we say "(α, u) is an anomalous G-action on A" and "(A, α, u) is an anomalous G-C*-algebra" interchangeably. Let Λ be a functor whose domain category is the category of C*-algebras (denoted **C*alg**). We say Λ is *invariant under approximate unitary equivalence* if $\Lambda(\alpha) = \Lambda(\theta)$ whenever $\alpha \approx_{a.u} \theta$ (see Definition 4.1 for notation). We also say that Λ restricted to a subcategory $\mathcal{C} \subset$ **C*alg** *is full on isomorphisms*, if whenever $\Phi \in \mathrm{Hom}(\Lambda(A), \Lambda(B))$ is an isomorphism for $A, B \in \mathcal{C}$, then there exists an isomorphism $\varphi : A \to B$ in $\mathcal{C}$ with $\Lambda(\varphi) = \Phi$. The sort of functors with these properties are those used in the classification of C*-algebras. For example, the functor K from the category of unital C*-algebras into the category consisting of pairs of an abelian group and a pointed abelian group defined at the level of objects by $A \mapsto K(A) = ((K_0(A), [1_A]), K_1(A))$ is invariant under approximate unitary equivalence. The functor K is also full on isomorphisms when restricted to the category of unital Kirchberg algebras satisfying the UCT (see [39]). Similarly, the functors KT_u and $\underline{K}T_u$ of [5] are invariant under approximate unitary equivalence and are full on isomorphisms when restricted to classifiable C*-algebras.

If Λ is invariant under unitary equivalence, an anomalous action (A, α, u) induces a G-action on $\Lambda(A)$ through the automorphisms $\Lambda(\alpha_g)$. If (A, α, u) and (B, β, v) are anomalous actions, we say that the induced actions $\Lambda(\alpha_g)$ and $\Lambda(\beta_g)$ are *conjugate* if there exists an isomorphism $\Phi : \Lambda(A) \to \Lambda(B)$ with $\Phi\Lambda(\alpha_g)\Phi^{-1} = \Lambda(\beta_g)$ for all $g \in G$. We denote this by $\Lambda(\alpha) \sim \Lambda(\beta)$.

Let (A, α, u) and (A, β, v) be two anomalous G-C*-algebras. We write $(\alpha, u) \simeq_\Lambda (\beta, v)$ if $(\alpha, u) \simeq (\beta, v)$ through an automorphism θ with $\Lambda(\theta) = \mathrm{id}_{\Lambda(A)}$. This notion recovers K-trivial cocycle conjugacy of Definition 2.5 when Λ is taken to be the functor consisting of $K_0 \oplus K_1$. Finally, if $\mathfrak{R}$ is a class of anomalous G-C*-algebras, we will say $\mathfrak{R}$ is *closed under conjugacy*, if whenever $(A, \alpha, u) \in \mathfrak{R}$ and $\varphi : A \to B$ is an isomorphism in **C*alg** then $(B, \varphi\alpha\varphi^{-1}, \varphi(u)) \in \mathfrak{R}$.

Lemma 5.1 *Let G be a group, $\mathcal{D}$ a strongly self-absorbing C*-algebra, and $\mathfrak{R}$ a class of anomalous G-C*-algebras that is closed under conjugacy. Let Λ be a functor with domain category the category of C*-algebras such that Λ is invariant under approximate unitary equivalence and is full on isomorphisms for C*-algebras in $\mathfrak{R}$. Suppose further that:*

- *(A1) there exists a G-action $(\mathcal{D}, s_G, 1)$ such that if $(A, \alpha, u) \in \mathfrak{R}$, then $(A, \alpha, u) \simeq (A \otimes \mathcal{D}, \alpha \otimes s_G, u \otimes 1)$ through an isomorphism that is approximately unitarily equivalent to $\mathrm{id}_A \otimes 1_{\mathcal{D}}$;*
- *(A2) if there exists a (G, ω) action in $\mathfrak{R}$ for some $\omega \in Z^3(G, \mathbb{T})$, then there exist a (G, ω) and $(G, \overline{\omega})$ action $(\mathcal{D}, s_G^{\omega}, u^{\omega})$ and $(\mathcal{D}, s_G^{\overline{\omega}}, u^{\overline{\omega}})$, respectively, such that $(\mathcal{D}, s_G^{\overline{\omega}}, u^{\overline{\omega}}) \otimes (\mathcal{D}, s_G^{\omega}, u^{\omega}) \simeq (\mathcal{D}, s_G, 1)$ and for any (G, ω)-action $(A, \alpha, u) \in \mathfrak{R}$, $(A, \alpha, u) \otimes (\mathcal{D}, s_G^{\overline{\omega}}, u^{\overline{\omega}}) \in \mathfrak{R}$;*
- *(A3) for cocycle actions $(A, \alpha, u), (B, \beta, v) \in \mathfrak{R}$, $\Lambda(\alpha) \sim \Lambda(\beta)$ if and only if $\alpha \simeq \beta$.*

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

Then, if (A,α,u) *and* (B,β,v) *in* $\mathfrak{R}$, $(A,\alpha,u)\simeq(B,\beta,v)$ *if and only if* $\Lambda(\alpha)\sim\Lambda(\beta)$ *and* $o(\alpha,u)=o(\beta,v)$.

With the same hypothesis but replacing (A3) with the condition that

(A3') for cocycle actions (A,α,u) *and* (A,β,v) *in* $\mathfrak{R}$, $(A,\alpha,u)\simeq_\Lambda(A,\beta,v)$ *if and only if* $\Lambda(\alpha_g)=\Lambda(\beta_g)$ *for all* $g\in G$,

then if (A,α,u) *and* (A,β,v) *in* $\mathfrak{R}$, $(A,\alpha,u)\simeq_\Lambda(A,\beta,v)$ *if and only if* $o(\alpha,u)=o(\beta,v)$ *and* $\Lambda(\alpha_g)=\Lambda(\beta_g)$ *for every* $g\in G$.

Proof First we show that if (A1)–(A3) hold and (A,α,u), (B,β,v) are anomalous actions in $\mathfrak{R}$, then $(A,\alpha,u)\simeq(B,\beta,v)$ if and only if $\Lambda(\alpha)\sim\Lambda(\beta)$ and $o(\alpha,u)=o(\beta,v)$. If $(A,\alpha,u)\simeq(B,\beta,v)$, it is clear that $o(\alpha,u)=o(\beta,v)$ and also that $\Lambda(\alpha)\sim\Lambda(\beta)$ as Λ is trivial when evaluated at inner automorphisms. We now turn to the converse. Suppose $\Lambda(\alpha)\sim\Lambda(\beta)$ and $o(\alpha,u)=o(\beta,v)$. First, note that this implies that also $\Lambda(\alpha\otimes\mathrm{id}_{\mathcal{D}})\sim\Lambda(\beta\otimes\mathrm{id}_{\mathcal{D}})$. Indeed, by (A1), let $\phi_A:A\to A\otimes\mathcal{D}$ and $\phi_B:B\to B\otimes\mathcal{D}$ be isomorphisms which are approximately unitarily equivalent to the first factor embeddings and $\Phi:\Lambda(A)\to\Lambda(B)$ be an isomorphism such that $\Phi\Lambda(\alpha_g)\Phi^{-1}=\Lambda(\beta_g)$ for $g\in G$. Note that $\Lambda(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})\Lambda(\phi_A)=\Lambda(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})\Lambda(\mathrm{id}_A\otimes 1_{\mathcal{D}})=\Lambda(\alpha_g\otimes 1_{\mathcal{D}})=\Lambda(\phi_A)\Lambda(\alpha_g)$ (and similarly replacing A by B). Hence, we compute that

$$\begin{aligned}\Lambda(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})\Lambda(\phi_A)\Phi\Lambda(\phi_B)^{-1}&=\Lambda(\phi_A)\Lambda(\alpha_g)\Phi\Lambda(\phi_B)^{-1}\\&=\Lambda(\phi_A)\Phi\Lambda(\beta_g)\Lambda(\phi_B)^{-1}\\&=\Lambda(\phi_A)\Phi\Lambda(\phi_B)^{-1}\Lambda(\beta_g\otimes\mathrm{id}_{\mathcal{D}}),\end{aligned}$$

it follows that $\Lambda(\phi_B)\Phi\Lambda(\phi_A)^{-1}$ conjugates $\Lambda(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})$ to $\Lambda(\beta_g\otimes\mathrm{id}_{\mathcal{D}})$ for all $g\in G$. Now, by hypothesis, we have that

$$\begin{aligned}(A,\alpha,u)&\overset{(A1)}{\simeq}(A\otimes\mathcal{D},\alpha\otimes s_G,u\otimes 1_{\mathcal{D}})\\&\overset{(A2)}{\simeq}(A\otimes(\mathcal{D}\otimes\mathcal{D}),\alpha\otimes(s_G^{\overline{\omega}}\otimes s_G^{\omega}),u\otimes(u^{\overline{\omega}}\otimes u^{\omega}))\\&=((A\otimes\mathcal{D})\otimes\mathcal{D},(\alpha\otimes s_G^{\overline{\omega}})\otimes s_G^{\omega},(u\otimes u^{\overline{\omega}})\otimes u^{\omega})\\&\overset{(A3),(A2)}{\simeq}((B\otimes\mathcal{D})\otimes\mathcal{D},(\beta\otimes s_G^{\overline{\omega}})\otimes s_G^{\omega},(v\otimes u^{\overline{\omega}})\otimes u^{\omega})\\&=(B\otimes(\mathcal{D}\otimes\mathcal{D}),\beta\otimes(s_G^{\overline{\omega}}\otimes s_G^{\omega}),v\otimes(u^{\overline{\omega}}\otimes u^{\omega}))\\&\overset{(A2)}{\simeq}(B\otimes\mathcal{D},\beta\otimes s_G,v\otimes 1_{\mathcal{D}})\\&\overset{(A1)}{\simeq}(B,\beta,v).\end{aligned}\tag{5.1}$$

Where in the third isomorphism we have used (A3) for the cocycle actions $(A\otimes\mathcal{D},\alpha_g\otimes s_G^{\overline{\omega}},u\otimes u^{\overline{\omega}})$ and $(B\otimes\mathcal{D},\beta_g\otimes s_G^{\overline{\omega}},v\otimes u^{\overline{\omega}})$. The reason we may apply (A3) in this setting is that $s_G^{\overline{\omega}}$ is approximately inner and hence our previous computation shows that $\Lambda(\alpha_g\otimes s_G^{\overline{\omega}})=\Lambda(\alpha_g\otimes\mathrm{id}_{\mathcal{D}})\sim\Lambda(\beta_g\otimes\mathrm{id}_{\mathcal{D}})=\Lambda(\beta_g\otimes s_G^{\overline{\omega}})$ as required for the application of (A3).

Now suppose that we replace condition (A3) with (A3'). We will show that under the hypothesis of the lemma, (A3') implies (A3). Therefore, the cocycle conjugacies in (5.1) still hold. Then we compute the isomorphisms that induce the cocycle

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

conjugacies in (5.1) and show that their composition is the identity after applying Λ. Let (A,α,u) and (B,β,v) be cocycle actions in $\mathfrak{R}$. Suppose $\Lambda(\alpha)\sim\Lambda(\beta)$. There exists an isomorphism $\Phi\in\mathrm{Hom}(\Lambda(A),\Lambda(B))$ such that $\Phi\Lambda(\beta_g)\Phi^{-1}=\Lambda(\alpha_g)$ for all $g\in G$. As Λ is full on isomorphisms, there exists a *-isomorphism $\varphi: B\to A$ with $\Lambda(\varphi)=\Phi$. Therefore, $\Lambda(\varphi\beta_g\varphi^{-1})=\Lambda(\alpha_g)$ for all $g\in G$. By (A3'), one has that $(A,\alpha,u)\simeq_\Lambda (A,\varphi\beta\varphi^{-1},\varphi(v))\simeq(B,\beta,v)$.

Set $A=B$ in (5.1). Reading from top to bottom in (5.1), denote by $\varphi_1,\varphi_2,\varphi_3,\varphi_4$, and φ_5 the isomorphisms inducing each of the conjugacies. Note that $\varphi_5=\varphi_1^{-1}$ and $\varphi_4=\varphi_2^{-1}$. By (A1), $\varphi_1\approx_{a.u}\mathrm{id}_A\otimes 1_{\mathcal{D}}$. Moreover, $\varphi_2\approx_{a.u}\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{D}}\otimes 1_{\mathcal{D}}$ by [47, Corollary 1.12]. Denote by φ the isomorphism inducing the cocycle conjugacy from $(A\otimes\mathcal{D},\alpha\otimes s_G^{\overline{\omega}},u\otimes u^{\overline{\omega}})$ to $(A\otimes\mathcal{D},\beta\otimes s_G^{\overline{\omega}},u\otimes u^{\overline{\omega}})$ which satisfies $\Lambda(\varphi)=\Lambda(\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{D}})$. We may use the functoriality of Λ and its invariance under approximate unitary equivalence to see that

$$\begin{aligned}\Lambda(\varphi_5\varphi_4\varphi_3\varphi_2\varphi_1)&=\Lambda(\mathrm{id}_A\otimes 1_{\mathcal{D}}\otimes 1_{\mathcal{D}})^{-1}\Lambda(\varphi\otimes\mathrm{id}_{\mathcal{D}})\Lambda(\mathrm{id}_A\otimes 1_{\mathcal{D}}\otimes 1_{\mathcal{D}})\\&=\Lambda(\mathrm{id}_A\otimes 1_{\mathcal{D}}\otimes 1_{\mathcal{D}})^{-1}\Lambda(\mathrm{id}_A\otimes\mathrm{id}_{\mathcal{D}}\otimes\mathrm{id}_{\mathcal{D}})\Lambda(\mathrm{id}_A\otimes 1_{\mathcal{D}}\otimes 1_{\mathcal{D}})\\&=\mathrm{id}_{\Lambda(A)}\,.\end{aligned}$$ ■

We now prove our classification theorems.

Theorem 5.2 *Let G be a finite group and A be a unital Kirchberg algebra satisfying the UCT with $A\cong A\otimes\mathbb{M}_{|G|^\infty}$. If (α,u), (β,v) are anomalous actions of G on A with the Rokhlin property, then $(\alpha,u)\simeq_K(\beta,v)$ if and only if $o(\alpha,u)=o(\beta,v)$ and $K_i(\alpha_g)=K_i(\beta_g)$ for all $g\in G$ and $i=0,1$.*

Proof We check that the hypothesis of Lemma 5.1 is satisfied. Let $\mathcal{D}=\mathbb{M}_{|G|^\infty}$, Λ be the functor given by the pointed K_0 group direct sum the K_1 group, i.e., $\Lambda(A)=((K_0(A),[1_A]),K_1(A))$, and $\mathfrak{R}$ the class of Rokhlin anomalous G-actions on unital Kirchberg algebras satisfying the UCT that absorb $\mathbb{M}_{|G|^\infty}$. That Λ is full on isomorphisms follows from [39]. Condition (A1) follows from Proposition 4.5. For any $\omega\in Z^3(G,\mathbb{T})$, we have actions $(\mathcal{D},s_G^\omega,u^\omega)$ as discussed in Section 3. That $(\mathcal{D},s_G^{\overline{\omega}},u^{\overline{\omega}})\otimes(\mathcal{D},s_G^\omega,u^\omega)\simeq(\mathcal{D},s_G,1)$ follows from [19, Theorem III.6] combined with [22, Lemma 3.12] as the actions $(\mathcal{D},s_G^\omega,u^\omega)$ have the Rokhlin property by Proposition 3.1. Therefore, (A2) is also satisfied. Finally, (A3') is satisfied by Izumi's classification result [23, Theorem 4.2] and that every cocycle action with the Rokhlin property is a unitary perturbation of a group action [23, Lemma 3.12]. ■

Theorem 5.3 *Let G be a finite group and A be a unital, simple, nuclear TAF-algebra in the UCT class such that $A\cong A\otimes\mathbb{M}_{|G|^\infty}$ and $(\alpha,u),(\beta,v)$ are anomalous actions on A with the Rokhlin property, then $(\alpha,u)\simeq_K(\beta,v)$ if and only if $o(\alpha,u)=o(\beta,v)$ and $K_i(\alpha_g)=K_i(\beta_g)$ for all $g\in G$ and $i=0,1$.*

Proof We apply Lemma 5.1 with $\mathcal{D}=\mathbb{M}_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $\mathbb{M}_{|G|^\infty}$-stable unital, simple, separable, nuclear TAF-algebras satisfying the UCT and Λ the functor consisting of the ordered, pointed K_0 functor direct sum K_1. First, Λ is full on isomorphisms by [34]. (A1) holds by Proposition 4.5. (A2) holds for the same reason as in the proof of Theorem 5.2. Condition (A3') follows from a combination of [23, Theorem 4.3] and [22, Lemma 3.12]. ■

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

Similarly, one may classify anomalous actions with the Rokhlin property on the Razak–Jacelon algebra $\mathcal{W}$.

Theorem 5.4 *Let G be a finite group and (α, u), (β, v) be anomalous G actions with the Rokhlin property on $\mathcal{W}$. Then $(\alpha, u) \simeq (\beta, v)$ if and only if $o(\alpha, u) = o(\beta, v)$.*

Proof We check the conditions of Lemma 5.1 with $\mathcal{D} = \mathbb{M}_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $\mathcal{W}$ and Λ the trivial functor. First, (A1) holds by Proposition 4.5. Moreover, (A2) holds as in the proof of Theorem 5.2. Finally, (A3) follows from [35, Corollary 3.7] as every cocycle action of a finite group on $\mathcal{W}$ is cocycle conjugate to a group action (this follows as $\mathcal{W} \cong \mathcal{W} \otimes M_{|G|}$ and hence [15, Remark 1.5] applies). ∎

In light of [5, Theorem B], it follows from [22, Theorem 3.5] that all Rokhlin anomalous actions of G on classifiable $\mathbb{M}_{|G|^\infty}$-stable C*-algebras are classified up to cocycle conjugacy by their induced action on the total invariant $\underline{K}T_u$ (see [5, Section 3]) and their anomaly.

Corollary 5.5 *Let G be a finite group. Let A be a unital, simple, separable, nuclear, $\mathbb{M}_{|G|^\infty}$-stable C*-algebra satisfying the UCT and (α, u), (β, v) be anomalous G-actions with the Rokhlin property on A. Then $(\alpha, u) \simeq (\beta, v)$ if and only if $\underline{K}T_u(\alpha) \sim \underline{K}T_u(\beta)$ and $o(\alpha, u) = o(\beta, v)$.*

Proof We apply Lemma 5.1 with $\mathcal{D} = \mathbb{M}_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $\mathbb{M}_{|G|^\infty}$-stable unital, simple, separable, nuclear C*-algebras satisfying the UCT and $\Lambda = \underline{K}T_u$. First, Λ is full on isomorphisms by [5, Theorem A]. (A1) holds by Proposition 4.5. (A2) holds as in the proof of Theorem 5.2. It remains to show (A3). By [22, Lemma 3.12], it suffices to show that for any two Rokhlin G-actions (A, α) and (B, β) such that $\underline{K}T_u(\alpha) \sim \underline{K}T_u(\beta)$ then $\alpha \simeq \beta$. This has been shown for simple, unital AH-algebras in [16, Theorem 3.8]. With [5, Theorem B] in hand, this also follows for arbitrary unital, simple, separable, nuclear, $\mathcal{Z}$-stable C*-algebras satisfying the UCT. Indeed, as $\underline{K}T_u$ is full on isomorphisms, there exists an isomorphism $\theta : A \to B$ such that $\underline{K}T_u(\theta\alpha_g\theta^{-1}) = \underline{K}T_u(\beta_g)$ for all $g \in G$. Therefore, it follows from [5, Theorem B] that $\theta\alpha_g\theta^{-1} \approx_{a.u} \beta_g$. Now, it follows immediately from [22, Theorem 3.5] that $\alpha \simeq \beta$. ∎

We illustrate another application of Lemma 5.1 to the classification of Rokhlin anomalous actions on a class of non-simple C*-algebras. Precisely, we can classify Rokhlin anomalous actions on inductive limits of one-dimensional NCCW complexes with trivial K_1-groups as a consequence of the classification results of [16, Section 3.3.1].

Theorem 5.6 *Let G be a finite group and A be a C*-algebra that can be written as an inductive limit of one-dimensional NCCW complexes with trivial K_1 groups satisfying $A \cong A \otimes M_{|G|^\infty}$. If (α, u), (β, v) are anomalous actions of G on A, then $(\alpha, u) \simeq (\beta, v)$ through an automorphism that is approximately inner if and only if $o(\alpha, u) = o(\beta, v)$ and $\mathrm{Cu}^\sim(\alpha_g) = \mathrm{Cu}^\sim(\beta_g)$ for all $g \in G$.*[8]

[8] See [16, Section 2.2] for the definition of the functor $\mathrm{Cu}^\sim$.

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

Proof We apply Lemma 5.1 with $\mathcal{D} = M_{|G|^\infty}$, $\mathfrak{R}$ the class of Rokhlin anomalous actions on $M_{|G|^\infty}$-stable C*-algebras that can be written as an inductive limit of one-dimensional NCCW complexes with trivial K_1 groups and $\Lambda = \mathrm{Cu}^\sim$. First, Λ is invariant under approximate unitary equivalence. Moreover, it is full on isomorphisms by [41, Theorem 1.0.1] (see also [41, Corollary 5.2.3]). Conditions (A1) and (A2) hold as in the proof of Theorem 5.2. Condition (A3') holds as a consequence of [16, Theorem 3.6] (note also that $M_{|G|}(A) \cong A$ so [15, Remark 1.5] applies). Now, it follows from Lemma 5.1 that any two Rokhlin anomalous actions $(\alpha, u), (\beta, v)$ of G on an inductive limit of one-dimensional NCCW complex satisfy $(\alpha, u) \simeq_{\mathrm{Cu}^\sim} (\beta, v)$. But any automorphism of an inductive limit of one-dimensional NCCW complexes with trivial K_1 groups that is the identity under $\mathrm{Cu}^\sim$ is approximately inner by [41, Theorem 1]. ∎

Remark 5.7 Note that, by [16, Theorem 5.2], the UHF-stability assumption in Theorem 5.6 is immediate for the following subclasses:

(i) unital C*-algebras that can be written as inductive limits of one-dimensional NCCW-complexes;
(ii) simple C*-algebras with trivial K_0-groups that can be written as inductive limits of one-dimensional NCCW-complexes;
(iii) C*-algebras that can be written as inductive limits of punctured-tree algebras.

We have shown a classification of anomalous actions on some classes of simple C*-algebras. Such a classification also implies a classification of G-kernels, we illustrate it by using Theorem 5.2, the same argument may also be used to rewrite the results of Theorem 5.4, Theorem 5.3, and Corollary 5.5. As in the case of group actions, we say two G-kernels $\overline{\alpha}$ and $\overline{\beta}$ on a C*-algebra A are K trivially conjugate if there exists an automorphism $\theta \in \mathrm{Aut}(A)$ with $K_i(\theta) = \mathrm{id}_{K_i(A)}$ and $\overline{\theta\alpha_g\theta^{-1}} = \overline{\beta_g}$ for all $g \in G$.

Corollary 5.8 *Let A be a unital Kirchberg algebra satisfying the UCT with $A \cong A \otimes M_{|G|^\infty}$ and $\overline{\alpha}, \overline{\beta}$ be G-kernels with the Rokhlin property on A. Then $\overline{\alpha}$ and $\overline{\beta}$ are K trivially conjugate if and only if $\mathrm{ob}(\overline{\alpha}) = \mathrm{ob}(\overline{\beta})$ and $K_i(\overline{\alpha}_g) = K_i(\overline{\beta}_g)$ for all $g \in G$ and $i = 0, 1$.*

Proof The forward direction is clear. To show the reverse direction, pick lifts (α, u) of $\overline{\alpha}$ and (β, v) of $\overline{\beta}$ such that $o(\alpha, u) = o(\beta, v)$. As (α, u) and (β, v) satisfy the hypothesis of Theorem 5.2, it follows that $(\alpha, u) \simeq (\beta, v)$ and so $\overline{\alpha}$ and $\overline{\beta}$ are conjugate. ∎

6 Applications

We start this section by giving an alternative construction of a (G, ω) action on the UHF algebra $M_{|G|^\infty}$ which is visibly compatible with a Bratteli diagram of $M_{|G|^\infty}$. This action is an AF-action in the sense of [11] and [6, Definition 4.8] (see also the discussion in [17, Section 6.1]). The existence of an AF ω-anomalous action on $M_{|G|^\infty}$ follows from an adaptation of the Ocneanu compactness argument to the C*-setting [37]. We build it explicitly below. Before we do so, let us recall the definition of an AF anomalous action.

Definition 6.1 Let A be a unital AF-C*-algebra and (α, u) be a (G, ω)-action on A. We say (α, u) is an *AF anomalous action* if there exists an inductive limit (A_n, φ_n)

consisting of finite-dimensional C*-algebras A_n with unital connecting maps φ_n and (G,ω) actions (α_n, u_n) on A_n such that:

(1) $\varphi_n \alpha_n = \alpha_{n+1}\varphi_n$, $\forall n \in \mathbb{N}$,
(2) $\varphi_n(u_n) = u_{n+1}, \forall n \in \mathbb{N}$,
(3) (A, α, u) is cocycle conjugate to $\varinjlim(A_n, \varphi_n, \alpha_n, u_n)$,

where $\varinjlim(A_n, \varphi_n, \alpha_n, u_n)$ is the C*-algebra $\varinjlim(A_n, \varphi_n)$ with the canonical anomalous action induced by the sequence (α_n, u_n) (see [17, Section 6.1] for more details).

Proposition 6.1 *Let G be a finite group and $\omega \in Z^3(G, \mathbb{T})$, then there exists an AF ω-anomalous G action with the Rokhlin property on $M_{|G|^\infty}$. We denote this action by θ_G^ω.*

Proof In this proof, we will use the symbols $g, h, k, x, y, x_i, y_i, s_i$ for $i \in \mathbb{N}$ to denote elements of the group G. Let $A_n = C(G) \otimes \bigotimes_{i=1}^{n-1} \mathcal{B}(l^2(G))$ for $n \in \mathbb{N}$, where by convention $A_1 = C(G)$. For $f \in C(G)$, let $M_f \in \mathcal{B}(l^2(G))$ be the multiplication operator by f. Consider the *-homomorphisms $\varphi_n : A_n \to A_{n+1}$ defined by $\varphi_n(f \otimes T) = 1 \otimes M_f \otimes T$ for $f \in C(G)$ and $T \in \bigotimes_{i=1}^{n-1} \mathcal{B}(l^2(G))$.

The inductive system (A_n, φ_n) has an inductive limit (we write the limit by A) which is known to be isomorphic to $\mathbb{M}_{|G|^\infty}$. Indeed, the Bratelli diagram of this AF-algebra is easily seen to be the complete bipartite graph on $|G|$-vertices, it is common knowledge that this coincides with the UHF-algebra of type $|G|^\infty$ (see [9, Example III.2.4] for the case $|G| = 2$). We construct a (G,ω) action on each finite-dimensional algebra A_n such that the actions commute with the inclusion maps φ_n. This will induce an AF ω-anomalous G action on $M_{|G|^\infty}$ by the universal property of the inductive limit (see [17, Section 6.1]).

To be precise, we construct a family of maps $\theta_n : G \to \mathrm{Aut}(A_n)$ and $u_n : G \times G \to U(A_n)$ such that:

(1) $\theta_n(g)\theta_n(h) = \mathrm{Ad}(u_n(g,h))\theta_n(gh)$,
(2) $\omega_{g,h,k} = \theta_n(g)(u_n(h,k))u_n(g,hk)u_n(gh,k)^* u_n(g,h)^*$,
(3) $\varphi_n(u_n(g,h)) = u_{n+1}(g,h)$,
(4) $\varphi_n\theta_n(g) = \theta_{n+1}(g)\varphi_n$,

for all $n \in \mathbb{N}$. To build this, we will consider the group actions $\theta_n' : G \to \mathrm{Aut}(A_n)$ defined by $\theta_n'(g) = \lambda_G(g) \otimes \bigotimes_{i=1}^{n-1} \mathrm{Ad}(\lambda_G)_g$ where λ_G is the left regular representation of G. Note that $\varphi_n\theta_n'(g) = \theta_{n+1}'(g)\varphi_n$. To take into account the anomaly, we will tweak θ_n' by suitable diagonal operators $d_n \in \mathrm{Aut}(A_n)$ and ensuring that (13) and (13) hold. To define d_n, we start by introducing some notation. Let $\delta_k \in C(G)$ be the point mass at k, i.e.,

$$\delta_k(g) = \begin{cases} 1, & \text{if } g = k, \\ 0, & \text{otherwise.} \end{cases}$$

We now let

$$\theta_n(g) = d_n(g)\theta_n'(g)$$

with $d_n(g)$ defined inductively

$$\begin{aligned} d_1(g) &= \mathrm{id}_{A_1}, \\ d_2(g)(\delta_k \otimes e_{x_1,y_1}) &= \omega_{x_1^{-1},g,g^{-1}k}\overline{\omega_{y_1^{-1},g,g^{-1}k}}(\delta_k \otimes e_{x_1,y_1}), \end{aligned}$$

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

and

$$\begin{aligned} d_n(g)(\delta_k \otimes e_{x_1,y_1} \otimes \cdots \otimes e_{x_{n-1},y_{n-1}}) \\ = \omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}} \overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}} \omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}} \omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}} \\ (d_{n-2}(g)(\delta_k \otimes e_{x_1,y_1} \cdots \otimes e_{x_{n-3},y_{n-3}}) \otimes e_{x_{n-2},y_{n-2}} \otimes e_{x_{n-1},y_{n-1}}) \end{aligned}$$

for all $n > 2$ with the convention that $x_0 = y_0 = k$. As we have defined $d_n(g)$ on a spanning set of A_n, $d_n(g)$ extend to linear maps from A_n to itself. In fact, each $d_n(g)$ is an endomorphism of A_n. First, it is clear that they preserve the *-operation. To show the multiplicativity, it is sufficient to check on a spanning set. We show this by induction. For the case $n = 2$, it is only nontrivial to check that

$$d_2(g)(\delta_k \otimes e_{x_1,y_1}) d_2(g)(\delta_k \otimes e_{y_1,y_2}) = d_2(g)(\delta_k \otimes e_{x_1,y_2}).$$

The left-hand side is given by

$$\begin{aligned} & d_2(g)(\delta_k \otimes e_{x_1,y_1}) d_2(g)(\delta_k \otimes e_{y_1,y_2}) \\ & = \omega_{x_1^{-1},g,g^{-1}k} \overline{\omega_{y_1^{-1},g,g^{-1}k}} \omega_{y_1^{-1},g,g^{-1}k} \overline{\omega_{y_2^{-1},g,g^{-1}k}} (\delta_k \otimes e_{x_1,y_2}) \\ & = \omega_{x_1^{-1},g,g^{-1}k} \overline{\omega_{y_2^{-1},g,g^{-1}k}} (\delta_k \otimes e_{x_1,y_2}), \end{aligned}$$

which coincides with the right-hand side. To show that $d_n(g)$ is multiplicative, for $n > 2$, it suffices to show that

$$\begin{aligned} d_n(g)(\delta_k \otimes e_{x_1,y_1} \otimes \cdots \otimes e_{x_{n-1},y_{n-1}}) d_n(g)(\delta_k \otimes e_{y_1,s_1} \otimes \dots . \otimes e_{y_{n-1},s_{n-1}}) \\ = d_n(g)(\delta_k \otimes e_{x_1,s_1} \otimes \dots . \otimes e_{x_{n-1},s_{n-1}}). \end{aligned}$$

This follows immediately from the induction hypothesis and a direct computation of the left-hand side (as in the case for $n = 2$). Notice that each $d_n(g)$ fixes elements of the form $\delta_k \otimes e_{x_1,x_1} \otimes e_{x_2,x_2} \cdots \otimes e_{x_{n-1},y_{n-1}}$.

To construct a (G, ω) action on the first stage A_1, we let $u_1(g, h)(k) = \omega_{k^{-1},g,h}$. That (θ_1, u_1) defines a (G, ω) action on $C(G)$ is a straightforward computation (this is computed in [3, Section 4]). We proceed to extend this action on A_1 to all of $M_{|G|^\infty}$ through the inductive limit. Let $u_n(g, h) = \varphi_{1,n}(u_1(g, h))$ and $\theta_n(g) = d_n(g)\theta'_n(g)$. For the remaining part of the proof, we check that (θ_n, u_n) satisfy (13)–(13) for all $n \in \mathbb{N}$. We will repeatedly use the 3-cocycle formula during the calculations, instead of commenting on this every time, we will instead color-code the parts of our equations to which we apply the 3-cocycle formula.

We start by showing (13). First,

$$\begin{aligned} \theta_n(g)\theta_n(h) &= d_n(g)\theta'_n(g)d_n(h)\theta'_n(h) \\ &= d_n(g)\theta'_n(g)d_n(h)\theta'_n(g)^{-1}\theta'_n(gh) \\ &= d_n(g)[g \cdot d_n(h)]\theta'_n(gh), \end{aligned}$$

denoting $g \cdot d_n(h) = \theta'_n(g)d_n(h)\theta'_n(g)^{-1}$. It is clear that (13) holds for all $n \in \mathbb{N}$ if and only if $d_n(g)[g \cdot d_n(h)]d_n(gh)^{-1} = \mathrm{Ad}(u_n(g, h))$ on A_n for all $n \in \mathbb{N}$. This holds trivially for $n = 1$. For $n = 2$, it follows from the 3-cocycle formula that

$$
\begin{aligned}
&d_2(g)[g\cdot d_2(h)]d_2(gh)^{-1}(\delta_{ghk}\otimes e_{x_1,y_1})\\
&= d_2(g)[g\cdot d_2(h)](\delta_{ghk}\otimes e_{x_1,y_1})\overline{\omega_{x_1^{-1},gh,k}}\omega_{y_1^{-1},gh,k}\\
&= d_2(g)(\delta_{ghk}\otimes e_{x_1,y_1})\overline{\omega_{x_1^{-1},gh,k}}\omega_{y_1^{-1},gh,k}\omega_{x_1^{-1}g,h,k}\overline{\omega_{y_1^{-1}g,h,k}}\\
&= (\delta_{ghk}\otimes e_{x_1,y_1}){\color{red}\overline{\omega_{x_1^{-1},gh,k}}\omega_{x_1^{-1}g,h,k}\omega_{x_1^{-1},g,hk}}{\color{blue}\overline{\omega_{y_1^{-1},g,hk}}\omega_{y_1^{-1},gh,k}\overline{\omega_{y_1^{-1}g,h,k}}}\\
&= (\delta_{ghk}\otimes e_{x_1,y_1})\omega_{g,h,k}\omega_{x_1^{-1},g,h}\overline{\omega_{g,h,k}\omega_{y_1^{-1},g,h}}\\
&= \mathrm{Ad}(\varphi_1(u_1(g,h))(\delta_{ghk}\otimes e_{x_1,y_1}).
\end{aligned}
$$

We now proceed with an inductive argument for arbitrary n. We assume that (13) holds for $n-2$, preforming a similar computation to the case $n=2$:

$$
\begin{aligned}
&d_n(g)[g\cdot d_n(h)]d_n(gh)^{-1}(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
&= \big(\mathrm{Ad}(u_{n-2}(g,h))((\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}\big)\\
&{\color{red}\overline{\omega_{x_{n-1}^{-1},gh,x_{n-2}}}\omega_{x_{n-1}^{-1}g,h,x_{n-2}}\omega_{x_{n-1}^{-1},g,hx_{n-2}}}{\color{blue}\omega_{x_{n-3}^{-1},gh,x_{n-2}}\overline{\omega_{x_{n-3}^{-1}g,h,x_{n-2}}\omega_{x_{n-3}^{-1},g,hx_{n-2}}}}\\
&{\color{red}\omega_{y_{n-1}^{-1},gh,y_{n-2}}}\overline{{\color{red}\omega_{y_{n-1}^{-1}g,h,y_{n-2}}\omega_{y_{n-1}^{-1},g,hy_{n-2}}}{\color{blue}\omega_{y_{n-3}^{-1},gh,y_{n-2}}}}{\color{blue}\omega_{y_{n-3}^{-1}g,h,y_{n-2}}\omega_{y_{n-3}^{-1},g,hy_{n-2}}}\\
&= \big(\mathrm{Ad}(u_{n-2}(g,h))((\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}\big)\\
&\omega_{g,h,x_{n-2}}\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{g,h,x_{n-2}}\omega_{x_{n-3}^{-1},g,h}\omega_{g,h,y_{n-2}}\omega_{y_{n-1}^{-1},g,h}}\omega_{g,h,y_{n-2}}\omega_{y_{n-3}^{-1},g,h}\\
&= \big(\mathrm{Ad}(u_{n-2}(g,h))(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}\big)\\
&\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{x_{n-3}^{-1},g,h}\omega_{y_{n-1}^{-1},g,h}}\omega_{y_{n-3}^{-1},g,h}\\
&= (\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-1},y_{n-1}})\omega_{x_{n-1}^{-1},g,h}\overline{\omega_{y_{n-1}^{-1},g,h}}\\
&= \mathrm{Ad}(u_n(g,h))(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{ghx_{n-2},ghy_{n-2}}\otimes e_{x_{n-1},y_{n-1}}).
\end{aligned}
$$

For (13), it suffices to show that $\varphi_n d_n(g) = d_{n+1}(g)\varphi_n$. For $n=1$,

$$
\begin{aligned}
d_2(g)\varphi_1(\delta_k) &= \sum_{r\in G} d_2(g)(\delta_r\otimes e_{k,k})\\
&= (1\otimes e_{k,k})\\
&= \varphi_1 d_1(g)(\delta_k)
\end{aligned}
$$

as d_1 is the identity map. The case $n=2$ follows too as

$$
\begin{aligned}
d_3(g)\varphi_2(\delta_k\otimes e_{x,y}) &= \sum_{r\in G} d_3(g)(\delta_r\otimes e_{k,k}\otimes e_{x,y})\\
&= (1\otimes e_{k,k}\otimes e_{x,y})\omega_{x^{-1},g,g^{-1}k}\overline{\omega_{y^{-1},g,g^{-1}k}}\\
&= \varphi_2 d_2(g)(\delta_k\otimes e_{x,y}).
\end{aligned}
$$

Assuming that the case $n-2$ holds, we now argue, by induction,

$$
\begin{aligned}
&d_{n+1}(g)\varphi_n(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-1},y_{n-1}})\\
&= d_{n+1}(g)(\varphi_{n-2}(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{x_{n-2},y_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
&= (d_{n-1}(g)\varphi_{n-2}(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{x_{n-2},y_{n-2}}\otimes e_{x_{n-1},y_{n-1}})\\
&\omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}}\overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}}\omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}}\omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}}\\
&= (\varphi_{n-2}d_{n-2}(g)(\delta_k\otimes e_{x_1,y_1}\cdots\otimes e_{x_{n-3},y_{n-3}})\otimes e_{x_{n-2},y_{n-2}}\otimes e_{x_{n-1},y_{n-1}})
\end{aligned}
$$

https://doi.org/10.4153/S0008414X2400018X Published online by Cambridge University Press

$$\omega_{x_{n-1}^{-1},g,g^{-1}x_{n-2}}\overline{\omega_{x_{n-3}^{-1},g,g^{-1}x_{n-2}}\omega_{y_{n-1}^{-1},g,g^{-1}y_{n-2}}}\omega_{y_{n-3}^{-1},g,g^{-1}y_{n-2}}$$
$$= \varphi_n d_n(g)(\delta_k \otimes e_{x_1,y_1} \cdots \otimes e_{x_{n-1},y_{n-1}}).$$

Condition (13) is immediate. It remains to show that (13) holds for arbitrary n. This follows from (13) for the case $n = 1$ and from (13). For $n \in \mathbb{N}$,

$$\begin{aligned}
&\theta_n(g)(u_n(h,k))u_n(g,hk)u_n(gh,k)^* u_n(g,h)^* \\
&= \theta_n(g)(\varphi_{1,n}(u_1(h,k)))\varphi_{1,n}(u_1(g,hk))\varphi_{1,n}(u_1(gh,k)^*)\varphi_{1,n}(u_1(g,h)^*) \\
&= \varphi_{1,n}(\theta_1(g)(u_1(h,k))u_1(gh,k)u_1(g,hk)^* u_1(g,h)^*) \\
&= \omega_{g,h,k}\varphi_{1,n}(1_{A_1}) \\
&= \omega_{g,h,k}.
\end{aligned}$$

This completes the construction of the AF anomalous action θ_G^ω.

To show that θ_G^ω has the Rokhlin property, we construct a family of Rokhlin projections. The projections $\delta_g \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}} \in Z(A_n)$ satisfy $\theta_n(g)(\delta_h \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}}) = \delta_{gh} \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}}$ and also $\sum_{g\in G} \delta_g \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}} = \mathrm{id}_{A_n}$. Therefore, the projections $p_g \in A_\infty$ with the nth coordinate given by $\varphi_{n,\infty}(\delta_g \otimes \mathrm{id}_{\mathcal{B}(l^2(G))^{\otimes n-1}})$ for $g \in G$ satisfy the conditions of Definition 2.3. ∎

Remark 6.2 In the case that $\omega = 1$, the construction in Proposition 6.1 greatly simplifies. Indeed, $d_n(g)$ is the identity automorphism and $u_n(g,h)$ is the unit for all $g, h \in G$ and $n \in \mathbb{N}$. Therefore, θ_G^1 restricts to the group action $\theta_n = \lambda_G \otimes \bigotimes_{i=0}^{n-1} \mathrm{Ad}(\lambda_G)$ on each A_n with λ_G the left regular representation. This action coincides with the infinite tensor product action s_G (see Section 4). To see this, consider the inductive system (B_n, ϕ_n) with $B_{2n-1} = A_n$, $B_{2n} = \bigotimes_{i=0}^{n} B(l^2(G))$ and $\phi_{2n-1}(f \otimes T) = M_f \otimes T$, $\phi_{2n}(S) = 1 \otimes S$ for all $n \in N$, $f \otimes T \in A_n$, and $S \in B_{2n}$. The even terms of the inductive system $(B_{2n}, \phi_{2n+1} \circ \phi_{2n})$ coincide with the inductive limit $(\bigotimes_{i=1}^{n} B(l^2(G)), M \mapsto \mathrm{id}_{B(l^2(G))} \otimes M)$. The odd terms $(B_{2n-1}, \phi_{2n} \circ \phi_{2n-1})$ coincide with the inductive system (A_n, φ_n) from the proof of Proposition 6.1. This allows to interpolate between $(\bigotimes_{i=1}^{n} B(l^2(G)), M \mapsto \mathrm{id}_{B(l^2(G))} \otimes M)$ and (A_n, φ_n). It is immediate that θ_G and s_G are conjugate. Moreover, it follows from Theorem 5.3 that θ_G^ω is cocycle conjugate to s_G^ω for any $\omega \in Z^3(G, \mathbb{T})$.

We end this paper by studying to what extent Rokhlin anomalous actions on AF-algebras are AF-actions and vice versa. To do this, we will require results of [6]. In [6], the authors associate an invariant with any AF-action F, of a fusion category $\mathcal{C}$, on an AF-algebra A. Vaguely, this invariant consists of the K_0-groups of all Q-system extensions of A by F and all natural maps between these extensions. The authors also show that any two AF-actions on AF-algebras A and B are equivalent if and only if their invariants are isomorphic. As observed in [6, Section 5.1], if the acting category $\mathcal{C}$ is *torsion-free* (see [1, Definition 3.7]), the invariant of [6] simplifies to just the module structure of $K_0(A)$ under the action of the fusion ring of $\mathcal{C}$. We apply this when the acting category is $\mathbf{Hilb}(G, \omega)$ and the action is induced by an anomalous action (α, u) as explained in [13, Proposition 5.6]. The fusion ring of $\mathbf{Hilb}(G, \omega)$ is $\mathbb{Z}[G]$, and the module structure of $K_0(A)$ is given by $K_0(\alpha_g)$.

Corollary 6.3 *Let G be a finite group and A a simple, unital AF-algebra such that $A \cong A \otimes M_{|G|^\infty}$. Let (α, u) be a (G, ω)-action on A such that $K_0(\alpha_g) = \mathrm{id}_A$ for all $g \in G$.*

If (α, u) has the Rokhlin property, then (α, u) is an AF-action. Moreover, if $[\omega|_H] \neq 0$ for any subgroup $H < G$, then the converse holds.

Proof If (α, u) is a (G, ω)-action with the Rokhlin property on an AF-algebra A, then by Theorem 5.3 it is cocycle conjugate to the AF ω-anomalous G-action $\mathrm{id}_A \otimes \theta_G^\omega$ on A. Therefore, (α, u) is AF.

We now consider the converse statement. An AF ω-anomalous G action (α, u) induces an AF-action of the fusion category $\mathbf{Hilb}(G, \omega)$ in the sense of [6] (to see how a (G, ω)-action induces a $\mathbf{Hilb}(G, \omega)$ action, see [13, Proposition 5.6], that this is AF is discussed [17, Remark 6.1.7]). By the hypothesis on ω, the fusion category $\mathbf{Hilb}(G, \omega)$ is torsion-free, so as $K_0(\alpha_g) = \mathrm{id}_A$ and $K_0(\mathrm{id}_A \otimes \theta_G^\omega) = \mathrm{id}_A$, then [6, Theorem A] yields that the AF ω-anomalous G actions induced by (α, u) and $\mathrm{id}_A \otimes \theta_G^\omega$ are cocycle conjugate. So (α, u) has the Rokhlin property. ∎

Remark 6.4 One may drop the hypothesis that $A \cong A \otimes M_{|G|^\infty}$ in Corollary 6.3 if one instead assumes that the anomaly ω of (α, u) is such that $[\omega]$ has order $|G|$. Indeed, it follows from [17, Corollary 5.4.4] that in this case A will automatically absorb $M_{|G|^\infty}$. Also, note that under this assumption on $[\omega]$, it is automatic that $[\omega|_H] \neq 0$ for any subgroup $H < G$.

The behavior observed in the converse of Corollary 6.3 is quite different from the behavior of group actions. It was already observed in [14] that there exist AF-actions of $\mathbb{Z}_2$ on M_{2^∞} which do not have the Rokhlin property.

Acknowledgements The author would like to thank Samuel Evington, Eusebio Gardella, André Henriques, Corey Jones, Ulrich Pennig, and Stuart White for comments and discussions that have been useful for this paper. An initial version of this work forms part of the author's Ph.D. thesis [17].

References

[1] Y. Arano and K. De Commer, *Torsion-freeness for fusion rings and tensor C*-categories*. J. Noncommut. Geom. **13**(2019), no. 1, 35–58.

[2] S. Barlak and G. Szabó, *Rokhlin actions of finite groups on UHF-absorbing C*-algebras*. Trans. Amer. Math. Soc. **369**(2017), no. 2, 833–859.

[3] P. Bouwknegt, K. Hannabuss, and V. Mathai, *Nonassociative tori and applications to T-duality*. Commun. Math. Phys. **264**(2006), no. 1, 41–69.

[4] R. C. Busby and H. A. Smith, *Representations of twisted group algebras*. Trans. Amer. Math. Soc. **149**(1970), 503–537.

[5] J. R. Carrión, J. Gabe, C. Schafhauser, A. Tikuisis, and S. White, *Classifying *-homomorphisms I: unital simple nuclear C^*-algebras*. Preprint, 2023, arXiv:2307.06480.

[6] Q. Chen, R. H. Palomares, and C. Jones, *K-theoretic classification of inductive limit actions of fusion categories on AF-algebras*. Commun. Math. Phys. **405**(2024), no. 83.

[7] A. Connes, *Outer conjugacy classes of automorphisms of factors*. Ann. Sci. École Norm. Sup. (4) **8**(1975), no. 3, 383–419.

[8] A. Connes, *Periodic automorphisms of the hyperfinite factor of type II_1*. Acta Sci. Math. **39**(1977), 39–66.

[9] K. R. Davidson. *C*-algebras by example*, Fields Institute Monographs, 6, American Mathematical Society, Providence, RI, 1996.

[10] S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups. II: group extensions with a non-abelian kernel*. Ann. of Math. **2**(1947), 326–341.

[11] G. A. Elliott and H. Su, *K-theoretic classification for inductive limit $\mathbb{Z}_2$ actions on AF algebras*. Canad. J. Math. **48**(1996), no. 5, 946–958.

[12] D. E. Evans and A. Kishimoto, *Trace scaling automorphisms of certain stable AF algebras*. Hokkaido Math. J. **26**(1997), no. 1, 211–224.
[13] S. Evington and S. Girón Pacheco, *Anomalous symmetries of classifiable C*-algebras*. Stud. Math. **270**(2023), no. 1, 73–101.
[14] T. Fack and O. Maréchal, *Sur la classification des automorphismes periodiques des C*-algebres UHF*. J. Funct. Anal. **40**(1981), no. 3, 267–301.
[15] J. Gabe and G. Szabó, *The dynamical Kirchberg–Phillips theorem*. Preprint, to appear in Acta. Math, 2023.
[16] E. Gardella and L. Santiago, *Equivariant $*$-homomorphisms, Rokhlin constraints and equivariant UHF-absorption*. J. Funct. Anal. **270**(2016), no. 7, 2543–2590.
[17] S. Girón Pacheco, *Anomalous symmetries of classifiable C*-algebras*. Ph.D. thesis, University of Oxford, 2023.
[18] R. H. Herman and V. F. R. Jones, *Period two automorphisms of C*-algebras*. J. Funct. Anal. **45**(1982), no. 2, 169–176.
[19] R. H. Herman and V. F. R. Jones, *Models of finite group actions*. Math. Scand. **52**(1983), no. 2, 312–320.
[20] R. H. Herman and A. Ocneanu, *Stability for integer actions on UHF C*-algebras*. J. Funct. Anal. **59**(1984), no. 1, 132–144.
[21] I. Hirshberg and W. Winter, *Rokhlin actions and self-absorbing C*-algebras*. Pacific J. Math. **233**(2007), no. 1, 125–143.
[22] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. I*. Duke Math. J. **122**(2004), no. 2, 233–280.
[23] M. Izumi, *Finite group actions on C*-algebras with the Rohlin property. II*. Adv. Math. **184**(2004), no. 1, 119–160.
[24] M. Izumi, *G-kernels of Kirchberg algebras*, to appear in Münster J. Math., 2023, arXiv:2309.03441.
[25] M. Izumi and H. Matui, *Poly-$\mathbb{Z}$ group actions on Kirchberg algebras II*. Invent. Math. **224**(2021), no. 3, 699–766.
[26] M. Izumi and H. Matui, *Poly-$\mathbb{Z}$ group actions on Kirchberg algebras I*. Int. Math. Res. Not. IMRN **16**(2021), 12077–12154.
[27] C. Jones, *Remarks on anomalous symmetries of C*-algebras*. Commun. Math. Phys. **388**(2021), no. 1, 385–417.
[28] V. F. R. Jones, *An invariant for group actions*. In: Algèbres d'opérateurs (Sém., les plans-Sur-Bex, 1978), Lecture Notes in Mathematics, 725, Springer, Berlin, 1979, pp. 237–253.
[29] V. F. R. Jones, *Actions of finite groups on the hyperfinite type II_1 factor*. Mem. Amer. Math. Soc. **28**(1980), no. 237, v + 70.
[30] V. F. R. Jones, *Index for subfactors*. Invent. Math. **11**(1990), 395–419.
[31] Y. Katayama and M. Takesaki, *Outer actions of a discrete amenable group on approximately finite dimensional factors. I. General theory*. In: Operator algebras, quantization, and noncommutative geometry, Contemporary Mathematics, 365, American Mathematical Society, Providence, RI, 2004, pp. 181–237.
[32] E. Kirchberg, *Exact C*-algebras, tensor products, and the classification of purely infinite algebras*. In: Proceedings of the International Congress of Mathematicians, vols. 1 and 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 943–954.
[33] H. Lin, *Classification of simple tracially AF C*-algebras*. Canad. J. Math. **53**(2001), no. 1, 161–194.
[34] H. Lin, *Classification of simple C*-algebras of tracial topological rank zero*. Duke Math. J. **125**(2004), no. 1, 91–119.
[35] N. Nawata, *Finite group actions on certain stably projectionless C*-algebras with the Rohlin property*. Trans. Amer. Math. Soc. **368**(2016), no. 1, 471–493.
[36] A. Ocneanu, *Actions of discrete amenable groups on von Neumann algebras*, Lecture Notes in Mathematics, 1138, Springer, Berlin, 1985.
[37] A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*. In: Operator algebras and applications, Vol. 2, London Mathematical Society Lecture Note Series, 136, Cambridge University Press, Cambridge, 1988, pp. 119–172.
[38] D. S. Ornstein and B. Weiss, *Ergodic theory of amenable group actions. I. The Rohlin lemma*. Bull. Amer. Math. Soc. (N.S.) **2**(1980), no. 1, 161–164.
[39] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*. Doc. Math. **5**(2000), 49–114.
[40] S. Popa, *Classification of amenable subfactors of type II*. Acta Math. **172**(1994), no. 2, 163–255.
[41] L. Robert, *Classification of inductive limits of 1-dimensional NCCW complexes*. Adv. Math. **231**(2012), no. 5, 2802–2836.

[42] V. A. Rokhlin, *A general measure-preserving transformation is not mixing*. Doklady Akad. Nauk SSSR (N.S.) **60**(1948), 349–351.
[43] L. Santiago, *Crossed products by actions of finite groups with the Rokhlin property*. Int. J. Math. **26**(2015), no. 7, Article no. 1550042, 31 pp.
[44] G. Szabó, *Strongly self-absorbing C*-dynamical systems, III*. Adv. Math. **316**(2017), 356–380.
[45] G. Szabó, *Strongly self-absorbing C*-dynamical systems*. Trans. Amer. Math. Soc. **370**(2018), no. 1, 99–130.
[46] G. Szabó, *Strongly self-absorbing C*-dynamical systems. II*. J. Noncommut. Geom. **12**(2018), no. 1, 369–406.
[47] A. S. Toms and W. Winter, *Strongly self-absorbing C*-algebras*. Trans. Amer. Math. Soc. **359**(2007), no. 8, 3999–4029.

Department of Mathematics, KU Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium
e-mail: sergio.gironpacheco@kuleuven.be