



Regularization of the Kepler Problem on the Three-sphere

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Abstract. In this paper we regularize the Kepler problem on S^3 in several different ways. First, we perform a Moser-type regularization. Then, we adapt the Ligon–Schaaf regularization to our problem. Finally, we show that the Moser regularization and the Ligon–Schaaf map we obtained can be understood as the composition of the corresponding maps for the Kepler problem in Euclidean space and the gnomonic transformation.

1 Introduction

It often happens that the flow associated with a vector field is incomplete. Famous examples are the Newtonian n -body problem, where singularities arise because of the existence of collision orbits, and the Kepler problem. Regularization of vector fields is a common procedure in the study of differential equations. There are two main approaches. In the first approach the incompleteness of the flow is removed by embedding it into a complete flow. The qualitative behavior of solutions of the set of singularities is the same for both vector fields. The second approach involves surgery and is usually called regularization by surgery or block-regularization. Roughly, the idea is to excise a neighborhood of the singularity from the manifold on which the vector field is defined and then to identify appropriate points on the boundary of the region.

Both approaches have been applied to the Kepler problem in Euclidean space. The second approach was first used for the Kepler problem by Easton [11], while the first approach has several variants. We mention only the most relevant for our work. As far as we know the regularization of the planar Kepler problem using the first approach was first discussed by Levi-Civita [17]. Another beautiful incarnation of the second approach, due to Moser [21], consists in showing that the flow of the n -dimensional Kepler problem on the surface of constant negative energy is conjugate to the geodesic flow on the unit tangent bundle of S^n . The main disadvantage of Levi-Civita and Moser's regularization methods is that they handle each energy level separately. This disadvantage is partially removed by a regularization procedure due to Ligon and Schaaf [18]. The Ligon–Schaaf regularization procedure allows us to handle all negative (resp., all positive) energy levels together. However, negative, positive, and zero energy levels still cannot be handled together with this procedure. The

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treatment of the regularization in the original article by Ligon and Schaaf requires laborious computations, and a somewhat simplified treatment of the Ligon–Schaaf regularization map is due to Cushman and Duistermaat [4] and Cushman and Bates [5]. Recently Marle [19], and Heckman, and de Laat [14] posted preprints on arXiv that give another simplified treatment by showing that the Ligon–Schaaf map can be understood as an adaptation of the Moser regularization map. Regularization by surgery handles all the energy level at once; however, this is a completely different approach that has other shortcomings. In particular, on the one hand it is difficult to find the attaching map, and on the other hand this kind of regularization is not too helpful in understanding the global flow of the system and the near-collision orbits.

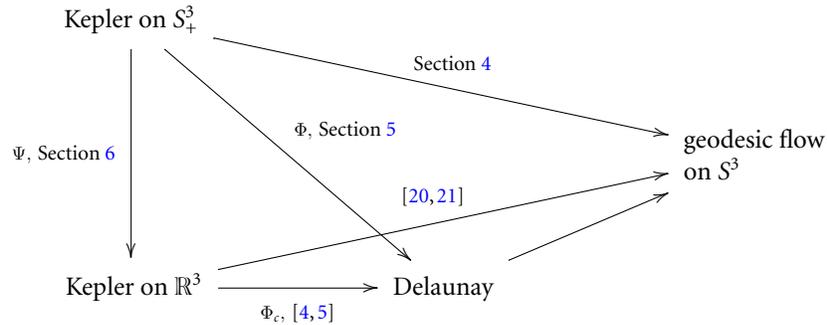
The Kepler problem on the three-sphere S^3 is the main topic of this paper. The Kepler problem and n -body problem on spaces of constant curvature are over a century old problems and recently have generated a good deal of scholarly interest. See [6–8, 22, 23] for some recent results and some history. See [8] for a more exhaustive survey of recent results.

The flow of the Kepler problem on the three-sphere S^3 is incomplete, and several procedures have been used to regularize its vector field. The surgery approach to regularization was applied in [23] to the Kepler problem on a class of surfaces of revolution that include the two-sphere. The Levi-Civita and Moser approaches were used in [12].

In this article we extend the regularization of Moser and of Ligon and Schaaf, and we study how they are connected. Other methods that could be extended to the Kepler problem on S^3 are the Kustaanheimo–Stiefel regularization [16], and the approach of using a larger group of symmetry as Souriau [25] does. Studying these issues and how the various regularizations are connected is an interesting problem, which is outside the scope of this paper. However, we plan to address these issues in a future article.

The paper is organized as follows. We start by deriving the equations of motion using the method of Dirac brackets (Section 2). In Section 3, we describe the integrals of motion of the system and derive the equations of the orbit. In Section 4, we describe the Moser regularization applied to the Kepler problem on S^3 . The details are given for the negative energy level sets, where the orbits lie completely in the upper hemisphere. The same approach applies to the part of any energy level set that lies in the upper hemisphere. Furthermore, we carry out the Ligon–Schaaf regularization directly (Section 5) and compare it with the case of Euclidean Kepler problem. It turns out that the Ligon–Schaaf regularization for the negative energy parts of the two systems are naturally related by the gnomonic map (Section 6). The gnomonic map takes the upper hemisphere to the full Euclidean space. As will become clear in Section 6, the negative energy orbits stay completely in the upper hemisphere, and the two Kepler systems are related by a rescaling of the induced map on the tangent bundle. More precisely, let Φ and Φ_c be the Ligon–Schaaf maps for the Kepler problem on S^3 and \mathbb{R}^3 , respectively, and let Ψ denote the map induced by the gnomonic projection, then $\Phi = \Phi_c \circ \Psi$. The map Ψ is not symplectic, which implies the non-symplecticness of the Ligon–Schaaf map for the spherical case. In the same section, it is also shown that the gnomonic transformation relates the Moser regularization of the two systems.

The relations uncovered in this paper are summarized in the following diagram:



2 Preliminaries

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product in \mathbb{R}^4 and $\| \cdot \|$ the Euclidean norm in \mathbb{R}^4 , and let \cdot , \times , and $|\cdot|$ be the usual dot product, vector product and Euclidean norm in \mathbb{R}^3 , respectively. Let $q = (q_0, \mathbf{q})$ with $\mathbf{q} = (q_1, q_2, q_3)$ and $v = (v_0, \mathbf{v})$ with $\mathbf{v} = (v_1, v_2, v_3)$ be canonical coordinates in $T\mathbb{R}^4$, with the symplectic 2-form $\omega = \sum_i dq_i \wedge dv_i$.

Consider the following Hamiltonian system $(H, T\mathbb{R}^4, \omega)$ with Hamiltonian

$$H(q, v) = \frac{1}{2} \langle v, v \rangle + V(q),$$

where the potential energy $V(q)$ is

$$V(q) = -\gamma \frac{q_0}{(1 - q_0^2)^{\frac{1}{2}}}.$$

This Hamiltonian describes a particle that moves in \mathbb{R}^4 under the influence of potential $V(q)$.

When the particle is constrained to move on the unit 3-sphere $S^3 \subset \mathbb{R}^4$, the system is restricted to the tangent bundle of the 3-sphere:

$$TS^3 = \{(q, v) \in T\mathbb{R}^4 | \langle q, q \rangle - 1 = 0 \text{ and } \langle q, v \rangle = 0\}.$$

The Hamiltonian vector field X_H of H on $T\mathbb{R}^4$ does not restrict to the Hamiltonian vector field $X_{H|_{TS^3}}$ of the constrained system. There are two approaches in computing $X_{H|_{TS^3}}$ using either a modified Hamiltonian function H^* (resulting in X_{H^*} , see Cushman [5]) or a modified Poisson bracket, the Dirac–Poisson bracket $\{ \cdot, \cdot \}^*$ (resulting in X_{H^*} , see Dirac [9, 10]) on $T\mathbb{R}^4$, such that the restriction of the resulting vector fields coincide with $X_{H|_{TS^3}}$. The two approaches are related, since

$$\{F, G\}^*|_{TS^3} = \{F^*, G^*\}|_{TS^3}.$$

The Dirac–Poisson bracket can be explicitly written as follows.

Lemma 2.1 The Dirac–Poisson structure $\{\cdot, \cdot\}^*|_{TS^3}$ is

	q_0	q_1	q_2	q_3	v_0	v_1	v_2	v_3
q_0	0	0	0	0	$1 - q_0^2$	$-q_0q_1$	$-q_0q_2$	$-q_0q_3$
q_1		0	0	0	$-q_0q_1$	$1 - q_1^2$	$-q_1q_2$	$-q_1q_3$
q_2			0	0	$-q_0q_2$	$-q_1q_2$	$1 - q_2^2$	$-q_2q_3$
q_3				0	$-q_0q_3$	$-q_1q_3$	$-q_2q_3$	$1 - q_3^2$
v_0					0	$q_1v_0 - q_0v_1$	$q_2v_0 - q_0v_2$	$q_3v_0 - q_0v_3$
v_1						0	$q_2v_1 - q_1v_2$	$q_3v_1 - q_1v_3$
v_2							0	$q_3v_2 - q_2v_3$
v_3								0

or

$$\{q_\alpha, v_\beta\}^*|_{TS^3} = \delta_{\alpha\beta} - q_\alpha q_\beta, \quad \{v_\alpha, v_\beta\}^*|_{TS^3} = L_{\alpha\beta}, \quad \{q_\alpha, q_\beta\}^*|_{TS^3} = 0,$$

where $\alpha, \beta = 0, 1, 2, 3$ and $L_{\alpha\beta} = q_\beta v_\alpha - q_\alpha v_\beta$.

Proof The phase space TS^3 is given as a subset of $T\mathbb{R}^4$ by the constraints

$$c_1(q, v) = \langle q, q \rangle - 1 = 0 \quad \text{and} \quad c_2(q, v) = \langle q, v \rangle = 0$$

The Dirac–Poisson brackets are given by the relation

$$\{F, G\}^* = \{F, G\} + \sum_{i,j} C_{ij} \{F, c_i\} \{G, c_j\},$$

where C_{ij} are the elements of the inverse of the matrix with entries $\{c_i, c_j\}$. In this case we have

$$C = \frac{1}{2\langle q, q \rangle} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The result follows from a computation. ■

Lemma 2.2 The constrained equations on TS^3 are

$$\dot{q} = v, \quad \dot{v} = -\nabla V(q) - (\langle v, v \rangle - \langle q, \nabla V(q) \rangle) q.$$

Proof We first compute the equations of motion for the Hamiltonian function H with respect to the Dirac–Poisson bracket

$$\dot{q} = \{q, H\}^* \quad \dot{v} = \{v, H\}^*,$$

then we restrict the resulting vector field X_H^* to TS^3 . ■

Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis for \mathbb{R}^4 . Then for our choice of the potential, the equations of motion take the form:

$$(2.1) \quad \begin{aligned} \dot{q} &= v \\ \dot{v} &= \frac{\gamma e_0}{(1 - q_0^2)^{3/2}} - (\langle v, v \rangle + \gamma \frac{q_0}{(1 - q_0^2)^{3/2}}) q. \end{aligned}$$

3 Conserved Quantities and the Equations of the Trajectory

It is easy to see that the Hamiltonian H and the angular momentum $\mu = \mathbf{q} \times \mathbf{v}$ are integrals of motion of the vector field $X_{H^*}|_{TS^3} = X_{H|_{TS^3}}$. There is another interesting integral

$$\mathbf{A} = \pi \times \mu - \gamma \frac{\mathbf{q}}{|\mathbf{q}|},$$

where $\pi = q_0 \mathbf{v} - v_0 \mathbf{q}$. The vector \mathbf{A} is a spherical generalization of the vector known as the *Runge–Lenz vector* in Euclidean space (see [15]). Priority for the Runge–Lenz vector in Euclidean space is sometimes attributed to Laplace, but appears to be due to Hermann and Bernoulli [13].

Note that, as in the case of the Kepler problem in Euclidean space, there is no universally accepted definition of the Runge–Lenz vector. The most common definition is given above, while the common alternative $\mathbf{e} = \mathbf{A}/\gamma$ is also called the *eccentricity vector*.

Proposition 3.1 *The Runge–Lenz vector is an integral of motion.*

Proof

$$\begin{aligned} \frac{d\mathbf{A}}{dt} &= \frac{d\pi}{dt} \times \mu - \gamma \frac{\dot{\mathbf{q}}}{|\mathbf{q}|} + \gamma \frac{(\mathbf{q} \cdot \dot{\mathbf{q}})\mathbf{q}}{|\mathbf{q}|^3} \\ &= (\dot{q}_0 \mathbf{v} + q_0 \dot{\mathbf{v}} - \dot{\mathbf{q}} v_0 - \mathbf{q} \dot{v}_0) \times \mu - \gamma \frac{\dot{\mathbf{q}}}{|\mathbf{q}|} + \gamma \frac{(\mathbf{q} \cdot \dot{\mathbf{q}})\mathbf{q}}{|\mathbf{q}|^3} \\ &= \left[v_0 \mathbf{v} - \left(\langle v, v \rangle q_0 + \gamma \frac{q_0^2}{|\mathbf{q}|^{3/2}} \right) \mathbf{q} - v_0 \mathbf{v} - \mathbf{q} \left(\frac{\gamma}{|\mathbf{q}|^{3/2}} - \langle v, v \rangle q_0 - \gamma \frac{q_0^2}{|\mathbf{q}|^{3/2}} \right) \right] \times \mu \\ &\quad - \gamma \frac{\mathbf{v}}{|\mathbf{q}|} + \gamma \frac{(\mathbf{q} \cdot \mathbf{v})\mathbf{q}}{|\mathbf{q}|^3}, \text{ using (2.1)} \\ &= -\frac{\gamma}{|\mathbf{q}|^{3/2}} (\mathbf{q} \times \mu) - \frac{\gamma}{|\mathbf{q}|^3} (\mathbf{v}(\mathbf{q} \cdot \mathbf{q}) - \mathbf{q}(\mathbf{q} \cdot \mathbf{v})) = \frac{\gamma}{|\mathbf{q}|^{3/2}} (\mathbf{q} \times \mu - \mathbf{q} \times \mu) = 0. \quad \blacksquare \end{aligned}$$

Remark 3.2 Analogous to the Euclidean case, there is another conserved vector, namely the binormal vector,

$$\mathbf{B} = \pi - \frac{\gamma}{|\mathbf{q}||\mu|^2} (\mu \times \mathbf{q}).$$

It can be shown that $\mathbf{A} = \mathbf{B} \times \mu$, and \mathbf{B} is “binormal”, because it is normal to both \mathbf{A} and μ .

Regard $\mathbf{q}, \mathbf{v}, \mu, \mathbf{A}, \mathbf{B}$ as vectors in \mathbb{R}^3 . Since $\mu = \mathbf{q} \times \mathbf{v}$ is a constant of motion, it points in a fixed direction. Moreover, it is orthogonal to \mathbf{q} and \mathbf{v} . Therefore, when moving along an orbit, \mathbf{q} lies on a plane P_2 (in \mathbb{R}^3) orthogonal to μ . Furthermore, it can be shown that $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{q} = 0$, $\mathbf{A} \cdot \mu = 0$, and $\mathbf{B} \cdot \mu = 0$. Hence, \mathbf{A} and \mathbf{B} form a basis for P_2 when both are non-zero (however, they are zero on circular orbits). Returning to \mathbb{R}^4 , let $P = \text{Span}(e_0, \mathbf{A}, \mathbf{B})$, then along an orbit, $q \in S^2 = P \cap S^3$. Hence,

each orbit must lie on a three-dimensional subspace. Moreover, since $\langle q, q \rangle = 1$, the orbit is a curve on the two-sphere S^2 . If ϕ is used to denote the angle between \mathbf{q} and the fixed direction of \mathbf{A} , then

$$(3.1) \quad \mathbf{A} \cdot \mathbf{q} = |\mathbf{A}||\mathbf{q}| \cos \phi = \mathbf{q} \cdot (\pi \times \mu) - \gamma \frac{\mathbf{q} \cdot \mathbf{q}}{|\mathbf{q}|}.$$

Permuting the terms in the scalar triple product and noting that $\pi = q_0\mathbf{v} - v_0\mathbf{q}$,

$$\mathbf{q} \cdot (\pi \times \mu) = \mu \cdot (\mathbf{q} \times \pi) = q_0\mu \cdot (\mathbf{q} \times \mathbf{v}) = q_0|\mu|^2.$$

Let $|\mathbf{q}| = r$ and $q_0 = z$. Rearranging equation (3.1) yields

$$(3.2) \quad \frac{z}{r} = \frac{\gamma}{|\mu|^2} \left(1 + \frac{|\mathbf{A}|}{\gamma} \cos \phi \right), \quad r^2 + z^2 = 1,$$

where (r, ϕ, z) are cylindrical coordinates on P . Write equation (3.2) in spherical coordinates (ρ, ϕ, θ) using the formulas $r = \rho \sin \theta$, $\phi = \phi$, and $z = \rho \cos \theta$, and we obtain

$$\frac{1}{\tan \theta} = \frac{\gamma}{|\mu|^2} \left(1 + \frac{|\mathbf{A}|}{\gamma} \cos \phi \right).$$

This recovers the known formula for the orbits of the Kepler problem on the sphere [3] (compare also with [22]). The orbits are always closed. Each orbit is obtained by intersecting a conical surface with vertex at the origin and a sphere, which reduces to a circle when $|\mathbf{A}| = 0$. In fact, the first of equations (3.2) is a conical surface (and, in particular, one nappe of a quadric conical surface for $|\mathbf{A}|/\gamma < 1$), since for $\phi = \phi_0$ it reduces to the equation of a line through the origin, and for $z = z_0$ it reduces to a conic section. The constant $\epsilon = |\mathbf{e}| = |\mathbf{A}|/\gamma$ is called the *eccentricity* of the orbit, which explains why \mathbf{e} is called the eccentricity vector. When $\epsilon = 0$, the orbits are circles, when $0 < \epsilon < 1$ they are curves corresponding to ellipses on the plane, $\epsilon = 1$ corresponding to parabolas, and $\epsilon > 1$ corresponding to hyperbolas. Note that the orbits are confined to the upper hemisphere when $0 < \epsilon < 1$, while it is not the case when $\epsilon \geq 1$. We refer the reader to [3,22] for a more detailed discussion of the orbits.

The Hamiltonian H can be rewritten as

$$H = \frac{1}{2}(|\pi|^2 + |\mu|^2) + V(q),$$

and the length of the eccentricity \mathbf{e} vector is

$$|\mathbf{e}|^2 = 1 + \frac{|\mu|^2}{\gamma^2} (2H - |\mu|^2).$$

There is another constant of motion,

$$E = H - \frac{|\mu|^2}{2} = \frac{1}{2}|\pi|^2 + V(q),$$

which satisfies the equation

$$(3.3) \quad |\mathbf{e}|^2 = 1 + 2\frac{E}{\gamma^2}|\mu|^2.$$

Define the *modified eccentricity vector* $\tilde{\mathbf{e}} = -\nu\mathbf{e}$, where

$$\nu = \frac{\gamma}{\sqrt{-2E}},$$

then the following hold:

$$\mu \cdot \tilde{\mathbf{e}} = 0, \quad \|\mu\|^2 + \|\tilde{\mathbf{e}}\|^2 = \nu^2 > 0,$$

where the second equation follows from (3.3). These relations are equivalent to

$$(\mu + \tilde{\mathbf{e}}) \cdot (\mu + \tilde{\mathbf{e}}) = (\mu - \tilde{\mathbf{e}}) \cdot (\mu - \tilde{\mathbf{e}}) = \nu^2 > 0,$$

which defines a smooth 4-dimensional manifold diffeomorphic to $S_\nu^2 \times S_\nu^2$.

4 Hodograph and Moser's Regularization

In this section, we describe the Moser's regularization for the Kepler problem on S^3 . The approach follows closely Milnor's work [20] for the Euclidean case. Without loss of generality, suppose that $\mu = (0, 0, \mu)$. Then $\mu = |\mu|$, $\mathbf{q} = (q_1, q_2, 0)$ and $\mathbf{v} = (v_1, v_2, 0)$. Furthermore, suppose that $\mathbf{A} = (|\mathbf{A}|, 0, 0) \neq 0$ and let ϕ be the angle formed by \mathbf{q} with \mathbf{A} (as in the equation of the trajectory). Since $\mathbf{A} = \mathbf{B} \times \mu$, it follows that $|\mathbf{B}| = |\mathbf{A}|/|\mu|(0, 1, 0)$. Then $\mu \times \mathbf{q} = |\mu||\mathbf{q}|(-\sin \phi, \cos \phi, 0)$, and from the definition of \mathbf{B} , it follows that

$$\begin{aligned} \pi &= \mathbf{B} + \frac{\gamma}{|\mu|}(\mu \times \mathbf{q}) = \frac{|\mathbf{A}|}{|\mu|}(0, 1, 0) + \frac{\gamma}{|\mu|}(-\sin \phi, \cos \phi, 0) \\ &= \frac{\gamma}{|\mu|}\left(-\sin \phi, \frac{|\mathbf{A}|}{\gamma} + \cos \phi, 0\right) \end{aligned}$$

Thus, for each orbit, π moves along a circle centered at $|\mathbf{A}|/|\mu|$, with radius $\gamma/|\mu|$ and lying in the plane through the origin that is orthogonal to μ .

Conversely, starting with a circle with the equation

$$\pi = \frac{\gamma}{|\mu|}\left(-\sin \phi, \frac{|\mathbf{A}|}{\gamma} + \cos \phi, 0\right),$$

a direct computation gives

$$\mu \cos \theta = q_0 \mu = \mathbf{q} \times \pi = \frac{\gamma}{|\mu|} \sin \theta \left(0, 0, 1 + \frac{|\mathbf{A}|}{\gamma} \cos \phi\right).$$

Hence, we recover the equation of the orbit in spherical coordinates

$$\frac{1}{\tan \theta} = \frac{\gamma}{|\mu|^2} \left(1 + \frac{|\mathbf{A}|}{\gamma} \cos \phi \right).$$

It follows that the hodocycle completely determines the corresponding Kepler orbit. We can now prove the following theorem.

Theorem 4.1 *Fixing some constant value of $E = H - |\mu|^2/2 < 0$, consider the space M_E^\pm of all the vectors π such that $\pi \cdot \pi > 2E$, together with a single improper point $\pi = \infty$. This space possesses one and only one Riemannian metric ds^2 , so that the arc-length parameter $\int ds$ along any circle $t \rightarrow \pi(t)$ is precisely equal to the parameter $\int dt/q_0|\mathbf{q}|$. This metric is smooth and complete, with constant curvature $-2E$, and its geodesics are precisely the circles or lines $t \rightarrow \pi(t)$ associated with Kepler orbits.*

Proof From (4.1) it is clear that if $E < 0$ the orbits must have $q_0 > 0$ and thus are limited to the upper hemisphere. In this case $\epsilon = |\mathbf{A}|/\gamma < 1$ and only the space M_E^+ is relevant. In fact $E = 1/2|\pi|^2 - \gamma q_0/(1 - q_0^2)^{1/2}$ and thus, if $E < 0$, $|\pi^2| > 2E$, since in such case $q_0 > 0$. However, if $E > 0$, things get more complicated.

Using the equation of motions it is easy to show that

$$\dot{\pi} = \dot{q}_0 \mathbf{v} + q_0 \dot{\mathbf{v}} - \dot{v}_0 \mathbf{q} - v_0 \dot{\mathbf{q}} = -\frac{\gamma \mathbf{q}}{|\mathbf{q}|^3},$$

and thus $|\dot{\pi}| = \frac{\gamma}{|\mathbf{q}|}$. Dividing this equation by the definition of the time rescaling $ds = \gamma/q_0|\mathbf{q}|dt$ and using the fact that

$$(4.1) \quad \frac{\gamma q_0}{|\mathbf{q}|} = \frac{1}{2}|\pi|^2 - \frac{1}{2}(2H - |\mu|^2)$$

we obtain

$$(4.2) \quad \left| \frac{d\pi}{ds} \right| = \frac{\gamma|q_0|}{|\mathbf{q}|} = \left| \frac{1}{2}|\pi|^2 - E \right|, \quad ds^2 = \frac{4d\pi \cdot d\pi}{(|\pi|^2 - (2H - |\mu|^2))^2} = \frac{4d\pi \cdot d\pi}{(|\pi|^2 - 2E)^2}$$

Thus, there is one and only one Riemannian metric on the spaces M_E^\pm that satisfies our condition, and it is given by formula (4.2).

To describe what happens in a neighborhood of infinity, we work with the inverted velocity coordinate $\mathbf{w} = \pi/|\pi|^2$ (see, for example, Milnor’s paper [20] for a discussion of inversion). Since the differential of \mathbf{w} is

$$d\mathbf{w} = \frac{(\pi \cdot \pi)d\pi - 2(d\pi \cdot \pi)\pi}{(\pi \cdot \pi)^2}$$

we have $d\mathbf{w} \cdot d\mathbf{w} = \frac{d\pi \cdot d\pi}{(\pi \cdot \pi)^2}$. Hence,

$$(4.3) \quad ds^2 = \frac{4d\mathbf{w} \cdot d\mathbf{w}}{(1 - 2E\mathbf{w} \cdot \mathbf{w})^2}.$$

The metrics given in equations (4.2) and (4.3) have constant curvature $-2E$. ■

Since $E < 0$, the corresponding metric space is a round 3-sphere and $t \rightarrow \pi(t)$ is thus a geodesic on a round 3-sphere.

5 Ligon–Schaaf Regularization

5.1 $\mathfrak{so}(4)$ Momentum Map

The components of the angular momentum and of an opportunely rescaled eccentricity vector form a Lie algebra under a Poisson bracket that is isomorphic to $\mathfrak{so}(4)$. This gives the momentum map of the Kepler problem on the sphere.

Let $\mathfrak{g} = \mathfrak{so}(4)$. For a suitably chosen basis $\{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$, the Lie bracket is

$$(5.1) \quad [X_i, X_j] = [Y_i, Y_j] = \epsilon_{ijk}X_k \quad \text{and} \quad [X_i, Y_j] = \epsilon_{ijk}Y_k$$

The identification $\mathfrak{g} \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ can be seen via substitution $\frac{1}{2}(X_i + Y_i)$ and $\frac{1}{2}(X_i - Y_i)$. The basis can be thought of as coordinate functions on \mathfrak{g}^* , then the Lie bracket defines a Poisson bracket $\{\cdot, \cdot\}_{\mathfrak{g}}$ on $C^\infty(\mathfrak{g}^*)$, which defines the Lie–Poisson structure on \mathfrak{g}^* .

We first write down the brackets of the components of μ and \mathbf{A} .

Lemma 5.1 *The brackets of the components of μ and \mathbf{A} are:*

$$(5.2) \quad \begin{aligned} \{\mu_i, \mu_j\}^*|_{TS^3} &= \epsilon_{ijk}\mu_k, & \{\mu_i, A_j\}^*|_{TS^3} &= \epsilon_{ijk}A_k, \\ \{A_i, A_j\}^*|_{TS^3} &= -2(H - |\mu|^2)\epsilon_{ijk}\mu_k \end{aligned}$$

In the Appendix, we give a sketch of the proof of the lemma above and of the proposition below.

Proposition 5.2 *Let $\mathfrak{g} = \mathfrak{so}(4)$, $C(H) = H + \sqrt{\gamma^2 + H^2}$, and*

$$\eta(|\mu|^2, H) = \frac{-|\mu|^2 + C(H)}{|\mathbf{A}|^2}$$

Then the following is the momentum map of the Kepler problem on the sphere

$$\rho = (\mu, \eta(|\mu|^2, H)\mathbf{A}): TS^3 \rightarrow \mathfrak{g}^*.$$

It is a Poisson map with respect to the bracket $\{\cdot, \cdot\}^$ on TS^3 and the Lie–Poisson bracket on \mathfrak{g}^* .*

5.2 Delaunay Vector Field

Let (x, y) be the coordinates on $T\mathbb{R}^4 \cong \mathbb{R}^4 \oplus \mathbb{R}^4$. Then $TS^3 \subset T\mathbb{R}^4$ is given by $\langle x, y \rangle = 0$ and $\langle x, x \rangle = 1$. Let $T^+S^3 = \{(x, y) \in TS^3 | y \neq 0\}$ be the tangent bundle of S^3 less its zero section and let $\tilde{\omega} = \omega|_{T^+S^3}$ be the restriction to T^+S^3 of the standard symplectic form ω on $T\mathbb{R}^4$. Consider the Delaunay Hamiltonian on T^+S^3 :

$$\mathcal{H} = -\frac{1}{2} \frac{\gamma^2}{\langle y, y \rangle},$$

which resembles the Kepler Hamiltonian written in Delaunay coordinates (see, for example, [1]). It is clear that \mathcal{H} is invariant under the standard action of $SO(4)$ on T^+S^3 .

The integral curves of the Delaunay vector field $X_{\mathcal{H}}$ satisfy

$$\frac{dx}{dt} = \frac{\gamma^2}{\langle y, y \rangle^2} y, \quad \frac{dy}{dt} = -\frac{\gamma^2}{\langle y, y \rangle} x.$$

It can be proved that the Delaunay vector field $X_{\mathcal{H}}$ is a time rescaling of the Hamiltonian vector field of the geodesic flow on the unit sphere. The space $\mathfrak{so}(4)^*$ can be naturally identified with $\bigwedge^2(\mathbb{R}^4)^*$. Under this identification, the momentum mapping of the standard action of $SO(4)$ on T^+S^3 is:

$$\mathcal{J}(x, y) = x \wedge y \in \bigwedge^2(\mathbb{R}^4)^* \cong \mathfrak{so}(4).$$

A more detailed description of these facts can be found in [5].

5.3 Regularization

All the orbits of the Kepler problem on the sphere that have $E < 0$ can be regularized at once, with a slightly modified Ligon–Schaaf map. Let

$$J: TS^3 \longrightarrow \mathfrak{so}(4): (q, v) \longmapsto (\mu, \tilde{\mathbf{c}}).$$

We begin our search for a Ligon–Schaaf map by noting that the image of \mathcal{J} is the same as the image of J (see [5] for a detailed study of the Delaunay vector field and its momentum map). This suggests that the two maps are somewhat related, even though J is not a momentum map. Note that this situation differs from the case of the Kepler problem in \mathbb{R}^3 , studied by Cushman [5], where the maps that are related are the momentum maps.

Theorem 5.3 *The smooth map $\Phi: (q, v) \mapsto (x, y)$ intertwines the momentum map \mathcal{J} and the map J ; that is, $\Phi^*\mathcal{J} = J$ if and only if*

$$(5.3) \quad \Phi: (q, v) \mapsto (x, y) = (\alpha \sin \varphi + \beta \cos \varphi, \nu(-\alpha \cos \varphi + \beta \sin \varphi)),$$

where

$$(5.4) \quad \alpha = (\alpha_0, \alpha) = \left(\frac{1}{\nu q_0} \mathbf{q} \cdot \pi, \frac{\mathbf{q}}{|\mathbf{q}|} - \frac{\mathbf{q} \cdot \pi}{\gamma q_0} \pi \right),$$

$$(5.5) \quad \beta = (\beta_0, \beta) = \left(\frac{|\mathbf{q}|}{\gamma q_0} \pi \cdot \pi - 1, \frac{|\mathbf{q}|}{\nu q_0} \pi \right),$$

where φ is an arbitrary smooth real function and $\nu = \frac{\gamma}{\sqrt{-2E}}$.

Proof Suppose Φ intertwines the momentum map \mathcal{J} and the map J . Write $x = (x_0, \mathbf{x}) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$, $y = (y_0, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Then $\Phi^*\mathcal{J} = J$ is equivalent to

$$(5.6) \quad \mathbf{x} \times \mathbf{y} = \mathbf{q} \times \mathbf{v},$$

$$\mathbf{x}_0 \mathbf{y} - y_0 \mathbf{x} = \frac{1}{\sqrt{-2E}} \left(\gamma \frac{\mathbf{q}}{|\mathbf{q}|} - \pi \times (\mathbf{q} \times \mathbf{v}) \right) = M\mathbf{q} + N\pi,$$

where

$$M = \frac{1}{\sqrt{-2E}} \left(\frac{\gamma}{|\mathbf{q}|} - \frac{\pi \cdot \pi}{q_0} \right) \quad \text{and} \quad N = \frac{1}{\sqrt{-2E}} \left(\frac{\pi \cdot \mathbf{q}}{q_0} \right).$$

Suppose $\mathbf{q} \times \mathbf{v} = \frac{1}{q_0}(\mathbf{q} \times \pi) \neq 0$, then it follows that \mathbf{x} and \mathbf{y} lie on the same plane of \mathbf{q} and π . Since \mathbf{q} and π are linearly independent, we obtain

$$(5.7) \quad \mathbf{x} = a\mathbf{q} + b\pi, \quad \mathbf{y} = c\mathbf{q} + d\pi.$$

Since $\mathbf{q} \times \mathbf{v} = 1/q_0(\mathbf{q} \times \pi) = \mathbf{x} \times \mathbf{y} = (ad - bc)(\mathbf{q} \times \pi)$, we find $ad - bc = 1/q_0$. Substituting (5.7) into (5.6) and using the linear independence of \mathbf{q} and π gives a set of linear equations for \mathbf{x}_0 and \mathbf{y}_0 . Since $ad - bc = 1/q_0$, these equations may be solved to give

$$x_0 = (aN - bM)q_0, \quad y_0 = (cN - dM)q_0.$$

Since $(x, y) \in T^+S^3$,

$$(5.8) \quad 1 = \mathbf{x} \cdot \mathbf{x} + x_0^2, \quad 0 = \mathbf{x} \cdot \mathbf{y} + x_0 y_0.$$

Substituting the expressions for \mathbf{x} and \mathbf{y} and the expressions for x_0 and y_0 above into (5.8) yields

$$(5.9) \quad 1 = a^2(\mathbf{q} \cdot \mathbf{q}) + \left(\frac{\pi \cdot \mathbf{q}}{\sqrt{-2E}} a + \frac{\gamma q_0}{|\mathbf{q}| \sqrt{-2E}} b \right)^2,$$

$$(5.10) \quad 0 = ac \left(\mathbf{q} \cdot \mathbf{q} + \frac{(\pi \cdot \mathbf{q})^2}{(-2E)} \right) + (ad + bc) \left(\gamma \frac{(\pi \cdot \mathbf{q}) q_0}{|\mathbf{q}| (-2E)} \right) + bd \left(\gamma^2 \frac{q_0^2}{|\mathbf{q}|^2 (-2E)} \right),$$

where we have used the identities

$$\mathbf{q} \cdot \mathbf{q} + N^2 q_0^2 = \mathbf{q} \cdot \mathbf{q} + \frac{(\pi \cdot \mathbf{q})^2}{(-2E)},$$

$$(\mathbf{q} \cdot \pi) - MN q_0^2 = \gamma \frac{(\pi \cdot \mathbf{q}) q_0}{|\mathbf{q}| (-2E)},$$

$$\pi \cdot \pi + M^2 q_0^2 = \gamma^2 \frac{q_0^2}{|\mathbf{q}|^2 (-2E)},$$

which follow from the definition of M and N and the identity

$$\pi \cdot \pi - \gamma \frac{q_0}{|\mathbf{q}|} = 2E + \gamma \frac{q_0}{|\mathbf{q}|}.$$

Multiplying (5.9) by c and (5.10) by $-a$ and adding the resulting equations gives

$$c = -a \frac{\gamma(\pi \cdot \mathbf{q})}{|\mathbf{q}|(-2E)} - b \frac{\gamma^2 q_0}{|\mathbf{q}|^2(-2E)}.$$

Similarly, multiplying (5.9) by d and (5.10) by $-a$ yields

$$d = a \left(\mathbf{q} \cdot \mathbf{q} + \frac{(\pi \cdot \mathbf{q})^2}{(-2E)} \right) \frac{1}{q_0} + b \frac{\gamma(\pi \cdot \mathbf{q})}{|\mathbf{q}|(-2E)}.$$

All solutions of (5.9) are parametrized by

$$a = \frac{1}{|\mathbf{q}|} \sin \varphi, \quad b = \frac{\sqrt{-2E}|\mathbf{q}|}{\gamma q_0} \cos \varphi - \frac{\pi \cdot \mathbf{q}}{q_0 \gamma} \sin \varphi,$$

where θ is an arbitrary function. Substituting the previous equations in the expressions for c and d gives

$$c = -\frac{\gamma}{|\mathbf{q}|\sqrt{-2E}} \cos \varphi, \quad d = \frac{|\mathbf{q}|}{q_0} \sin \varphi + \frac{(\pi \cdot \mathbf{q})}{q_0 \sqrt{-2E}} \cos \varphi.$$

Conversely, a calculation shows that a map Φ of the form (5.3) intertwines the momentum map \mathcal{J} and the map J . ■

Corollary 5.4 Φ intertwines E and the Delaunay Hamiltonian; that is, $\Phi^* \mathcal{H} = E$.

Proof We have that

$$(\Phi^* \mathcal{H})(q, \nu) = -\frac{1}{2} \frac{\gamma^2}{\langle y, y \rangle} = -\frac{1}{2} \frac{(-2E)\gamma^2}{\gamma^2 |\mathbf{q}|^2 - \alpha \cos \varphi + \beta \sin \varphi} = E(q, \nu),$$

since $\langle \alpha, \alpha \rangle = 1$, $\langle \beta, \beta \rangle = 1$, $\langle \alpha, \beta \rangle = 0$, as the following shows:

$$\begin{aligned} \langle \alpha, \alpha \rangle &= 1 - 2 \frac{(\pi \cdot \mathbf{q})^2}{|\mathbf{q}| \gamma q_0} + \frac{(\pi \cdot \mathbf{q})^2 (\pi \cdot \pi)}{\gamma^2 q_0^2} + \frac{(\pi \cdot \mathbf{q})^2}{\nu^2 q_0^2} \\ &= 1 + 2 \frac{(\pi \cdot \mathbf{q})^2}{\gamma^2 q_0^2} \left(\frac{\pi \cdot \pi}{2} - \frac{\gamma q_0}{|\mathbf{q}|} \right) + \frac{(-2E)(\pi \cdot \mathbf{q})^2}{\gamma^2 q_0^2} = 1 \end{aligned}$$

$$\begin{aligned} \langle \beta, \beta \rangle &= 1 + \frac{(\pi \cdot \pi)^2 |\mathbf{q}|^2}{\gamma^2 q_0^2} - 2 \frac{(\pi \cdot \pi) |\mathbf{q}|}{\gamma q_0} + \frac{|\mathbf{q}|^2 (\pi \cdot \pi)}{\nu^2 q_0^2} \\ &= 1 + 2 \frac{|\mathbf{q}|^2 (\pi \cdot \pi)}{\gamma^2 q_0^2} \left(\frac{\pi \cdot \pi}{2} - \frac{\gamma q_0}{|\mathbf{q}|} + \frac{(-2E)}{2} \right) = 1 \end{aligned}$$

$$\langle \alpha, \beta \rangle = -\frac{\pi \cdot \mathbf{q}}{\nu q_0} + \frac{|\mathbf{q}|}{\nu \gamma q_0^2} (\pi \cdot \mathbf{q})(\pi \cdot \pi) + \frac{\pi \cdot \mathbf{q}}{\nu q_0} - \frac{|\mathbf{q}|}{\nu \gamma q_0^2} (\pi \cdot \mathbf{q})(\pi \cdot \pi) = 0. \quad \blacksquare$$

The next result is a computation we will use to understand the relationship between the Kepler vector field X_H and the Delaunay vector field $X_{\mathcal{J}\mathcal{L}}$.

Lemma 5.5 *The derivatives of α and β along the flow generated by the vector field X_H are*

$$\frac{d\alpha}{dt} = \frac{\sqrt{-2E}}{q_0|\mathbf{q}|} \beta, \quad \frac{d\beta}{dt} = -\frac{\sqrt{-2E}}{q_0|\mathbf{q}|} \alpha.$$

Proof Let $t \rightarrow (q(t), v(t))$ be an integral curve of the Kepler vector field X_H . Let $\alpha = \alpha(q(t), v(t))$ and $\beta = \beta(q(t), v(t))$, where $\alpha = (\alpha_0, \alpha)$ and $\beta = (\beta_0, \beta)$ are given by (5.4) and (5.5), respectively. Recall that $\pi = q_0\mathbf{v} - v_0\mathbf{q}$ so that $\mathbf{v} = 1/q_0(\pi + v_0\mathbf{q})$. Moreover a simple computation shows that $d\pi/dt = -\gamma\mathbf{q}/|\mathbf{q}|^3$. With these equations in mind, together with the expressions for X_H , we compute

$$\begin{aligned} \frac{d\alpha_0}{dt} &= \frac{1}{\nu q_0} (\dot{\pi} \cdot \mathbf{q} + \pi \cdot \mathbf{v}) - \frac{1}{\nu q_0^2} (\pi \cdot \mathbf{q}) \dot{q}_0 \\ &= \frac{1}{\nu q_0} \left(-\frac{\gamma}{|\mathbf{q}|} + \frac{\pi \cdot \pi}{q_0} \right) + \frac{1}{\nu q_0^2} (\pi \cdot \mathbf{q}) v_0 - \frac{1}{\nu q_0^2} (\pi \cdot \mathbf{q}) v_0 \\ &= \frac{1}{\nu q_0 |\mathbf{q}|} \left(\frac{\pi \cdot \pi}{q_0 \gamma} - 1 \right) = \frac{1}{\nu q_0 |\mathbf{q}|} \beta_0 \end{aligned}$$

and

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{\mathbf{v}}{|\mathbf{q}|} - \frac{(\mathbf{q} \cdot \mathbf{v})}{|\mathbf{q}|^3} \mathbf{q} - \frac{(\dot{\pi} \cdot \mathbf{q})}{\gamma q_0} \pi - \frac{(\pi \cdot \dot{\mathbf{q}})}{\gamma q_0} \pi - \frac{(\pi \cdot \mathbf{q})}{\gamma q_0} \dot{\pi} - \frac{(\pi \cdot \mathbf{q})\pi}{\gamma q_0^2} v_0 \\ &= -\frac{2\pi}{\gamma q_0^2} \left(\frac{\pi \cdot \pi}{2} - \frac{\gamma q_0}{|\mathbf{q}|} \right) \\ &= \frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \left(\frac{|\mathbf{q}|}{\nu q_0} \pi \right) = \frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \beta. \end{aligned}$$

Similarly, to verify the expression for $\frac{d\beta}{dt}$, we compute

$$\begin{aligned} \frac{d\beta_0}{dt} &= 2 \frac{(\pi \cdot \dot{\pi})|\mathbf{q}|}{\gamma q_0} - \frac{(\pi \cdot \pi)|\mathbf{q}|}{\gamma q_0^2} v_0 + \frac{(\pi \cdot \pi)(\mathbf{q} \cdot \mathbf{v})}{q_0 \gamma |\mathbf{q}|} \\ &= -2 \frac{(\pi \cdot \mathbf{q})}{q_0 |\mathbf{q}|^2} - \frac{(\pi \cdot \pi)|\mathbf{q}|}{\gamma q_0^2} v_0 + \frac{(\pi \cdot \pi)}{\gamma q_0^2 |\mathbf{q}|} (\mathbf{q} \cdot \pi) + \frac{(\pi \cdot \pi)}{q_0^2 \gamma |\mathbf{q}|} (\mathbf{q} \cdot \mathbf{q}) \\ &= 2 \frac{\pi \cdot \mathbf{q}}{q_0^2 |\mathbf{q}| \gamma} \left(\frac{\pi \cdot \pi}{2} - \frac{\gamma q_0}{|\mathbf{q}|} \right) = -\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \left(\frac{\sqrt{-2E}(\pi \cdot \mathbf{q})}{\gamma q_0} \right) \\ &= -\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \alpha_0 \end{aligned}$$

and

$$\begin{aligned} \frac{d\beta}{dt} &= \frac{|\mathbf{q}|\dot{\pi}}{\nu q_0} - \frac{|\mathbf{q}|\pi}{\nu q_0^2} v_0 + \frac{(\mathbf{q} \cdot \mathbf{v})\pi}{\nu q_0 |\mathbf{q}|} \\ &= \sqrt{-2E} \left(\frac{|\mathbf{q}|\dot{\pi}}{\gamma q_0} - \frac{|\mathbf{q}|\pi}{\gamma q_0^2} v_0 + \frac{(\mathbf{q} \cdot \pi)\pi}{\gamma q_0^2 |\mathbf{q}|} + \frac{|\mathbf{q}|^2 \pi}{\gamma q_0^2 |\mathbf{q}|} v_0 \right) \\ &= -\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \left(\frac{\mathbf{q}}{|\mathbf{q}|} - \frac{(\mathbf{q} \cdot \pi)\pi}{\gamma q_0} \right) = -\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} \alpha. \quad \blacksquare \end{aligned}$$

Remark 5.6 Note that, with the time reparametrization given by $dt = q_0 |\mathbf{q}| dt'$, the equations for α and β , given in the previous lemma, can be rewritten as

$$\frac{d\alpha}{dt'} = \sqrt{-2E} \beta, \quad \frac{d\beta}{dt'} = -\sqrt{-2E} \alpha.$$

These are formally equivalent to the equations of motion of an harmonic oscillator of Hamiltonian

$$H_{\text{har}} = \frac{1}{2} \sqrt{-2E} (\beta^2 + \alpha^2),$$

with the standard symplectic form, where (α, β) are coordinates on $T^*\mathbb{R}^4$. However, in our case, α and β satisfy the constraints that define T^*S^3 : $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$ and $\langle \alpha, \beta \rangle = 0$. The equations for the harmonic oscillator on $T^*\mathbb{R}^4$, of Hamiltonian H_{har} , automatically preserve T^*S^3 and give the equations of a reparametrization of the geodesic geodesic flow on S^3 when restricted to this invariant (symplectic) submanifold.

We recall some standard terminology that we will use to describe the relation between the Kepler vector field and the Delaunay vector field.

Definition 5.7 Let X and Y be vector fields on the manifolds M and N , and let $g_t^X: M \rightarrow M$ and $g_t^Y: N \rightarrow N$ be the corresponding flows. We say that X and Y (and the corresponding flows) are C^k -conjugate if there is a C^k -diffeomorphism $\Phi: M \rightarrow N$ such that $\Phi(g_t^X(x)) = g_t^Y(\Phi(x))$. We say that X and Y (and the corresponding flows) are C^k -equivalent if there is a C^k -diffeomorphism $\Phi: M \rightarrow N$ that maps the orbits of g^X onto the orbits of g^Y and preserves the direction of time. That is, there is a family of monotone increasing diffeomorphisms $\tau_x: \mathbb{R} \rightarrow \mathbb{R}$ such that $\Phi(g_t^X(x)) = g_{\tau_x(t)}^Y(\Phi(x))$.

Remark 5.8 Note that differentiating $\Phi(g_t^X(x)) = g_{\tau(x,t)}^Y(\Phi(x))$ with respect to t yields

$$(5.11) \quad \Phi_* X(x) = \left(\frac{\partial \tau_x}{\partial t} \right) Y(\Phi(x)).$$

Hence, if this equation is satisfied, under the conditions of the definition above, the two vector fields are C^k equivalent.

The next theorem finally describes the relationship between the Kepler and the Delaunay vector fields.

Theorem 5.9 *The Kepler vector field X_H and the Delaunay vector field $X_{\mathcal{J}\mathcal{C}}$ are C^∞ -equivalent with maps*

$$\Phi(q, v) = (\alpha \sin \varphi + \beta \cos \varphi, \nu(-\alpha \cos \varphi + \beta \sin \varphi))$$

and

$$\tau_q = \int_{t_0}^t \frac{dt}{q_0^2(t)},$$

where $\varphi = \frac{1}{\nu q_0} \mathbf{q} \cdot \pi (= \alpha_0)$.

Proof Differentiating

$$(x(t), y(t)) = \Phi(q(t), v(t)) = (\alpha \sin \varphi + \beta \cos \varphi, \nu(-\alpha \cos \varphi + \beta \sin \varphi))$$

with respect to t along a trajectory $(q(t), v(t))$ of X_H and using Lemma 5.5 yields

$$\begin{aligned} \frac{dx}{dt} &= \frac{d\alpha}{dt} \sin \varphi + \alpha \cos \varphi \frac{d\varphi}{dt} + \frac{d\beta}{dt} \cos \varphi - \beta \sin \varphi \frac{d\theta}{dt} \\ &= \left(\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} - \frac{d\varphi}{dt} \right) (\beta \sin \varphi - \alpha \cos \varphi) = \frac{\sqrt{-2E}}{\gamma} \left(\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} - \frac{d\varphi}{dt} \right) y \\ &= \frac{1}{\nu} \left(\frac{\gamma}{\nu q_0 |\mathbf{q}|} - \frac{d\varphi}{dt} \right) y \end{aligned}$$

and

$$\begin{aligned} \frac{dy}{dt} &= \nu \left(-\frac{d\alpha}{dt} \cos \varphi + \alpha \sin \varphi \frac{d\theta}{dt} + \frac{d\beta}{dt} \sin \varphi + \beta \cos \varphi \frac{d\theta}{dt} \right) \\ &= -\nu \left(\frac{\sqrt{-2E}}{q_0 |\mathbf{q}|} - \frac{d\varphi}{dt} \right) (\alpha \sin \varphi + \beta \cos \varphi) \\ &= -\nu \left(\frac{\gamma}{\nu q_0 |\mathbf{q}|} - \frac{d\varphi}{dt} \right) x. \end{aligned}$$

Taking $\varphi = \frac{1}{\nu q_0} (\mathbf{q} \cdot \pi)$ implies that

$$\begin{aligned} \frac{d\varphi}{dt} - \frac{\gamma}{\nu q_0 |\mathbf{q}|} &= \frac{1}{\nu} \left(\frac{\mathbf{v} \cdot \pi}{q_0} + \frac{\mathbf{q} \cdot \dot{\pi}}{q_0} - \frac{\mathbf{q} \cdot \pi}{q_0^2} \dot{q}_0 \right) - \frac{\gamma}{\nu q_0 |\mathbf{q}|} \\ &= \frac{1}{\nu} \left(\frac{\pi \cdot \pi}{q_0^2} + \nu_0 \frac{\mathbf{q} \cdot \pi}{q_0^2} - \frac{\gamma}{q_0 |\mathbf{q}|} - \nu_0 \frac{\mathbf{q} \cdot \pi}{q_0^2} \right) - \frac{\gamma}{\nu q_0 |\mathbf{q}|} \\ &= \frac{1}{\nu q_0^2} \left(\pi \cdot \pi - 2 \frac{\gamma q_0}{|\mathbf{q}|} \right) = -\frac{\gamma^2}{\nu^3 q_0^2}, \end{aligned}$$

and thus by Corollary 5.4 we obtain

$$\Phi_*(X_H) = \begin{pmatrix} \frac{\gamma^2}{q_0^2 \langle y, y \rangle^2} y \\ -\frac{\gamma^2}{q_0^2 \langle y, y \rangle} x \end{pmatrix} = \frac{1}{q_0^2} X_{\mathcal{H}}.$$

Let $\tau_q = \int_{t_0}^t \frac{dt}{q_0^2(t)}$, then (5.11) holds, and hence the two vector fields are smoothly equivalent. ■

Note that the vector field of the Kepler problem in \mathbb{R}^3 is C^k -conjugate to the one of the Delaunay vector fields ([4, 5]). And the diffeomorphism Φ_c intertwining the two vector fields is a symplectomorphism. However, the vector field X_H of the Kepler problem in S^3 is C^k -equivalent but not C^k -conjugate to the Delaunay vector field $X_{\mathcal{H}}$.

6 Gnomonic Transformation

The computations in the previous section are analogous to those of [4, 5]. However, there is a more direct link between the Kepler problem on S^3 and in Euclidean space. We first recall a classical result due to Appell and Serret [2, 24], relating the Kepler problem on the upper hemisphere S^3_+ to that on \mathbb{R}^3 via the gnomonic transformation.

On the phase space $T_0\mathbb{R}^3 = (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ with coordinates (\mathbf{Q}, \mathbf{V}) and symplectic form $\omega_3 = \sum_{i=1}^3 dQ_i \wedge dV_i$ consider the Kepler Hamiltonian

$$H_K(\mathbf{Q}, \mathbf{V}) = \frac{1}{2} \mathbf{V} \cdot \mathbf{V} - \frac{\gamma}{|\mathbf{Q}|},$$

where \cdot is the Euclidean inner product on \mathbb{R}^3 and $|\mathbf{Q}|$ is the length of the vector \mathbf{Q} . The integral curves of the Hamiltonian vector field X_{H_K} on $T_0\mathbb{R}^3$ satisfy the equations

$$\dot{\mathbf{Q}} = \mathbf{V} \quad \dot{\mathbf{V}} = -\gamma \frac{\mathbf{Q}}{|\mathbf{Q}|^3},$$

which (for $\gamma > 0$) describe the motion of a particle of mass 1 about the origin under the influence of the Newtonian gravity. The momentum map of this system is

$$J_K = \left(\mathbf{Q} \times \mathbf{V}, -\frac{\gamma}{\sqrt{-2H_K}} \left(\frac{1}{\gamma} \mathbf{V} \times (\mathbf{Q} \times \mathbf{V}) - \frac{\mathbf{Q}}{|\mathbf{Q}|} \right) \right).$$

This is a classical system, and a detailed analysis can be found in [5] or any good mechanics book.

As before, S^3 is the unit sphere in \mathbb{R}^4 . We now consider the gnomonic projection, which is the projection onto the tangent plane at the north pole from the center of the sphere

$$q = (q_0, \mathbf{q}) \mapsto \mathbf{Q} := \frac{\mathbf{q}}{\sqrt{1 - |\mathbf{q}|^2}} = \frac{\mathbf{q}}{q_0}.$$

The induced map on the tangent space at \mathbf{q} is given by

$$\nu = (v_0, \mathbf{v}) \mapsto \frac{\mathbf{v}}{\sqrt{1 - |\mathbf{q}|^2}} + \frac{(\mathbf{q} \cdot \mathbf{v})\mathbf{q}}{(1 - |\mathbf{q}|^2)^{\frac{3}{2}}} = \frac{1}{q_0} \mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{v}}{q_0^3} \mathbf{q} = \frac{1}{q_0} \mathbf{v} - \frac{v_0}{q_0^2} \mathbf{q} = \frac{1}{q_0^2} \pi.$$

Recall that

$$H(q, v) = \frac{1}{2} \langle v, v \rangle - \gamma \frac{q_0}{(1 - q_0^2)^{\frac{1}{2}}}.$$

On the lower hemisphere, $q_0 < 0 \Rightarrow H > 0$ and it follows that negative energy orbits completely lie in the upper hemisphere.

Theorem 6.1 *The Kepler vector field X_H on S^3 (restricted to the upper hemisphere) and the Kepler vector field X_{H_K} on \mathbb{R}^3 are C^∞ -equivalent with maps*

$$\Psi: (q, v) \mapsto (\mathbf{Q}, \mathbf{V}) = \left(\frac{\mathbf{q}}{q_0}, \pi \right) \quad \text{and} \quad \tau_q = \int_{t_0}^t \frac{dt}{q_0^2(t)}.$$

The map Ψ together with $t \mapsto \tau_q$ is called the gnomonic transformation.

Proof Differentiating $(\mathbf{Q}(t), \mathbf{V}(t)) = \Psi(\mathbf{q}(t), \mathbf{v}(t))$ with respect to t along a trajectory $(\mathbf{q}(t), \mathbf{v}(t))$ of X_H and using Lemma 5.5 yields

$$\frac{d\mathbf{Q}}{dt} = \frac{1}{q_0} \mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{v}}{q_0^3} \mathbf{q} = \frac{1}{q_0^2} \mathbf{V}$$

and

$$\begin{aligned} \frac{d\mathbf{V}}{dt} &= q_0 \frac{d\mathbf{v}}{dt} + \frac{|\mathbf{v}|^2}{q_0} \mathbf{q} + \frac{(\mathbf{q} \cdot \frac{d\mathbf{v}}{dt})}{q_0} \mathbf{q} + \frac{(\mathbf{q} \cdot \mathbf{v})^2}{q_0^3} \mathbf{q} \\ &= - \left(|\mathbf{v}|^2 + \frac{(\mathbf{q} \cdot \mathbf{v})^2}{q_0^2} + \gamma \frac{q_0}{(1 - q_0^2)^{3/2}} \right) (q_0^2 + (\mathbf{q} \cdot \mathbf{q})) \frac{\mathbf{q}}{q_0} + \left(|\mathbf{v}|^2 + \frac{(\mathbf{q} \cdot \mathbf{v})^2}{q_0^2} \right) \frac{\mathbf{q}}{q_0} \\ &= -\gamma \frac{\mathbf{q}}{|\mathbf{q}|^3} = -\gamma \frac{\mathbf{Q}}{|\mathbf{Q}|^3} (1 + |\mathbf{Q}|^2) = -\frac{\gamma}{q_0} \frac{\mathbf{Q}}{|\mathbf{Q}|^3}, \end{aligned}$$

where we used the fact that $q_0^2 v_0^2 = (\mathbf{q} \cdot \mathbf{v})^2$. Consequently we obtain

$$\Psi_*(X_H) = \begin{pmatrix} \frac{\mathbf{v}}{q_0} \\ -\gamma \frac{\mathbf{Q}}{q_0 |\mathbf{Q}|^3} \end{pmatrix} = \frac{1}{q_0^2} X_{H_K}.$$

Let

$$\tau_q = \int_{t_0}^t \frac{dt}{q_0^2(t)},$$

then (5.11) holds, and hence the two vector field are smoothly equivalent. ■

The map Ψ of Theorem 6.1 has properties that are analogous to the map Φ . In fact it intertwines E and the Kepler Hamiltonian in Euclidean space, and it also intertwines the momentum map J_K of the Kepler problem in Euclidean space with the map J . We prove these properties below.

Proposition 6.2 *Consider the smooth map $\Psi: (\mathbf{q}, \mathbf{v}) \rightarrow (\mathbf{Q}, \mathbf{V})$*

- (i) Ψ intertwines E and the Kepler Hamiltonian in Euclidean space, that is, $\Psi^* H_K = E$.

(ii) Ψ intertwines the momentum map J_K and the map J , that is $\Psi^* J_K = J$.

Proof To prove the first part of the proposition we compute $\Psi^* H_K$. A simple computation yields

$$\begin{aligned} (\Psi^* H_K)(q, v) &= \frac{1}{2} \mathbf{V} \cdot \mathbf{V} - \frac{\gamma}{|\mathbf{Q}|} = \frac{1}{2} (q_0^2 |\mathbf{v}|^2 + 2q_0^2 v_0^2 + |\mathbf{q}|^2 v_0^2) - \frac{\gamma q_0}{\sqrt{1 - q_0^2}} \\ &= \frac{1}{2} \pi \cdot \pi - \frac{\gamma q_0}{\sqrt{1 - q_0^2}} = E(q, v), \end{aligned}$$

since

$$\mathbf{V} = q_0 \mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{v}}{q_0} \mathbf{q} = q_0 \mathbf{v} - \frac{q_0 v_0}{q_0} \mathbf{q} = \pi$$

and

$$\frac{\gamma}{|\mathbf{Q}|} = \gamma \frac{q_0}{|\mathbf{q}|} = \gamma \frac{q_0}{\sqrt{1 - q_0^2}}.$$

We prove the second part by computing $\Psi^* J_K$. A computation gives

$$(\Psi^* J_K)(q, v) = \left(\mathbf{Q} \times \mathbf{V}, -\frac{\gamma}{\sqrt{-2H_K}} \left(\frac{1}{\gamma} \mathbf{V} \times (\mathbf{Q} \times \mathbf{V}) - \frac{\mathbf{Q}}{|\mathbf{Q}|} \right) \right) = (\mu, \tilde{\mathbf{e}}) = J(q, v)$$

since

$$\begin{aligned} \mathbf{Q} \times \mathbf{V} &= \mathbf{q} \times \left(\mathbf{v} + \frac{(\mathbf{q} \times \mathbf{v})}{1 - |\mathbf{q}|^2} \mathbf{q} \right) = \mathbf{q} \times \mathbf{v} = \mu, \\ -\frac{\mathbf{Q}}{|\mathbf{Q}|} + \frac{1}{\gamma} \mathbf{V} \times (\mathbf{Q} \times \mathbf{V}) &= -\frac{\mathbf{q}}{\sqrt{1 - q_0^2}} + \frac{1}{\gamma} \pi \times \mu = \mathbf{e}, \end{aligned}$$

and $\Psi^* H_K = E$ by part (i) of the proposition. ■

Recall that if (M, ω_M) and (N, ω_N) are symplectic manifolds and $f: M \rightarrow N$ is a diffeomorphism, then f is symplectic if and only if for all $h \in C^\infty(N)$,

$$(6.1) \quad f^* X_h = X_{h \circ f};$$

see [1] for a proof. With this in mind we show that Ψ is not symplectic.

Proposition 6.3 *The map Ψ is not symplectic.*

Proof To show that the map is not symplectic it is enough to find a Hamiltonian for which (6.1) is not satisfied. Let $h = H_K$, and let $f = \Psi$, then it can be shown that (6.1) is equivalent to writing

$$X_{H_K}(\Psi(q, v)) = \Psi_* X_{H_K \circ \Psi}(q, v) = \Psi_* X_E(q, v).$$

The left-hand side of the equation is

$$(X_{H_K}(\Psi(q, v))) = \begin{pmatrix} \mathbf{V} \\ -\gamma \frac{\mathbf{Q}}{|\mathbf{Q}|^3} \end{pmatrix} = \begin{pmatrix} q_0 \mathbf{v} + (\mathbf{q} \cdot \mathbf{v}) q_0^{-1} \mathbf{q} \\ -\gamma \frac{q_0^2}{|\mathbf{q}|^3} \end{pmatrix}.$$

To compute the right-hand side note that

$$\{\mu_i, q_j\}^*|_{TS^3} = \epsilon_{ijk}q_k \quad \text{and} \quad \{\mu_i, \nu_j\}^*|_{TS^3} = \epsilon_{ijk}\nu_k$$

(see the Appendix), and thus $\{\mathbf{q}, |\mu|^2\}^*|_{TS^3} = \mu \times \mathbf{q}$ and $\{\mathbf{v}, |\mu|^2\}^*|_{TS^3} = \mu \times \mathbf{v}$. Consequently

$$X_E^{(\mathbf{q}, \mathbf{v})} = \left(\begin{matrix} \{\mathbf{q}, H - |\mu|^2/2\}^*|_{TS^3} \\ \{\mathbf{v}, H - |\mu|^2/2\}^*|_{TS^3} \end{matrix} \right) X_H^{(\mathbf{q}, \mathbf{v})} - \frac{1}{2} \begin{pmatrix} \mu \times \mathbf{q} \\ \mu \times \mathbf{v} \end{pmatrix},$$

where $X_E^{(\mathbf{q}, \mathbf{v})}$ denotes the part of the vector field corresponding to

$$\left\{ \mathbf{q}, H - \frac{1}{2}|\mu|^2 \right\}^*|_{TS^3} \quad \text{and} \quad \left\{ \mathbf{v}, H - \frac{1}{2}|\mu|^2 \right\}^*|_{TS^3}.$$

Differentiating $\mathbf{Q}(t)$ with respect to t along a trajectory of X_E yields

$$\begin{aligned} \frac{d\mathbf{Q}}{dt} &= \frac{\mathbf{v} - \frac{1}{2}\mu \times \mathbf{q}}{\sqrt{1 - |\mathbf{q}|^2}} + \frac{\mathbf{q}[\mathbf{q} \cdot (\mathbf{v} - \frac{1}{2}\mu \times \mathbf{q})]}{(1 - |\mathbf{q}|^2)^{3/2}} \\ &= \frac{1}{q_0} (q_0 \mathbf{v} + (\mathbf{q} \cdot \mathbf{v})q_0^{-1} \mathbf{q}) + \frac{1}{2q_0} (-|\mathbf{q}|^2 \mathbf{v} + (\mathbf{q} \cdot \mathbf{v})\mathbf{q}). \end{aligned}$$

This gives the first three components of $\Psi_* X_E$. Comparing with the first three components of $X_{H_K}(\Psi(q, \nu))$ yields

$$-\frac{|\mathbf{q}|^2}{q_0^2} (q_0 \mathbf{v} + (\mathbf{q} \cdot \mathbf{v})q_0^{-1} \mathbf{q}) = \frac{1}{2q_0} (-|\mathbf{q}|^2 \mathbf{v} + (\mathbf{q} \cdot \mathbf{v})\mathbf{q}).$$

If \mathbf{v} is non-zero, one can compare the coefficients of \mathbf{v} . This comparison yields the impossibility $1 = \frac{1}{2}$, and hence the identity (6.1) is not satisfied for $h = H_K$. ■

Remark 6.4 The non-symplecticness of the map Ψ can be seen alternatively as follows. It can be shown that Ψ is the composition of the tangent lift with a scaling $\kappa: TS_+^3 \rightarrow TS_+^3$ of the fiber direction by the factor of $q_0^2 = 1 - |\mathbf{q}|^2$. As shown below, κ is not symplectic with respect to the standard symplectic structure, which implies that the map Ψ is not symplectic either:

$$\kappa^* d\mathbf{q} \wedge d\mathbf{v} = d\mathbf{q} \wedge d((1 - |\mathbf{q}|^2)\mathbf{v}) = (1 - |\mathbf{q}|^2)d\mathbf{q} \wedge d\mathbf{v} - 2 \sum_{i,j=1}^3 v_i q_j dq_i \wedge dq_j \neq d\mathbf{q} \wedge d\mathbf{v}.$$

Here, \mathbf{q} is used as coordinates on S_+^3 , via the projection along the q_0 direction.

6.1 Relation to the Ligon–Schaaf Regularization for the Kepler Problem in \mathbb{R}^3

Recall the description of the Ligon–Schaaf regularization for the Kepler problem on \mathbb{R}^3 as given in [4,5]. The symplectomorphism Φ_c intertwining the vector fields of the Kepler problem and one of the Delaunay vector fields is given by

$$\Phi_c : (\mathbf{Q}, \mathbf{V}) \mapsto (x_c, y_c) = (\alpha_c \sin \varphi_c + \beta_c \cos \varphi_c, \nu_c(-\alpha_c \cos \varphi_c + \beta_c \sin \varphi_c)),$$

where $\nu_c = \frac{\gamma}{\sqrt{-2H_K}}$, $\varphi_c = \alpha_{c,0}$ and

$$\begin{aligned} \alpha_c &= (\alpha_{c,0}, \alpha_c) = \left(\frac{1}{\nu_c} \mathbf{V} \cdot \mathbf{Q}, \frac{\mathbf{Q}}{|\mathbf{Q}|} - \frac{\mathbf{V} \cdot \mathbf{Q}}{\gamma} \mathbf{V} \right), \\ \beta_c &= (\beta_{c,0}, \beta_c) = \left(\frac{|\mathbf{Q}|}{\gamma} (\mathbf{V} \cdot \mathbf{V}) - 1, \frac{|\mathbf{Q}|}{\nu_c} \mathbf{V} \right). \end{aligned}$$

This calculation proves the following proposition.

Proposition 6.5 $\Phi_c \circ \Psi = \Phi$

A straightforward consequence of this proposition is the following corollary.

Corollary 6.6 Φ is not symplectic.

Proof Suppose Φ is symplectic. Then we can rewrite the proposition as $\Psi = \Phi_c^{-1} \circ \Phi$. Since Φ_c^{-1} and Φ are symplectic, it follows that Ψ is symplectic. This is in contradiction with Proposition 6.3. ■

6.2 Relation to Moser’s Regularization for the Kepler Problem in \mathbb{R}^3

First we recall [20, Theorem 2] for the Kepler problem in \mathbb{R}^3 . For a given constant value of $h = H_K$, define the space $M_h = \{\mathbf{V} \mid \mathbf{V} \cdot \mathbf{V} > 2h\} \cup \{\infty\}$. Define also a Riemannian metric

$$ds^2 = \frac{4d\mathbf{V} \cdot d\mathbf{V}}{(\mathbf{V} \cdot \mathbf{V} - 2h)^2}$$

on M_h . The arc-length parameter $\int ds$ of this metric along any velocity circle $\tau \rightarrow \mathbf{V}(\tau)$ is equal to the parameter $\int d\tau/|\mathbf{Q}|$, where τ denotes the time.

Proposition 6.7 *The Moser’s regularization for the Kepler problems in \mathbb{R}^3 and the upper hemisphere are related by the gnomonic transformation.*

Proof The gnomonic transformation is given by $(\mathbf{Q}, \mathbf{V}) = \Psi(q, \nu) = (\mathbf{q}/q_0, \pi)$ and $d\tau = dt/q_0^2$ along the trajectory $q(t)$. It is clear that Ψ maps M_E^+ to M_E and the metrics correspond. The arc-length parameter becomes

$$\int \frac{d\tau}{|\mathbf{Q}|} = \int \frac{\frac{dt}{q_0^2}}{\left| \frac{\mathbf{q}}{q_0} \right|} = \int \frac{dt}{q_0 |\mathbf{q}|}. \quad \blacksquare$$

Appendix

Proof of Lemma 5.1 We verify only the third equation in (5.2). Using Lemma 2.1 we obtain

$$\{\mu_i, q_j\}^*|_{TS^3} = \epsilon_{ijk}q_k, \quad \{\mu_i, v_j\}^*|_{TS^3} = \epsilon_{ijk}v_k, \quad \{\mu_i, \pi_j\}^*|_{TS^3} = \epsilon_{ijk}\pi_k$$

and

$$\begin{aligned} \{\pi_i, \pi_j\}^*|_{TS^3} &= q_i v_j - q_j v_i, \quad \{\mu_i, |\mathbf{q}|^2\}^*|_{TS^3} = 0, \\ \left\{ \pi_i, \frac{1}{|\mathbf{q}|} \right\}^*|_{TS^3} &= \frac{q_0 q_i}{|\mathbf{q}|^3}, \quad \{\pi_i, q_j\}^*|_{TS^3} = 0. \end{aligned}$$

Using the bilinearity property of Poisson brackets, expand

$$\{A_i, A_j\} = \left\{ \epsilon_{lmi}\pi_l\mu_m - \gamma \frac{q_i}{|\mathbf{q}|}, \epsilon_{pqj}\pi_p\mu_q - \gamma \frac{q_j}{|\mathbf{q}|} \right\}$$

to obtain

$$\{A_i, A_j\}^*|_{TS^3} = -2(H - |\mu|^2)\epsilon_{ijk}\mu_k.$$

It helps to recall the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{km}\delta_{jl} - \delta_{jm}\delta_{kl}$. ■

Proof of Proposition 5.2. Let $\tilde{\mathbf{E}} = \eta(|\mu|^2, H)\mathbf{A}$, and let $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$, then the proposition amounts to showing that $\{\mu_1, \mu_2, \mu_3, \tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ satisfies (5.1) with the Poisson bracket $\{\cdot, \cdot\}^*$.

We start with (5.2) and compute the brackets of the components of $\tilde{\mathbf{E}}$. For $\eta = \eta(|\mu|^2, H)$,

$$\begin{aligned} \{\eta A_i, \eta A_j\}^*|_{TS^3} &= \eta^2 \{A_i, A_j\}^*|_{TS^3} + \eta \{A_i, \eta\}^*|_{TS^3} A_j + \eta \{\eta, A_j\}^*|_{TS^3} A_i \\ \{\eta, A_i\}^*|_{TS^3} &= \epsilon_{ijk} A_k \frac{\partial \eta}{\partial \mu_i} \delta_{jl} = \epsilon_{ilk} A_k \frac{\partial \eta}{\partial |\mu|^2} \frac{\partial |\mu|^2}{\partial \mu_i} = -2\epsilon_{ikl} \mu_i A_k \frac{\partial \eta}{\partial |\mu|^2} \\ &= -2 \frac{\partial \eta}{\partial |\mu|^2} (\mu \times \mathbf{A})_i. \end{aligned}$$

Consequently,

$$\{\eta A_i, \eta A_j\}^*|_{TS^3} = \eta^2 \{A_i, A_j\}^*|_{TS^3} + 2\eta \frac{\partial \eta}{\partial |\mu|^2} ((\mu \times \mathbf{A})_i A_j - (\mu \times \mathbf{A})_j A_i).$$

Simple computations show that

$$\begin{aligned} \{\eta A_i, \eta A_j\}^*|_{TS^3} &= -2\eta^2(H - |\mu|^2)\epsilon_{ijk}\mu_k - \frac{\partial \eta^2}{\partial |\mu|^2} |\mathbf{A}|^2 \epsilon_{ijk}\mu_k \\ &= \left[-2\eta^2(H - |\mu|^2) - \frac{\partial \eta^2}{\partial |\mu|^2} (\gamma^2 + 2H|\mu|^2 - (|\mu|^2)^2) \right] \epsilon_{ijk}\mu_k \end{aligned}$$

Since $|\mathbf{A}|^2 = \gamma^2 + |\mu|^2(2H - |\mu|^2)$, we have that $\frac{\partial|\mathbf{A}|^2}{\partial|\mu|^2} = 2(H - |\mu|^2)$. Therefore,

$$(6.2) \quad \{\tilde{E}_i, \tilde{E}_j\}^*|_{TS^3} = \{\eta A_i, \eta A_j\}^*|_{TS^3} = -\frac{\partial|\eta\mathbf{A}|^2}{\partial|\mu|^2} \epsilon_{ijk}\mu_k = -(\epsilon_{ijk}\mu_k) \frac{\partial}{\partial|\mu|^2} \cdot |\tilde{\mathbf{e}}|^2$$

Lastly, we determine the function η . Equation (6.2) implies that η has to satisfy

$$\frac{\partial|\eta\mathbf{A}|^2}{\partial|\mu|^2} = -1$$

Integrating, we obtain $\eta^2|\mathbf{A}|^2 = -|\mu|^2 + C(H)$, and if $|\mathbf{A}|^2 \neq 0$, we have

$$\eta^2 = \frac{-|\mu|^2 + C(H)}{|\mathbf{A}|^2}$$

where $C(H)$ is an arbitrary function of H . For $|\mathbf{A}|^2 = 0$ we have $|\mu|^2 = C(H)$. Substituting in the expression $|\mathbf{A}|^2 = \gamma^2 + |\mu|^2(2H - |\mu|^2)$ yields

$$\gamma^2 + C(H)(2H - C(H)) = 0.$$

Then the positive solution gives $C(H) = H + \sqrt{\gamma^2 + H^2}$. ■

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