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METRIZATION OF RANKED SPACES

BY

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ABSTRACT. K. Kunugi introduced the notion of ranked space as a generalization of that of metric spaces, (see [6]). In this note we define a metrizability of ranked spaces and study conditions under which a ranked space is metrizable.

Introduction. K. Kunugi introduced the notion of ranked space as a generalization of metric spaces (see [6]). In this note we define metrizability of ranked spaces and study conditions under which a ranked space is metrizable. Throughout this note, the term "ranked space" will mean a ranked space of indicator ω_0. (ω_0 is the first nonfinite ordinal).

1. **Preliminaries.** We define ranked space. Let R be a non-empty set such that, to every point p of R, there corresponds a non-empty family $\mathcal{V}(p)$ whose elements are subsets of R, denoted by $V(p)$, $U(p)$, etc. which are called preneighborhoods of p. Suppose that, for every p of R, every preneighborhood $V(p)$ in $\mathcal{V}(p)$ satisfies the following condition:

(A) (Axiom (A) of Hausdorff [5]) $V(p)\ni p$. Define $\mathcal{V}=\bigcup\{\mathcal{V}(p);\ p\in R\}$.

Then the space R is said to be a ranked space if for every $n\in N$ (throughout this note, N is the set $\{0,1,2,\dots\}$), there is associated a subfamily of $\mathcal{V}$, denoted by $\mathcal{V}_n$, satisfying the following axiom:

(a) For every $p\in R$, every $V(p)\in\mathcal{V}(p)$ and every $n\in N$, we can find a $U(p)$ such that:
   (1) $U(p)\subset V(p)$, and
   (2) $U(p)$ belongs to some $\mathcal{V}_m$ with $m\geq n$.

A preneighborhood belonging to $\mathcal{V}_n$ is said to have rank n. Preneighborhoods of p with rank n are written $V(p,n)$, $U(p,n)$, etc. Moreover we assume that R is a preneighborhood of every point with rank 0. A ranked space is a non-empty set R with those families $\mathcal{V}$, $\mathcal{V}_n$ ($n\in N$), which is written $(R,\mathcal{V},\mathcal{V}_n)$ (briefly, $(R,\mathcal{V})$). In a ranked space $(R,\mathcal{V})$ a sequence of preneighborhoods $\{V_i(p_i,n_i)\}$ (briefly, $\{V_i\}$) is called a fundamental (or more precisely $\mathcal{V}$-fundamental) sequence if the three conditions below are fulfilled.

(1) $V_0(p_0,n_0)\supset V_1(p_1,n_1)\supset\cdots\supset V_i(p_i,n_i)\supset\cdots$,

(2) $n_0\leq n_1\leq\cdots\leq n_i\leq\cdots$, $\quad 0\leq n_i<\infty \quad \lim n_i=\infty$ as $i\to\infty$.

(3) For every $n\in N$, there exists an $i\in N$ such that $i\geq n$, $p_i=p_{i+1}$ and $n_i<n_{i+1}$.

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In particular, $\{V_i(p_i, n_i)\}$ is called a fundamental sequence of center p, if $p_i = p$ for all i. A sequence $\{p_i\}$ in R is called a Cauchy sequence if there exists a fundamental sequence of preneighborhoods $\{V_i(q_i, n_i)\}$ such that for every V_i there exists a j with the property that $p_k \in V_i$ for all $k \geq j$. In this case, $\{V_i\}$ is called a defining sequence of the Cauchy sequence $\{p_i\}$. A sequence $\{p_i\}$ in R is said to ortho- (or r-) [resp. para- (or π-)] converge to p if $\{p_i\}$ is a Cauchy sequence for which we can find a defining sequence $\{V_i(p, n_i)\}$ [resp. $\{V_i(q_i, n_i)\}$ such that $p \in \bigcap_{i \in N} V_i(q_i, n_i)$.] We denote this by $p \in \{r - \lim p_i\}$ [resp. $p \in \{\pi - \lim p_i\}$].

A ranked space is said to be complete, if for every fundamental sequence $\{V_i\}$ we have $\bigcap_{i \in N} V_i \neq \emptyset$.

For two fundamental sequences $\{V_i\}$ and $\{U_i\}$ we write $\{V_i\} > \{U_i\}$ to mean that for every V_i, there exists a U_j such that $V_i \supset U_j$ and $\{V_i\}$ and $\{U_i\}$ are said to be equivalent if $\{V_i\} > \{U_i\}$ and $\{V_i\} < \{U_i\}$.

Two ranked spaces $(R, \mathcal{V})$ and $(R, \mathcal{U})$ are said to be equivalent (with respect to fundamental sequence) if for every $\mathcal{V}$-fundamental sequence $\{V_i(p, n_i)\}$ [resp. $\{V_i(q_i, n_i)\}$] there exists an equivalent $\mathcal{U}$-fundamental sequence $\{U_i(p, m_i)\}$ [resp. $\{U_i(r_i, m_i)\}$] and for every $\mathcal{U}$-fundamental sequence $\{U_i(p, n_i)\}$ [resp. $\{U_i(q_i, n_i)\}$] there exists an equivalent $\mathcal{V}$-fundamental sequence $\{V_i(p, m_i)\}$ [resp. $\{V_i(r_i, m_i)\}$].

2. **Metrization of ranked spaces.** A ranked space satisfies the axiom (1) and (2) of class (L) of Fréchet (see [4]) if we take r-convergence as the notion of limit. But in general, it is not a topological space. We define metrizability of ranked spaces. First we prove the following Proposition.

PROPOSITION 1. *In two equivalent ranked spaces $(R, \mathcal{V})$ and $(R, \mathcal{U})$, $r(\pi)$-convergence and completeness are identical.*

Proof. If $\{p_i\}$ is r-convergent to p in $(R, \mathcal{V})$, there exists a defining sequence $\{V_i(p, n_i)\}$ such that for every $V_i(p, n_i)$ a k can be found with the property that $p_{k'} \in V_i(p, n_i)$ for all $k' \geq k$. From equivalence of $(R, \mathcal{V})$ and $(R, \mathcal{U})$, for $\{V_i(p, n_i)\}$ there exists an equivalent $\mathcal{U}$-fundamental sequence $\{U_i(p, m_i)\}$. For every $U_i(p, m_i)$, there exists a $V_{i'}(p, n_{i'})$ such that $U_i(p, m_i) \supset V_{i'}(p, n_{i'})$. Therefore for every $U_i(p, m_i)$ there exists k such that $k \leq k'$ implies $p_{k'} \in U_i(p, m_i)$. Hence $\{p_i\}$ is r-convergent to p in $(R, \mathcal{U})$. If $\{P_i\}$ is r-convergent to p in $(R, \mathcal{U})$, then it is r-convergent to p in $(R, \mathcal{V})$. Similarly we can prove the case of π-convergence.

Let $(R, \mathcal{V})$ be complete. Then for every $\mathcal{U}$-fundamental sequence $\{U_i(p_i, n_i)\}$ there exists an equivalent $\mathcal{V}$-fundamental sequence $\{V_i(q_i, m_i)\}$. Therefore for every $U_i(p_i, n_i)$, there exists $V_{i'}(q_{i'}, m_{i'})$ such that $U_i(p_i, n_i) \supset V_{i'}(q_{i'}, m_{i'})$. Since $(R, \mathcal{V})$ is complete, we have $\bigcap_{i \in N} V_i \ni p$. Therefore we have $\bigcap_{i \in N} U_i \ni p$, hence $(R, \mathcal{U})$ is complete.

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Similarly if $(R, \mathcal{U})$ is complete, we have $(R, \mathcal{V})$ is complete.

DEFINITION 1. Consider a metric space (R, d), where we shall use (R, d) to stand for a metric space R with distance function d. Let $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots \to 0$ as $n \to \infty$. If for all $p \in R$ and $n \in N$, $S(p, \lambda_n) = \{q \mid d(p, q) \leq \lambda_n\}$ is taken as a preneighborhood of p with rank n, then R becomes a ranked space and is called a ranked metric space. If we let $U^*(p, n) = S(p, 2^{-n})$, $\mathcal{U}_n^* = \{U^*(p, n) : p \in R\}$ and $\mathcal{U}^* = \cup\{\mathcal{U}_n^* : n \in N\}$, then (R, U^*, U_n^*) is a ranked metric space.

DEFINITION 2. A ranked space $(R, \mathcal{V})$ is said to be metrizable if we can define a distance function d in R such that the ranked metric space $(R, \mathcal{U}^*)$ obtained from the metric space (R, d) is equivalent to the ranked space $(R, \mathcal{V})$.

PROPOSITION 2. *A ranked space $(R, \mathcal{V})$ is metrizable if and only if there exists an equivalent ranked space $(R, \mathcal{U}, \mathcal{U}_n)$ with the following property.*

For every point $p \in R$ and every $n \in N$, preneighborhood with rank n consists of only one preneighborhood and is denoted by $U(p, n)$. Let $\mathcal{U}_n = \{U(p, n) : p \in R\}$, $\mathcal{U} = \cup\{\mathcal{U}_n : n \in N\}$ and suppose that $\{\mathcal{U}_n : n \in N\}$ satisfies the following conditions.

(1) *For every $n \in N$ and every $p \in R$, we have $U(p, n) \supset U(p, n+1)$.*

(2) *For every pair p, q of R and every $n \in N$, we have*

  (i) *$U(p, n) \ni q$ implies $U(q, n) \ni p$.*

  (ii) *$U(p, n) \cap U(q, n) \neq \emptyset$ implies $U(p, n-1) \ni q$.*

(3) *For every p of R and every sequence of preneighborhoods such that $U(p, 0) \supset U(p, 1) \supset \cdots \supset U(p, n) \supset \cdots$, $\bigcap_{i \in N} U(p, n)$ consists of p alone.*

Proof. If for any two points p and q of R, there exists $U(p, n)$ that contains q, but for every $m \geq n+1$ there exists no $U(p, m)$ that contains q, we put $\rho(p, q) = 2^{-n}$. If for every n, there exists a $U(p, n)$ that contains q, we put $\rho(p, q) = 0$. We shall prove $\rho(p, q)$ determines a distance function. Because,

(i) From the definition of $\mathcal{U}_n$, $\rho(p, p) = 0$. Suppose $\rho(p, q) = 0$. Then we have for every n, $U(p, n) \ni q$. Since $U(p, n) \ni p, q$, by condition (3) we have $p = q$.

(ii) From 2 (i) we have $\rho(p, q) = \rho(q, p)$.

(iii) For any points p, q and r of R if we have $\rho(p, q) \leq 2^{-n}$ and $\rho(q, r) \leq 2^{-n}$, then there exists $U(p, n)$ and $U(r, n)$ which contain q. Therefore we have $U(p, n) \cap U(r, n) \ni q$. From condition (2) (ii) we have $U(p, n-1) \ni r$. Therefore $\rho(p, r) \leq 2^{-(n-1)}$. From Chittenden's Theorem [2] this function ρ determines a distance function. With this distance function d the metric space R is denoted by (R, d).

From (R, d) we have the ranked metric space $(R, \mathcal{U}^*)$. The two ranked spaces $(R, \mathcal{U})$ and $(R, \mathcal{U}^*)$ have the same preneighborhoods for every point of R and every rank n. Evidently $(R, \mathcal{U})$ and $(R, \mathcal{U}^*)$ are equivalent. Therefore $(R, \mathcal{V})$ and $(R, \mathcal{U}^*)$ are equivalent.

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Conversely if $(R, \mathcal{V})$ is metrizable, then we can define a metric function d in R such that the ranked metric space $(R, \mathcal{U}^*, \mathcal{U}_n^*)$ obtained from the metric space (R, d) is equivalent with $(R, \mathcal{V})$. Evidently $\{\mathcal{U}_n^* : n \in N\}$ satisfies the above three conditions (1), (2) and (3).

Applications. By the method of ranked space we can prove certain well known metrization theorems as follows.

ALEKSANDROV–URYSOHN'S THEOREM [1]. *In order that a T_1-space X be metrizable it is necessary and sufficient that there exists a countable sequence of open coverings $\mathcal{M}_0, \mathcal{M}_1, \ldots,$ satisfying:*

(1) *For all $n \in N$, $\mathcal{M}_{n+1} \ni M_1, M_2$ and $M_1 \cap M_2 \neq \varnothing$ imply there exist $M \in \mathcal{M}_n$ such that $M_1 \cup M_2 \subset M$.*

(2) *For every point x of X, if $M_n \in \mathcal{M}_n$ contains x for all $n \in N$, then $\{M_n : n \in N\}$ is a neighborhood base of x.*

Proof. We may assume for every n, $\mathcal{M}_n$ is a refinement of $\mathcal{M}_{n-1}$ (where $\mathcal{M}_n$ is a refinement of $\mathcal{M}_{n-1}$ means for any set $M_n \in \mathcal{M}_n$ there exists a set $M_{n-1} \in \mathcal{M}_{n-1}$ such that $M_n \subset M_{n-1}$) and M_0 consists of X alone. For every x of X, put $U(x, n) = St(x, \mathcal{M}_n)$, where $St(x, \mathcal{M}_n)$ means the union of the sets M of $\mathcal{M}_n$ such that $x \in M$, and call it a preneighborhood of x with rank n. Put $\mathcal{U}_n = \{U(x, n) : x \in X\}$ and $\mathcal{U} = \cup\{\mathcal{U}_n : n \in N\}$. Suppose that $\{U(x, n) : n \in N\}$ is not a neighborhood base of x. Then for any open set O such that $O \ni x$ and every $n \in N$ we have $U(x, n) \not\subset O$. Therefore for every $n \in N$ there exists an $M'_n \in \mathcal{M}_n$ such that $M'_n \ni x$ and $M_{n'} \not\subset O$. Hence $\{M'_n : n \in N\}$ is not a neighborhood base at x, which is a contradiction of (2). Therefore $\{U(x, n) : n \in N\}$ is a neighborhood base in the topological space X and $(X, \mathcal{U}, \mathcal{U}_n)$ is a ranked space such that r-convergence and convergence in the topological sense are identical. $\{\mathcal{U}_n : n \in N\}$ clearly satisfies the condition of Proposition 2. Therefore the ranked space $(X, \mathcal{U})$ is metrizable.

FRINK'S THEOREM [3]. *A T_1-space X is metrizable if and only if there exists a countable open neighborhood base $\{V_i(x) : i \in N\}$ for each point x in X which satisfies the following condition:*

For each point x in X and each number i there exists a number $j = j(x, i)$ such that $V_j(x) \cap V_j(y) \neq \varnothing$ implies $V_j(y) \subset V_i(x)$.

To prove this theorem set $W_i(x) = \bigcap_{j \le i} V_j(x)$. Take an arbitrary point x in X and an arbitrary number i. Set $j_1 = j(x, 1), \ldots, j_i = j(x, i)$. If $j_0 = \max\{j_1, \ldots, j_i\}$, then, as can easily be seen, $W_{j_0}(x) \cap W_{j_0}(y) \neq \varnothing$ implies $W_{j_0}(y) \subset W_i(x)$. Therefore we assume without loss of generality that $V_0(x) = X$ for any point x and the original $\{V_i(x)\}$ is monotone: $V_0(x) \supset V_1(x) \supset V_2(x) \supset \cdots$.

For any point x let $1(x) = 1 < 2(x) = j(x, 1(x)) < 3(x) = j(x, 2(x)) < \cdots$. Set $U_i(x) = V_{i(x)}(x)$, $\mathcal{U}_i = \{U_i(x) : x \in X\}$ $i = 0, 1, \ldots$. and $P(x, i) = St(x, \mathcal{U}_i)$ $i = 0, 1, 2, \ldots$. Then $\{P(x, i) : i \in N\}$ forms a neighborhood base of x. We call

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$P(x, i)$ a preneighborhood of x with rank i. Set $\mathcal{P}_i = \{P(x, i): x \in X\}$ and $\mathcal{P} = \cup\{\mathcal{P}_i : i \in N\}$. Moreover we assume X is a preneighborhood of every point with rank 0. Then $(X, \mathcal{P}, \mathcal{P}_i)$ is a ranked space. Let us show $(X, \mathcal{P}, \mathcal{P}_i)$ satisfies the condition of Proposition 2.

Evidently

(1) $P(x, i) \supset P(x, i+1)$ for $i \in N$.

(2) (i) Since $P(x, i) = St(X, \mathcal{U}_i)$, $P(x, i) \ni y$ implies $P(y, i) \ni x$.

(ii) Suppose $P(x, i) \cap P(y, i) \ni z$. Then there exist $U_i(a) \in \mathcal{U}_i$ such that $U_i(a) \ni x, z$, and $U_i(b) \in \mathcal{U}_i$ such that $U_i(b) \ni y, z$. $U_i(a) \cap U_i(b) \neq \emptyset$ implies $U_{i-1}(a) \supset U_i(b) \ni y, z$ and $U_{i-1}(a) \supset U_i(a) \ni x, z$. Since $P(x, i-1) \supset U_{i-1}(a)$, $P(x, i-1) \ni y$.

Since $\{P(x, i): i \in N\}$ is a neighborhood base in the topological sense and X is a T_1-space, we have $\bigcap_{i \in N} P(x, i) = \{x\}$. Therefore $(X, \mathcal{P}, \mathcal{P}_i)$ is metrizable such that r-convergence and convergence in the topological sense are identical.

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