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On the Error Term in Duke's Estimate for the Average Special Value of L-Functions

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Abstract. Let $\mathcal{F}$ be an orthonormal basis for weight 2 cusp forms of level N. We show that various weighted averages of special values $L(f \otimes \chi, 1)$ over $f \in \mathcal{F}$ are equal to $4\pi c + O(N^{-1+\epsilon})$, where c is an explicit nonzero constant. A previous result of Duke gives an error term of $O(N^{-1/2} \log N)$.

Introduction

Let N be a positive integer, and let $\mathcal{F}$ be a basis for $S_2(\Gamma_0(N))$ which is orthonormal for the Petersson inner product. Let χ be a Dirichlet character.

In [2], Duke proves the estimate

$$\sum_{f \in \mathcal{F}} a_1(f) L(f \otimes \chi, 1) = 4\pi + O(N^{-1/2} \log N) \tag{1}$$

in case N is prime and χ is unramified at N, using the Petersson formula and the Weil bounds on Kloosterman sums.

In this note, we will sharpen the error term in Duke's estimate to $O(N^{-1+\epsilon})$. At the same time, we observe that his techniques generalize to arbitrary N and χ, and to the situation where a_1 is replaced by an arbitrary a_m.

We have in mind an application to the problem of finding all primitive solutions to the generalized Fermat equation

$$A^4 + B^2 = C^p \tag{2}$$

In [3], we show how to associate to a solution of (2) an elliptic curve over $\mathbb{Q}[i]$ with an isogeny to its Galois conjugate and a non-surjective mod p Galois representation. Such curves are parametrized by rational points on a certain modular curve X; following Mazur's method, we can place strong constraints on $X(\mathbb{Q})$ by exhibiting a quotient of the Jacobian of X with Mordell–Weil rank 0. This problem, in turn, reduces via the theorem of Kolyvagin and Logachev to proving the existence of a new form f on level p^2 or $2p^2$ such that the image of f under a certain Hecke operator has an L-function with non-vanishing special value. We can then derive from Duke's estimate that (2) has no solutions for $p > 2 \cdot 10^5$. Using the sharper estimate derived here, we find in [3] that (2) has no solutions for $p \geq 211$.

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Theorem Statements

In this section we state various versions of our estimate. If f is a modular form, we always use $a_m(f)$ to denote the Fourier coefficients of the q-expansion of f:

$$f = \sum_{m=0}^{\infty} a_m(f) q^m.$$

As above, we denote by $\mathcal{F}$ a Petersson-orthonormal basis for $S_2(\Gamma_0(N))$.

Write (a_m, L_χ) for the sum

$$\sum_{f\in\mathcal{F}} a_m(f) L(f \otimes \chi, 1)$$

and let q be the conductor of χ.

We obtain a rather complicated bound for (a_m, L_χ), which we state below.

Theorem 1 *Suppose $N \geq 400$, $N \nmid q$ and let σ be a real number with $q^2/2\pi \leq \sigma \leq Nq/\log N$. Then we can write*

$$(a_m, L_\chi) = 4\pi\chi(m) e^{-2\pi m/\sigma N \log N} - E^{(3)} + E_3 - E_2 - E_1 + (a_m, B(\sigma N \log N))$$

where

- $|(a_m, B(\sigma N \log N))| \leq 30(400/399)^3 \exp(2\pi) q^2 m^{3/2} N^{-1/2} d(N) N^{-2\pi\sigma/q^2}$;
- $|E_1| \leq (16/3)\pi^3 m^{3/2} \sigma \log N e^{-N/2\pi m\sigma \log N}$;
- $|E_2| \leq (8/9)\pi^5 \zeta^2(7/2) m^{5/2} \sigma^2 N^{-3/2} \log^2 N$;
- $|E_3| \leq (8/3)\zeta^2(3/2)\pi^3 \sigma m^{3/2} N^{-1/2} \log N d(N) e^{-N/2\pi m\sigma \log N}$;
- $|E^{(3)}| \leq 16\pi^3 m \sum_{c>0, N|c} \min[\frac{2}{\pi}\phi(q) c^{-1} \log c, \frac{1}{6}\sigma N \log N m^{1/2} c^{-3/2} d(c)]$.

Proof Immediate from Propositions 5, 6, 7, 9, 10. ■

If q, m are considered as constants, the bound above simplifies considerably.

Corollary 2

$$(a_m, L_\chi) = 4\pi\chi(m) e^{-2\pi m/\sigma N \log N} + O(N^{-1+\epsilon})$$

where the implied constants depend only on m, q, and ϵ.

Proof The only thing to check is that the bound on $|E^{(3)}|$ is of order at most $N^{-1+\epsilon}$; one checks this by fixing some cutoff X, say $X = N^3$, and observing that both $\sum_{0<c<X, N|c} c^{-1} \log c$ and $N \log N \sum_{c>X, N|c} c^{-3/2} d(c)$ are $O(N^{-1+\epsilon})$. ■

The "true behavior" of (a_m, L_χ) is less clear. One might for instance ask: what is the true asymptotic behavior of $(a_m, L_\chi) - 4\pi\chi(m)$ as N grows with m, q held fixed? More generally, what is the shape of the region in m, q, N-space for which (a_m, L_χ) is close to $4\pi\chi(m)$? One might, for instance, define $f_\delta(N)$ to be the smallest

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integer such that $|(a_m, L_\chi) - 4\pi\chi(m)| \leq \delta$ for all $m \leq f(N)$. Duke's approach shows that $f_\delta(N) \gg N^{1/2}$, whereas the present results show that $f_\delta(N) \gg N^{3/5}$. (Remark: further expansion of the Bessel function in Taylor series will give $f_\delta(N) \gg N^{1-\epsilon}$, with a constant depending on q, ϵ.) Similarly, one could try to optimize the dependence on q in order to get a result that applied when q is large compared to N.

Proof of the Main Result

We begin by recalling the Petersson trace formula.

Lemma 3 (Petersson trace formula) *Let m, n be positive integers, and let $\mathcal{F}$ be an orthonormal basis for $S_2(\Gamma_0(N))$.*

Then

$$\frac{1}{4\pi\sqrt{mn}} \sum_{f\in\mathcal{F}} a_m(f)a_n(f) = \delta_{mn} - 2\pi \sum_{\substack{c>0 \\ c=0 \ (\mathrm{mod}\ N)}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c) \tag{3}$$

where $S(m,n;c)$ is the Kloosterman sum for $\Gamma_0(N)$, and J_1 is the J-Bessel function.

Proof See [4, Th. 3.6]. ■

We can and do assume that $\mathcal{F}$ consists of eigenforms for T_p for all $p \nmid N$, and for w_N.

The Petersson product on $S_2(\Gamma_0(N))$ induces an inner product on the dual space $S_2(\Gamma_0(N))^\vee$. With respect to this product, the left-hand side of (3) is $\frac{1}{4\pi\sqrt{mn}}(a_m, a_n)$.

Lemma 3 immediately gives a bound on the size of (a_m, a_n).

Lemma 4 *We have the bound*

$$|(a_m, a_n) - 4\pi\sqrt{mn}\delta_{mn}| \leq 8\zeta^2(3/2)\pi^2(m,n)^{1/2} mnN^{-3/2} d(N).$$

Proof Applying the Weil bound

$$|S(m,n;c)| \leq (m,n,c)^{1/2} d(c) c^{1/2}$$

and the fact that $|J_1(x)| \leq x/2$ yields

$$\begin{aligned}
&|4\pi\sqrt{mn} \sum_{\substack{c>0 \\ c=0 \ (\mathrm{mod}\ N)}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c)| \\
&\qquad \leq 4\pi\sqrt{mn} \sum_{\substack{c>0 \\ c=0 \ (\mathrm{mod}\ N)}} c^{-1/2} d(c)(m,n)^{1/2}(2\pi\sqrt{mn}/c) \\
&\qquad = 8\pi^2(m,n)^{1/2} mn \sum_{\substack{c>0 \\ c=0 \ (\mathrm{mod}\ N)}} c^{-3/2} d(c).
\end{aligned}$$

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Now the sum over c is equal to

$$\sum_{b>0}(Nb)^{-3/2}d(Nb)$$

which is bounded above by

$$N^{-3/2}d(N)\sum_{b>0}b^{-3/2}d(b)=\zeta^2(3/2)N^{-3/2}d(N).$$

This yields the desired result. ■

Let L_χ be the element of $S_2(\Gamma_0(N))^\vee$ which sends each cusp form f to the special value $L(f\otimes\chi,1)$. Then the value to be estimated is precisely (L_χ,a_m). In order to estimate this product via the Petersson formula, it is necessary to approximate L_χ as a sum of Fourier coefficients. We accomplish this via the standard approximation to $L_\chi(f)$ by a rapidly converging series [5].

We define a linear functional $A(x)$ on $S_2(\Gamma_0(N))$ by the rule

$$A(x)(f)=\sum_{n\geq 1}\chi(n)a_n(f)n^{-1}e^{-2\pi n/x}.$$

Then A is a good approximation to the functional L_χ when x becomes large. Let $B(x)=A(x)-L_\chi$. Let M be an integer such that $f\otimes\chi$ is a cuspform on $\Gamma_1(M)$ for all $f\in\mathcal{F}$.

By the functional equation for $L(f\otimes\chi,s)$, we have

$$B(x)(f)=\sum_{n\geq 1}a_n(w_M(f\otimes\chi))n^{-1}e^{-2\pi nx/M}.$$

When x is on the order of $N\log N$, then $B(x)$ is a short sum, and we want to show it is negligible. The only difficulty is bounding the Fourier coefficients of $w_M(f\otimes\chi)$. This is difficult only in case the conductor of χ has common factors with N, in which case $f\otimes\chi$ is not necessarily an eigenform for any W-operator, even when f is a new form, see [1].

A crude bound will be enough for us. We define an "average cuspform"

$$g=\sum_{f\in\mathcal{F}}a_m(f)(f\otimes\chi).$$

Then

$$a_n(g)=\chi(n)(a_m,a_n)$$

and it follows from Lemma 4 that

$$|a_n(g)|\leq(8\zeta^2(3/2)\pi^2m^{3/2}N^{-3/2}d(N))n$$

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for all $n \neq m$, while

$$|a_m(g)| \leq 4\pi\sqrt{mn} + (8\zeta^2(3/2)\pi^2 m^{3/2} N^{-3/2} d(N))n$$

when $m = n$.

We have that

$$\begin{aligned}(a_m, B(x)) &= \sum_{f\in\mathcal{F}} a_m(f) \sum_{n>0} a_n(w_M(f\otimes\chi))n^{-1}e^{-2\pi nx/M}\\ &= \sum_{n>0} a_n(w_M g)n^{-1}e^{-2\pi nx/M},\end{aligned}$$

so it remains to bound the Fourier coefficients of the single form $w_M g$. Write c for the constant $8\zeta^2(3/2)\pi^2 m^{3/2} N^{-3/2} d(N)$.

If τ is a point in the upper half plane, we have

$$\begin{aligned}|g(\tau)| \leq \sum_{n>0} |a_n e^{2\pi i\tau}| &= \sum_{n>0} |a_n| \exp(-2\pi \operatorname{Im}(n\tau))\\ &\leq \sum_{n>0} cn \exp(-2\pi \operatorname{Im}(n\tau)) + 4\pi m \exp(-2\pi \operatorname{Im}(m\tau))\\ &\leq c(2\pi \operatorname{Im}(\tau))^{-2} + 4\pi m.\end{aligned}$$

Choose a positive real constant α. The Fourier coefficient $a_n(w_M g)$ can be expressed as

$$\begin{aligned}(4)\qquad &\int_0^1 w_M g(\alpha i + t)\exp(-2\pi i n(\alpha i + t))\,dt\\ &\qquad = \int_0^1 M^{-1}(\alpha i + t)^{-2} g(-1/M(\alpha i + t))\exp(-2\pi i n(\alpha i + t))\,dt.\end{aligned}$$

Now $\operatorname{Im}((-1/M(\alpha i + t))) = M^{-1}\alpha|\alpha i + t|^{-2}$. So it follows from (4) that

$$\begin{aligned}|a_n(w_M g)| &\leq \int_0^1 M^{-1}|\alpha i + t|^{-2}[c(2\pi)^{-2}M^2\alpha^{-2}|\alpha i + t|^4 + 4\pi m]\exp(2\pi n\alpha)\,dt\\ &= cM(2\pi)^{-2}\exp(2\pi i n\alpha)\alpha^{-2}\int_0^1 |\alpha i + t|^2\,dt\\ &\quad + 4\pi m M^{-1}\exp(2\pi n\alpha)\int_0^1 |\alpha i + t|^{-2}\,dt\\ &\leq cM(2\pi)^{-2}\exp(2\pi n\alpha)\alpha^{-2}(\alpha^2 + 1) + 4\pi m M^{-1}\exp(2\pi n\alpha)\alpha^{-2}.\end{aligned}$$

Now setting $\alpha = 1/n$ yields

$$|a_n(w_M g)| \leq cM(2\pi)^{-2}\exp(2\pi)(1 + n^2) + 4\pi\exp(2\pi)mM^{-1}n^2.$$

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We now use the very rough bound $1 + n^2 \leq n^2(n+1)$ to obtain

$$\begin{aligned}|(a_m, B(x))| &= |\sum_{n>0} a_n(w_M g) n^{-1} e^{-2\pi nx/M}| \\ &\leq [cM(2\pi)^{-2}\exp(2\pi) + 4\pi m M^{-1}\exp(2\pi)] \sum_{n>0} n(n+1)e^{-2\pi nx/M} \\ &= \exp(2\pi)(cM(2\pi)^{-2} + 4\pi m M^{-1}) \\ &\qquad \times (2\exp(-2\pi x/M))(1-\exp(-2\pi x/M))^{-3}.\end{aligned}$$

Now M can be taken to be q^2N where q is the conductor of χ. Let σ be a constant to be fixed later, and set $x = \sigma N \log N$. Finally, suppose $N > 400$ and suppose $\sigma > q^2/2\pi$. First of all, we observe that under the hypothesis on N,

$$\begin{aligned}cM(2\pi)^{-2} + 4\pi m M^{-1} &= 2\zeta^2(3/2)q^2 m^{3/2} N^{-1/2} d(N) + 4\pi m q^{-2} N^{-1} \\ &\leq 15 q^2 m^{3/2} N^{-1/2} d(N).\end{aligned}$$

Also,

$$1 - \exp(-2\pi x/M) = 1 - \exp(-2\pi\sigma \log N/q^2) \leq 1 - 400^{-2\pi\sigma/q^2} \leq 400/399.$$

So, in all, we have proved the following.

Proposition 5 *Suppose $N \geq 400$ and $\sigma > q^2/2\pi$. Then*

$$|(a_m, B(\sigma N \log N))| \leq 30(400/399)^3 \exp(2\pi) q^2 m^{3/2} N^{-1/2} d(N) N^{-2\pi\sigma/q^2}.$$

In other words, we have shown that the error in approximating (a_m, L_χ) by $(a_m, A(x))$ is bounded by a function decreasing quickly in N, if x is chosen on the order of $q^2 N \log N$.

We now turn to the analysis of $(a_m, A(\sigma N \log N))$.

First of all, we have

$$\begin{aligned}(a_m, A(\sigma N \log N)) &= \sum_{f\in\mathcal{F}} a_m(f) \sum_{n>0} \chi(n) a_n(f) n^{-1} e^{-2\pi n/\sigma N \log N} \\ &= \sum_{n>0} \chi(n)(a_m, a_n) n^{-1} e^{-2\pi n/\sigma N \log N}\end{aligned}$$

which, by Lemma 3, equals

$$4\pi\chi(m)e^{-2\pi m/\sigma N \log N} - 8\pi^2\sqrt{m}\sum_{n>0}\chi(n)n^{-1/2}e^{-2\pi n/\sigma N \log N} \sum_{\substack{c>0 \\ c\equiv 0 \ (\mathrm{mod}\ N)}} c^{-1} S(m,n;c) J_1(4\pi\sqrt{mn}/c).$$

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We split the latter sum into two ranges; write

$$E^{(1)} = 8\pi^2\sqrt{m}\sum_{n>0}\chi(n)n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{\substack{c>2\pi\sqrt{mn}\\ c\equiv 0\ (\mathrm{mod}\ N)}} c^{-1}S(m,n;c)J_1(4\pi\sqrt{mn}/c)$$

and

$$E_1 = 8\pi^2\sqrt{m}\sum_{n>0}\chi(n)n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{\substack{0<c\le 2\pi\sqrt{mn}\\ c\equiv 0\ (\mathrm{mod}\ N)}} c^{-1}S(m,n;c)J_1(4\pi\sqrt{mn}/c).$$

We claim E_1 decreases quickly with N. First, recall that $|J_1(a)| \le \min(1, a/2)$ for all real a. So

$$|E_1| \le 8\pi^2\sqrt{m}\sum_{n>0}n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{0<Nb\le 2\pi\sqrt{mn}}(Nb)^{-1}S(m,n;Nb).$$

Note that the inner sum in $|E_1|$ has nonzero terms only when $n > (N/2\pi\sqrt{m})^2$. In this range, the exponential decay takes over. We observe that $|S(m,n;Nb)| \le m^{1/2}(Nb)^{1/2}d(Nb) < 2\sqrt{m}Nb$, so we can bound E_1 by

$$\begin{aligned}
|E_1| &\le 8\pi^2\sqrt{m}\sum_{n>(N/2\pi\sqrt{m})^2} n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{0<Nb\le 2\pi\sqrt{mn}} 2\sqrt{m}\\
&\le 8\pi^2\sqrt{m}\sum_{n>(N/2\pi\sqrt{m})^2} n^{-1/2}e^{-2\pi n/\sigma N\log N}(2\sqrt{m})(2\pi\sqrt{mn}/N)\\
&= 32\pi^3N^{-1}m^{3/2}\sum_{n>(N/2\pi\sqrt{m})^2} e^{-2\pi n/\sigma N\log N}\\
&\le 32\pi^3N^{-1}m^{3/2}e^{-N/2\pi m\sigma\log N}(1-e^{-2\pi/\sigma N\log N})^{-1}.
\end{aligned}$$

We now simplify this bound under assumptions on N and σ.

Proposition 6 *Suppose $N \ge 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_1| \le (16/3)\pi^3m^{3/2}\sigma\log Ne^{-N/2\pi m\sigma\log N}.$$

Proof This amounts to the observation that $\sigma N\log N \ge 300$, from which it follows that

$$(1-e^{-2\pi/\sigma N\log N})^{-1} \le (1/6)\sigma N\log N. \qquad \blacksquare$$

We now consider the sum $E^{(1)}$ over the range where n is small compared to c. In this range, we use the Taylor approximation

$$|J_1(a) - a/2| \le (1/16)a^3. \tag{5}$$

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So we can write $E^{(1)} = E^{(2)} + E_2$, where

$$E^{(2)} = 8\pi^2\sqrt{m}\sum_{n>0}\chi(n)n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}} c^{-1}S(m,n;c)(2\pi\sqrt{mn}/c).$$

We claim E_2 decreases with N. For we have by (5) that

$$|E_2| \le 8\pi^2\sqrt{m}\sum_{n>0}n^{-1/2}e^{-2\pi n/\sigma N\log N}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}} c^{-1}S(m,n;c)(1/16)(4\pi\sqrt{mn}/c)^3$$

$$= 32\pi^5 m^2\sum_{n>0}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}} ne^{-2\pi n/\sigma N\log N}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}} c^{-4}S(m,n;c).$$

We now use the Weil bound $|S(m,n;c)| \le m^{1/2}c^{1/2}d(c)$ to get

$$|E_2| \le 32\pi^5 m^{5/2}\sum_{n>0}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}} ne^{-2\pi n/\sigma N\log N}c^{-7/2}d(c)$$

$$\le 32\pi^5 m^{5/2}\sum_{n>0}\sum_{b>0} ne^{-2\pi n/\sigma N\log N}N^{-7/2}d(N)b^{-7/2}d(b)$$

$$\le 32\pi^5 m^{5/2}N^{-7/2}d(N)\zeta^2(7/2)\sum_{n>0} ne^{-2\pi n/\sigma N\log N}.$$

So we can write

$$|E_2| \le 32\pi^5\sqrt{3}\zeta(3)m^{5/2}N^{-7/2}e^{-2\pi/\sigma N\log N}(1-e^{-2\pi/\sigma N\log N})^{-2}.$$

Proposition 7 *Suppose $N > 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_2| \le (8/9)\pi^5\zeta^2(7/2)m^{5/2}\sigma^2 N^{-3/2}\log^2 N.$$

Proof Another use of the bound $(1-e^{-2\pi/\sigma N\log N})^{-1} \le (1/6)\sigma N\log N$. ■

We now come to $E^{(2)}$, which is the main term of the error

$$|(a_m, L_\chi) - 4\pi\chi(m)e^{-2\pi m/\sigma N\log N}|.$$

Recall from above that

$$E^{(2)} = 16\pi^3 m\sum_{n>0}\sum_{\substack{c>2\pi\sqrt{mn}\\ c=0\,(\mathrm{mod}\ N)}}\chi(n)e^{-2\pi n/\sigma N\log N}c^{-2}S(m,n;c).$$

Applying the Weil bound to $S(m,n;c)$ yields the estimate $E^{(2)} = O(N^{-1/2}\log N)$ which appears in [2]. We want to exploit cancellation between the Kloosterman sums in order to improve Duke's bound on $E^{(2)}$.

For simplicity, we carry this out under assumptions on the size of N and σ. For the remainder of this section, assume that

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- $N \geq 400$;
- $q^2/2\pi \leq \sigma \leq Nq/\log N$.

Recall that under these hypotheses

$$\sigma N \log N \geq (1/2\pi) 400 \log 400 > 300.$$

First of all, we will need a simple bound on the modulus of $1 - e^z$.

Lemma 8 *Let z be a complex number with $|\operatorname{Im} z| \leq \pi$ and $-2\pi/30 \leq \operatorname{Re} z \leq 0$. Then*

$$(1/2)|z| \leq |1 - e^z| \leq |z|.$$

Proof The extrema of $|1 - e^z|/|z|$ lie on the boundary of the rectangular region under consideration; now a consideration of the derivatives of $|1 - e^z|/|z|$ on each of the four edges of the region shows that the extrema are at the corners. Computation of the values of $|1 - e^z|/|z|$ gives the result. ■

Write

$$E^{(3)} = 16\pi^3 m \sum_{n>0} \sum_{\substack{c>0 \\ c=0 \,(\mathrm{mod}\ N)}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m, n; c)$$

and

$$E_3 = 16\pi^3 m \sum_{n>0} \sum_{\substack{c \leq 2\pi\sqrt{mn} \\ c=0 \,(\mathrm{mod}\ N)}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m, n; c).$$

So $E^{(2)} = E^{(3)} - E_3$.

The sum E_3, like E_1, is supported in the region where exponential decay dominates. To be precise, the inner sum in E_3 has nonzero terms only when

$$n \geq (c/2\pi\sqrt{m})^2 \geq N^2/4\pi^2 m.$$

It follows that

$$\begin{aligned} |E_3| &\leq 16\pi^3 m \sum_{n>N^2/4\pi^2 m} \sum_{\substack{c>0 \\ c=0 \,(\mathrm{mod}\ N)}} e^{-2\pi n/\sigma N \log N} m^{1/2} c^{-3/2} d(c) \\ &\leq 16\zeta^2(3/2)\pi^3 m^{3/2} (N^{-3/2} d(N)) e^{-N/2\pi m\sigma \log N} (1 - e^{-2\pi/\sigma N \log N})^{-1}. \end{aligned}$$

Using the lower bounds on N and σ, we obtain

Proposition 9 *Suppose $N > 400$ and $\sigma > q^2/2\pi$. Then*

$$|E_3| \leq (8/3)\zeta^2(3/2)\pi^3 \sigma m^{3/2} N^{-1/2} \log N d(N) e^{-N/2\pi m\sigma \log N}.$$

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It now remains only to bound the main term

$$E^{(3)} = 16\pi^3 m \sum_{n>0} \sum_{\substack{c>0 \\ c=0 \pmod N}} \chi(n) e^{-2\pi n/\sigma N \log N} c^{-2} S(m,n;c).$$

We can write

$$E^{(3)} = 16\pi^3 m \sum_{\substack{c>0 \\ c=0 \pmod N}} c^{-2} S(c) \tag{6}$$

where

$$\begin{aligned} S(c) &= \sum_{n>0} \chi(n) e^{-2\pi n/\sigma N \log N} S(m,n;c) \\ &= \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} \sum_{n>0} \chi(n) e^{-2\pi n/\sigma N \log N} e\left(\frac{mx+ny}{c}\right) \end{aligned}$$

where $e(z) = e^{2\pi i z}$ and $y \in (Z/c\mathbb{Z})^*$ is the multiplicative inverse of x.

For ease of notation, write $A = \sigma N \log N$, and for each integer y write $\epsilon_y = 2\pi(-1/A + yi/c)$. Then

$$\begin{aligned} |S(c)| &\leq \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} \left| \sum_{n>0} \chi(n) e^{-2\pi n/A} e\left(\frac{ny}{c}\right) \right| \\ &= \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} \left| \sum_{\alpha=1}^{q} \chi(\alpha) e^{-2\pi\alpha/A} e\left(\frac{\alpha y}{c}\right) \sum_{\nu \geq 0} e^{2\pi q\nu/A} e\left(\frac{q\nu y}{c}\right) \right| \\ &= \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} \left| \sum_{\alpha=1}^{q} \chi(\alpha) e^{-2\pi\alpha/A} e\left(\frac{\alpha y}{c}\right) (1 - e^{2\pi q(-1/A + iy/c)})^{-1} \right| \\ &= \sum_{y \in (\mathbb{Z}/c\mathbb{Z})^*} \left| (1 - e^{q\epsilon_y})^{-1} \sum_{\alpha=1}^{q} \chi(\alpha) e^{\alpha\epsilon_y} \right| \\ &\leq \sum_{y \in (\mathbb{Z}/c\mathbb{Z})^*} |(1 - e^{q\epsilon_y})|^{-1} \left| \sum_{\alpha=1}^{q} \chi(\alpha) e^{\alpha\epsilon_y} \right|. \end{aligned}$$

We have the trivial bound $|\sum_{\alpha=1}^{q} \chi(\alpha) e^{\alpha\epsilon_y}| \leq \phi(q)$. (This bound can be sharpened to $O(\sqrt{q}\log q)$ if one wishes to improve the dependence on q.) We now estimate $\sum_y |(1 - e^{q\epsilon_y})^{-1}|$. For each y, let $f(y)$ be the unique integer congruent to qy modulo c with $|f(y)| \leq c/2$. By our assumption that $N \not| q$, we have $f(y) \neq 0$. Then by Lemma 8 one has

$$|(1 - e^{q\epsilon_y})^{-1}| < \frac{c}{\pi |f(y)|}.$$

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Now the values of $|f(y)|$ range over the integers a between 1 and $c/2$ such that $(a,c)=(q,c)$, each of which arises from at most $2(q,c)$ values of y. So we have

$$\sum_{y\in(\mathbb{Z}/c\mathbb{Z})^*}\left|(1-e^{q\epsilon_y})^{-1}\right| \le \frac{2(q,c)c}{\pi}\Big[\frac{1}{(q,c)}+\frac{1}{2(q,c)}+\cdots+\frac{1}{r(q,c)}\Big]$$
$$=(2c/\pi)\Big[1+\frac{1}{2}+\cdots+\frac{1}{r}\Big]$$

where r is the largest integer such that $r(q,c)\le c/2$. The value of $(2c/\pi)[1+\ldots+1/r]$ is largest when $(q,c)=1$; in that case it is bounded above by

$$(2c/\pi)[\log(c/2)+\gamma+2/c],$$

where γ is Euler's constant. Since $c>400$, the above expression is bounded by $(2/\pi)c\log c$. So, in all, one has

$$|S(c)| < (2/\pi)\phi(q)c\log c. \tag{7}$$

We observe as well that, from the Weil bound, we have

$$|S(c)| \le \sum_{n>0} e^{-2\pi n/A}m^{1/2}c^{1/2}d(c) \le m^{1/2}c^{1/2}d(c)(1-e^{-2\pi/A})^{-1}.$$

Recall from the proof of Proposition 6 that $(1-e^{-2\pi/A})^{-1}\le(1/6)A$ under our conditions on N and σ. So

$$|S(c)| \le (1/6)Am^{1/2}c^{1/2}d(c). \tag{8}$$

In particular, we immmediately have the following proposition:

Proposition 10 *Suppose $N\ge 400$, $N\nmid q$, and $\sigma>q^2/2\pi$. Then*

$$|E^{(3)}| \le 16\pi^3 m \sum_{\substack{c>0\\ c\equiv 0 \ (\mathrm{mod}\ N)}} \min\Big[\frac{2}{\pi}\phi(q)c^{-1}\log c, \frac{1}{6}\sigma N\log N m^{1/2}c^{-3/2}d(c)\Big].$$

This completes the proof of Theorem 1.

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