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QUASI-DUAL-CONTINUOUS MODULES

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Abstract

Quasi-dual-continuous modules, which generalize the concept of dual-continuous modules, are studied Mohamed, Müller and Singh had obtained some decomposition theorems and their partial converses, for dual-continuous modules. It is shown that these results can be extended to quasi-dual-continuous modules. Further, a short proof of a decomposition theorem for quasi-dual-continuous modules established recently by Oshiro is given. Some more structure theorems for such modules are established. Finally, quasi-dual-continuous covers are studied, and duals for results of Müller and Rizvi are derived.

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Consider the following conditions on a module M_R.

(D_1) For any submodule A of M, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $A \cap M_2$ is small in M_2.

(D_2) If for any submodule N of M, M/N is isomorphic to a summand of M, then N is a summand of M.

(D_3) If for two summands A, B of M, $M = A + B$ holds, then $A \cap B$ is a summand of M.

(D_4) If for two summands A, B of M, $M = A + B$ holds and $A \cap B$ is small in M, then $M = A \oplus B$.

Utumi [18] studied continuous rings. The concept of continuous rings was extended to that of continuous modules by Jeremy [5] and by Mohamed and Bouhy [10]. Since the conditions (D_1) and (D_2) are dual to those defining

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continuous modules, a module satisfying (D_1) and D_2) was called a dual-continuous (in short d-continuous) module, by Mohamed and Singh [13]. In [11] and [13] Mohamed, Müller and Singh established a decomposition theorem for d-continuous modules. Then dual continuous modules, and modules satisfying (D_1) only, were further studied by Abdul-Karim, Mohamed, Müller and Singh in [8], [9], [12], [16], [17]. Jeremy [5] defined the concept of quasi-continous modules. Dualizing it, we call a module M_R satisfying conditions (D_1) and (D_3) *quasi-dual-continous* (in short qd-continuous). Now [13, Lemma 3.6] shows that condition (D_2) implies (D_3); so any d-continuous module is qd-continuous. In Section 1 we show that most of the techniques or results given for d-continuous modules in [13] hold for qd-continuous modules. Recently Oshiro [15] has introduced the concept of semi-perfect and quasi-semi-perfect modules. These concepts are precisely the same as that of d-continuous modules and qd-continuous modules respectively. He has established a decomposition theorem for qd-continuous modules which improves upon that for d-continuous modules established in [11] and [13]. In Section 2, we give a short proof of this theorem. Other interesting results for qd-continuous modules are in Propositions 2.8 and 2.9. We extend [12, Theorems 2.2 and 2.4] to qd-continuous modules. In Section 3, qd-continuous covers are studied.

The notations and terminology used in [13] are also used here. Thus for the definition of a small submodule, d-complement of a submodule, local module and other undefined terms we refer to [13]. A module M_R is said to be *supplemented* if for any submodule A of M, any submodule B, such that $M = A + B$, contains a d-complement of A. Supplemented modules are precisely the perfect modules defined by Miyashita [7]. A nonzero module M is said to be hollow if every proper submodule of M is small in M. Clearly any indecomposable module satisfying (D_1) is a hollow module. A decomposition $M = \Sigma_A \oplus M_\alpha$ of a module M as a direct sum of nonzero submodules $(M_\alpha)_{\alpha\in A}$ is said to complement summands (complement maximal summands) in case for every (every maximal) summand K of M there exists a subset $B \subseteq A$ with $M = (\Sigma_B \oplus M_\beta) \oplus K$. For properties of such decompositions we refer to Anderson and Fuller [1]. For the definition and properties of M-projective modules, where M is any module, we refer to Azumaya [2].

1. Some general results

Proposition 1.1. *Under condition* (D_1), *the conditions* (D_3) *and* (D_4) *are equivalent.*

Proof. It is clear that (D_3) implies (D_4). Assume (D_4) and let A and B be summands of M such that $M = A + B$. By (D_1), $M = M_1 \oplus M_2$ such that

$M_1 \subset A \cap B$ and $A \cap B \cap M_2 \subsetneq_s M$. Now $B = M_1 \oplus B \cap M_2$. Hence $B \cap M_2$ is a summand of M. Also

$$M = A + B = A + (M_1 \oplus B \cap M_2) = A + B \cap M_2.$$

As A and $B \cap M_2$ are summands of M and $A \cap B \cap M_2 \subsetneq_s M$, we get $(A \cap B) \cap M_2 = 0$. Hence $M = (A \cap B) \oplus M_2$, and the result follows.

The above proposition shows that quasi-semi-perfect modules as defined by Oshiro are exactly the *qd*-continuous modules.

The following is easy to prove.

PROPOSITION 1.2. *Any summand of a module M satisfying any condition* (D_i) *also satisfies* (D_i). *In particular a summand of a qd-continuous module is qd-continuous.*

In [13, Lemma 3.6] it was proved that a module with condition (D_2) satisfies (D_3). It is obvious that Lemma 3.6 in [13] also holds for *qd*-continuous modules. Then a number of results were proved using only condition (D_1) and Lemma 3.6. Therefore these results hold for *qd*-continuous modules. In particular Proposition 3.7, Corollary 3.9, Proposition 4.1 and Corollary 4.2 in [13] give respectively the following four results.

PROPOSITION 1.3. *A qd-continuous module M is supplemented (perfect in the sense of Miyashita* [7]), *and every d-complement submodule of M is a summand.*

COROLLARY 1.4. *Let* M_1 *be a summand of a qd-continuous module M. If* M_2 *is a d-complement of* M_1, *then* $M = M_1 \oplus M_2$.

PROPOSITION 1.5. *If* $A \oplus B$ *is qd-continuous, then A is B-projective.*

COROLLARY 1.6. *If* $M \times M$ *is qd-continuous, then M is quasi-projective.*

It was pointed out in the proof of [13, Theorem 2.3] that a quasi-projective module always satisfies (D_2). Hence for a quasi-projective module, the notions of *qd*-continuity and *d*-continuity coincide.

In [5, Definition 3.2], Jeremy mentioned that a module M is quasi-continuous if and only if $M = A \oplus B$ for any two submodules A and B which are complements of each other. The following dual result is an easy consequence of Proposition 1.3 and Corollary 1.4.

PROPOSITION 1.6. *A module M is qd-continuous if and only if M is supplemented and* $M = A \oplus B$ *for any two submodules A and B which are d-complements of each other.*

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2. Decomposition theorems

Mohamed, Müller and Singh [11] and [13] proved the following decomposition theorem for d-continuous modules.

THEOREM 2.1. *A d-continuous module M has a decomposition, unique up to isomorphism,* $M = \sum_{i\in I} \oplus A_i \oplus N$ *where each* A_i *is a local module and* $N = \operatorname{Rad} N$.

Recently, Oshiro [15] obtained a decomposition theorem for qd-continuous modules which improves the above theorem. The following comprises Theorem 3.5, Theorem 3.10 and Corollary 3.11 in [15].

THEOREM 2.2 (Oshiro). *A qd-continuous module M has a decomposition* $M = \sum_{i\in I} \oplus H_i$ *where each* H_i *is a hollow module; further, this decomposition complements summands.*

In this section we give a short and simplified proof of Oshiro's theorem. We also give some partial converses of this theorem, which extend analogous results for d-continuous modules due to Mohamed and Müller [11, 12].

We need the following three results.

LEMMA 2.3. *Let* $M = M_1 \oplus M_2$ *be a qd-continuous module, and* $\pi_i\colon M \to M_i$ *be the associated projections. If* $\pi_2 N \subsetneq_s M_2$ *for some summand N of M, then* $N \cap M_2 = 0$ *and* $N \oplus M_2$ *is a summand.*

PROOF. Let $S = \pi_1 N$. By (D_1), $M_1 = A \oplus B$ such that $A \subset S$ and $S \cap B \subsetneq_s M$. Let π denote the projection $A \oplus B \oplus M_2 \to B$. Then $\pi N = \pi\pi_1 N = \pi S = S \cap B \subsetneq_s M$. Now $N \cap (B \oplus M_2) \subset \pi N \oplus \pi_2 N \subsetneq_s M$. Since $M = N + (B \oplus M_2)$, we get by (D_3) that $N \cap (B \oplus M_2) = 0$. Hence $M = N \oplus B \oplus M_2$, proving the result.

PROPOSITION 2.4. *The union of any chain of summands of a qd-continuous module M is a summand of M.*

PROOF. Let $\{N_\alpha\}$ be a chain of summands of M and let $N = \bigcup_\alpha N_\alpha$. By ($D_1$), $M = M_1 \oplus M_2$ such that $M_1 \subset N$ and $N \cap M_2 \subsetneq_s M$. Let π_2 be the projection $M_1 \oplus M_2 \to M_2$. Then $\pi_2 N = N \cap M_2$. For any α, $\pi_2 N_\alpha \subset \pi_2 N \subsetneq_s M$. It follows by Lemma 2.3 that $N_\alpha \cap M_2 = 0$. Consequently $N \cap M_2 = 0$ and $N \oplus M_2 = M$.

LEMMA 2.5. *Let M be a qd-continuous module. For every nonzero $x \in M$, there exists a decomposition $M = M_1 \oplus M_2$ such that M_2 is hollow and $x \notin M_1$.*

PROOF. By Zorn's Lemma and Proposition 2.4, we can find a summand M_1 of M maximal with the property $x \notin M_1$. Write $M = M_1 \oplus M_2$. If M_2 is not hollow, then it contains a nonzero summand by (D_1). Let $M_2 = A \oplus B$. Then $M = M_1 \oplus A \oplus B$. Now maximality of M_1 implies that $x \in M_1 \oplus A$ and $x \in M_1 \oplus B$. However this implies $x \in M_1$, a contradiction. Hence M_2 is hollow.

PROOF OF THEOREM 2.2. Let M be a *qd*-continuous module. By Zorn's Lemma and Proposition 2.4, we can find a maximal direct sum $N = \sum_{i\in I} \oplus H_i$ of hollow summands H_i such that N is a summand of M. Then $N = M$ by Proposition 1.2 and Lemma 2.5. Hence $M = \sum_{i\in I} \oplus H_i$.

Let A be a summand of M. Again by Zorn's Lemma and Proposition 2.4, we can find a maximal subset J of I such that $A \cap \sum_{j\in J} \oplus H_j = 0$ and $K = A \oplus \sum_{j\in J} H_j$ is a summand of M. If possible, assume that $K \neq M$. Then by Lemma 2.5, $M = T \oplus H$, where H is a nonzero hollow summand and $K \subset T$. Let π be the projection $T \oplus H \to H$. If $\pi H_\alpha = H$ for some $\alpha \in I$, then $M = T + H_\alpha$. As $T \cap H_\alpha \subsetneq_s M$, we get by (D_3) that $T \cap H_\alpha = 0$. So that $M = T \oplus H_\alpha$. However this contradicts the maximality of J. Therefore, $\pi H_i \neq H$ for every $i \in I$. Let $\{i_1, i_2, \ldots, i_n\}$ be a finite subset of I and let

$$L = H_{i_1} \oplus H_{i_2} \oplus \cdots \oplus H_{i_n}.$$

Then

$$\pi L \subset \pi H_{i_1} + \pi H_{i_2} + \cdots + \pi H_{i_n}.$$

As H is hollow, we get $\pi L \subsetneq_s H$. Then it follows by Lemma 2.3 that $L \cap H = 0$. This proves that $(\sum_{i\in I} \oplus H_i) \cap H = 0$. Consequently $H = 0$, a contradiction. Hence $K = M$, and the result follows.

REMARK. Let $M = \sum_{i\in I} \oplus H_i = \sum_{j\in J} \oplus K_j$ be any two decompositions of a *qd*-continuous module M into hollow submodules. Since these decompositions complement summands, by Anderson and Fuller [1, Theorem 12.4] the two decompositions are equivalent, in the sense that there exist a bijection $\sigma: I \to J$ such that $H_i \cong K_{\sigma(i)}$ for every $i \in I$.

We now prove some more results which are related to the decomposition of *qd*-continuous modules.

PROPOSITION 2.6. *Let M be a qd-continuous module, and B a d-complement of a submodule A of M. If C is a summand of M contained in A, then $C \cap B = 0$ and $C \oplus B$ is a summand of M.*

PROOF. By Proposition 1.3, $M = A' \oplus B$ for some $A' \subset A$. Let π denote the projection $A' \oplus B \to B$. Then $\pi C \subset \pi A = A \cap B \subsetneq_s M$. Hence the result follows by Lemma 2.3.

The following is an immediate consequence of the above proposition.

THEOREM 2.7 (Oshiro [14]). *Let $\{N_\alpha\}_{\alpha \in I}$ be an independent family of submodules of a qd-continuous module M. If for every finite subset F of I, $\Sigma_{\alpha \in F} \oplus N_\alpha$ is a summand of M, then $\Sigma_{\alpha \in I} \oplus N_\alpha$ is a summand.*

PROOF. Let $A = \Sigma_{\alpha \in I} \oplus N_\alpha$ and B be a d-complement of A. Then $M = A \oplus B$ by Proposition 2.6.

The following extends [13, Proposition 4.5].

PROPOSITION 2.8. *Let M be a qd-continuous module. Let N be any summand and A be a hollow summand of M. Then either $N \cap A = 0$ and $N \oplus A$ is a summand of M, or, $N + A = N \oplus S$ for some small submodule S of M and A is isomorphic to a summand of N.*

PROOF. Write $M = N \oplus L$. Then $N + A = N \oplus [(N + A) \cap L]$ yields $(N + A) \cap L \cong A/(A \cap N)$. Consequently as A is hollow. $(N + A) \cap L$ is indecomposable. Two cases arise.

Case I. $(N + A) \cap L$ is not small in M. By (D_1), $(N + A) \cap L$ contains a nonzero summand of M. Consequently $(N + A) \cap L$ itself being indecomposable, is a summand of M. This in turn gives that $N + A$ is a summand of M. By condition (D_3), $N \cap A$ is a summand of M. However A indecomposable and $A \not\subset N$ yield $N \cap A = 0$ and so $N \oplus A$ is a summand of M.

Case II. $S = (N + A) \cap L \subsetneq_s M$. Write $M = A \oplus A'$. Then

$$M = (N + A) + A' = N + (N + A) \cap L + A' = N + A'.$$

By (D_3), $N \cap A'$ is a summand of M. So write $N = N' \oplus (N \cap A')$. Then $M = N' \oplus A'$, and $A \cong N'$. This completes the proof.

As a consequence we get the following result which extends [12, Lemma 2.3].

PROPOSITION 2.9. *Let $\{N_\alpha\}_{\alpha \in I}$ be a set of mutually non-isomorphic hollow summands of a qd-continuous module M. Then $\Sigma_{\alpha \in I} N_\alpha$ is direct and is a summand of M.*

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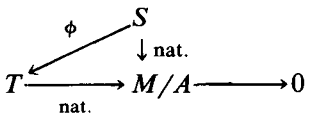
PROOF. By the above proposition $\sum_{\alpha \in F} N_\alpha$ is direct and is a summand of M, for every finite subset F of I. The result now follows by Theorem 2.7.

LEMMA 2.10. *Let $M = S \oplus T = A + T$ such that S is T-projective. Then $M = S' \oplus T$ where $S' \subset A$.*

PROOF. The hypothesis gives the following commutative diagram:



Let $S' = \{x - \phi(x): x \in S\}$. Then $S' \subset A$ and $M = S' \oplus T$.

THEOREM 2.11. *Let $M = \sum_{i=1}^{n} \oplus M_i$ such that M_i is hollow and M_j-projective whenever $i \neq j$. Then M is qd-continuous.*

PROOF. Let $\pi_i: M \to M_i$ be the associated projections.

(i) First consider a non-small submodule B of M. As $B \subset \sum_{i=1}^{n} \oplus \pi_i B$, and each M_i is hollow, we get $\pi_k B = M_k$ for some $k \in \{1, 2, \ldots, n\}$. Then $M = B + \sum_{i \neq k} \oplus M_i$. As M_k is $(\sum_{i \neq k} \oplus M_i)$-projective by [3, Proposition 1.16], using Lemma 2.10, we get $M = M_k' \oplus \sum_{i \neq k} \oplus M_i$, $M_k' \subset B$. Thus any non-small submodule of M contains a hollow summand of M.

(ii) Next, let $M = H \oplus K$ where H is indecomposable. By the above argument, there exists $\alpha \in \{1, 2, \ldots, n\}$ such that $M = H \oplus \sum_{i \neq \alpha} \oplus M_i$. As $H \cong M_\alpha$, H is hollow. Also $K \cong \sum_{i \neq \alpha} \oplus M_i$ implies that H is K-projective.

Let π denote the projection $H \oplus K \to H$. Then $H = \sum_{i=1}^{n} \pi M_i$. Since H is hollow, $H = \pi M_\beta$ for some $\beta \in \{1, 2, \ldots, n\}$. Then $M = M_\beta + K$. Applying Lemma 2.10, we get $M = H' \oplus K$, $H' \subset M_\beta$. As M_β is indecomposable, $H' = M_\beta$. Hence $M = M_\beta \oplus K$. This proves that the decomposition $M = \sum_{i=1}^{n} \oplus M_i$ complements maximal summands.

(iii) Let N be a submodule of M. If N is not small in M, then it contains a hollow summand H_1 of M, by (i). Write $M = H_1 \oplus T_1$. Then by (ii), $M = M_{i_1} \oplus T_1$ for some $i_1 \in \{1, 2, \ldots, n\}$. If $N \cap T_1$ is not small in M, then $N \cap T_1$ contains a hollow summand H_2 of M. Then $M = H_1 \oplus H_2 \oplus T_2 = M_{i_1} \oplus M_{i_2} \oplus T_2$. Repeating the process and noting that this can continue for at most n steps we get $M = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus T_k$ such that $\sum_{i=1}^{k} \oplus H_i \subset N$ and $N \cap T_k$ is small in M. This proves that M satisfies condition (D_1).

(iv) Let $M = C \oplus D$. By (iii) $C = C_1 \oplus C_2 \oplus \cdots \oplus C_t$ for some hollow submodules C_i. Then as the decomposition $M = \sum_{i=1}^{n} \oplus M_i$ complements maximal summands, we get $M = M_{i_1} \oplus M_{i_2} \oplus \cdots \oplus M_{i_t} \oplus D$. Thus $D \cong \sum_{j \notin F} \oplus M_j$ where $F = \{i_1, i_2, \ldots, i_t\}$. Then by [3, Proposition 1.16] C is D-projective.

(v) Let A and B be summands of M such that $M = A + B$. Write $M = B' \oplus B$. Then B' is B-projective by (iv). Then by Lemma 2.10, $M = A' \oplus B$ such that $A' \subset A$. Hence $A = A' \oplus A \cap B$, proving that $A \cap B$ is a summand of M. Thus condition (D_3) holds.

THEOREM 2.12. *Let* $M = \sum_{i \in I} \oplus A_i$ *such that* A_i *is local and* A_j*-projective for* $i \neq j$, *and* Rad $M \underset{s}{\subsetneq} M$. *Then* M *is qd-continuous.*

PROOF. That M satisfies condition (D_1) follows as in [12, Theorem 2.4]. Let $M = C \oplus D$. Then by Warfield [19, Theorem 1], there exist two disjoint sets J and K such that $I = J \cup K$ and $C \cong \sum_{i \in J} \oplus A_i$, $D \cong \sum_{i \in K} \oplus A_i$. Since each A_i is cyclic, it follows by [2, Propositions 1 and 5] that C is D-projective. Then condition (D_3) follows as in Theorem 2.11.

REMARK. Consider any free module $F = \sum_{i=1}^{\infty} \oplus R_i$, $R_i \cong R_R$, a discrete valuation ring of rank one. Clearly each R_i is R_j-projective. However F is not *qd*-continuous, as Rad F is not small in F.

3. Covers and *d*-continuous modules

We start with the following general result.

LEMMA 3.1. *Let* M *be a qd-continuous module. If* $M = \sum_{i \in I} M_i$ *is an irredundant sum of indecomposable submodules* M_i, *then* $M = \sum_{i \in I} \oplus M_i$.

PROOF. That the sum $\sum_{i \in I} M_i$ is irredundant implies that no M_i is small in M. Then M_i contains a summand of M by (D_1). As M_i is indecomposable, M_i is a summand of M. So M_i is hollow. Let F be a finite subset of I. Let K be a maximal subset of F such that $\sum_{i \in K} M_i$ is direct and is a summand of M. Suppose that $K \neq F$. Let $\alpha \in F$ such that $\alpha \notin K$. By Proposition 2.8, we have $(\sum_{i \in K} \oplus M_i) + M_\alpha = (\sum_{i \in K} \oplus M_i) + S$, for some small submodule S of M. However this implies that $M = \sum_{i \neq \alpha} M_i$, which is a contradiction to the irredundancy of the sum. Therefore $K = F$ and $\sum_{i \in F} M_i$ is direct. This completes the proof.

Next we prove the dual of [14, Theorem 4].

THEOREM 3.2. *Let* A_1 *and* A_2 *be two submodules of a qd-continuous module* M. *Let* Q_1 *and* Q_2 *be summands of* M *admitting epimorphisms* π_i: $Q_i \to M/A_i$ *with* Ker $\pi_i \underset{s}{\subsetneq} Q_i$, $i = 1, 2$. *If* $M/A_1 \cong M/A_2$, *then* $Q_1 \cong Q_2$.

Proof. Let $K_i = \operatorname{Ker} \pi_i$, $i = 1, 2$. Then $Q_1/K_1 \cong Q_2/K_2$. As Q_2 is *qd*-continuous, $Q_2 = \sum_{i \in I} \oplus B_i$ where each B_i is a nonzero hollow submodule of Q_2. Let $\overline{Q}_i$ denote Q_i/K_i. Let θ be an isomorphism of $\overline{Q}_2$ onto $\overline{Q}_1$. We have $\overline{Q}_1 = \sum_{i \in I} \overline{A}_i$, where $\theta(\overline{B}_i) = \overline{A}_i$. Let A_i be the full inverse image of $\overline{A}_i$ in Q_1. It is clear that $\sum_{i \in I} A_i$ is irredundant.

As Q_1 is *qd*-continuous, $Q_1 = M_i \oplus M_i'$ such that $M_i \subset A_i$ and $S_i = M_i' \cap A_i$ is small in Q_1. Hence $A_i = M_i \oplus S_i$. Now $\overline{A}_i \cong \overline{B}_i$ is hollow. This implies that $\overline{A}_i = \overline{S}_i$ or $\overline{A}_i = \overline{M}_i$. However $\overline{A}_i = \overline{S}_i$ implies $A_i = S_i + K_1 \subsetneq_s Q_1$, which is a contradiction of the irredundancy of the $\sum_{i \in I} A_i$. So $\overline{A}_i = \overline{M}_i$, and hence $A_i = M_i + K_1$. Then

$$Q_1 = \sum_{i \in I} A_i = \sum_{i \in I} (M_i + K_1) = \sum_{i \in I} M_i + K_1.$$

As $K_1 \subsetneq_s Q_1$, we get $Q_1 = \sum_{i \in I} M_i$. It is also clear that the sum $\sum_{i \in I} M_i$ is irredundant.

We claim that M_i is hollow. Assume that $M_i = X + Y$. Then $\overline{A}_i = \overline{M}_i = \overline{X} + \overline{Y}$. As $\overline{A}_i$ is hollow, $\overline{A}_i = \overline{X}$ or $\overline{A}_i = \overline{Y}$. Let us assume that $\overline{A}_i = \overline{X}$. Then $A_i = X + K_1$ and hence

$$Q_1 = M_i \oplus M_i' = A_i + M_i' = X + K_1 + M_i' = X \oplus M_i'.$$

This implies that $X = M_i$. Similarly $\overline{A}_i = \overline{Y}$ implies that $Y = M_i$. This proves our claim.

It now follows by Lemma 3.1 that $Q_1 = \sum_{i \in I} \oplus M_i$. Let $\alpha \in I$. As B_α and M_α are hollow summands of M, it follows by Proposition 2.8 that $M_\alpha \cong B_\alpha$ or $M_\alpha + B_\alpha$ is direct and is a summand of M. In the latter case M_α is B_α-projective by Propositions 1.2 and 1.5. Thus there exists a homomorphism $g\colon M_\alpha \to B_\alpha$ such that the following diagram is commutative:

$$\begin{array}{ccc} M_\alpha & \overset{\text{nat.}}{\to} & \overline{M}_\alpha = \overline{A}_\alpha \\ g\downarrow & & \downarrow \cong \\ B_\alpha & \overset{\text{nat.}}{\to} & \overline{B}_\alpha \end{array}$$

Since B is hollow, g is onto. As B_α is M_α-projective, g splits. Then g is an isomorphism as M_α is hollow. Thus one has $M_\alpha \cong B_\alpha$ in either case. Hence

$$Q_1 = \sum_{i \in I} \oplus M_i \cong \sum_{i \in I} \oplus B_i = Q_2.$$

Corollary 3.3. *Let A_1 and A_2 be submodules of a qd-continuous module M. Let Q_1 and Q_2 be d-complements of A_1 and A_2 respectively. If $M/A_1 \cong M/A_2$ then $Q_1 \cong Q_2$.*

PROOF. As Q_i is a d-complement of A_i, Q_i is a summand of M by Proposition 1.3 and $A_i \cap Q_i \subsetneq_s M$. Now

$$Q_1/(A_1 \cap Q_1) \cong M/A_1 \cong M/A_2 \cong Q_2/(A_2 \cap Q_2).$$

Hence the result follows by the above theorem.

For any factor module M/A of a qd-continuous module M, a summand Q of M is called a *cover of* M/A *in* M if there exists an epimorphism π: $Q \to M/A$ with $\operatorname{Ker} \pi \subsetneq_s Q$. Theorem 3.2 shows that any two covers in M of a factor module of M are isomorphic.

Theorem 3.2 has the following

COROLLARY 3.4. *A qd-continuous module M is d-continuous if and only if every epimorphism $M \to M$ with small kernel is an isomorphism.*

PROOF. Necessity is obvious.

To prove sufficiency, consider any summand B of M and any epimorphism f: $M \to B$. Let $K = \operatorname{Ker} f$. Write $M = P \oplus Q$ such that $P \subset K$ and $K \cap Q \subsetneq_s M$. Let $f^* = f|Q$. Then f^*: $Q \to B$ is an epimorphism, and $\operatorname{Ker} f^* = K \cap Q \subsetneq_s M$. Also $M = A \oplus B$ for some submodule A of M. Now $M/K \cong B \cong M/A$. Then, by Corollary 3.3, the d-complements of K and A are isomorphic; that is $Q \cong B$. Now

$$M = P \oplus Q \xrightarrow{1 \oplus f^*} P \oplus B \cong P \oplus Q = M.$$

This gives an epimorphism g: $M \to M$ with $\operatorname{Ker} g = \operatorname{Ker} f^* \subsetneq_s M$. By assumption, g is an isomorphism. Hence $K \cap Q = \operatorname{Ker} f^* = 0$. So $M = K \oplus Q$ and f splits. Hence M is d-continuous.

We apply the above theorem to determine when a qd-continuous module is d-continuous.

THEOREM 3.5. *Let M be a qd-continuous module. Then M is d-continuous if and only if every hollow summand of M is d-continuous.*

PROOF. Necessity follows by Proposition 1.2. Conversely, assume that every hollow summand of M is d-continuous. By Theorem 2.2, $M = \sum_{i \in I} \oplus M_i$ where each M_i is hollow. Let f: $M \to M$ be an epimorphism such that $\operatorname{Ker} f \subsetneq_s M$. Then $M = \sum_{i \in I} f(M_i)$ is an irredundant sum of hollow submodules $f(M_i)$. It follows by Lemma 3.1 that $M = \sum_{i \in I} \oplus f(M_i)$. Again by Theorem 2.2, $f(M_i) \overset{\theta}{\cong} M_j$ for some $j \in I$. Let $f^* = f|M_i$. Then as M_i is d-continuous and M_j is M_i-projective for $j \neq i$, the epimorphism $\theta f^*\colon M_i \to M_j$ splits. Since M_i is hollow, θf^* is an isomorphism, and hence f^* is an isomorphism. Consequently f is an isomorphism. The result now follows by the above corollary.

LEMMA 3.6. *Let M' be a qd-continuous module, and f be an epimorphism of any module M onto M' with* $\operatorname{Ker} f \underset{s}{\subsetneq} M$. *Then* $\operatorname{Ker} f$ *is invariant under every idempotent endomorphism of M.*

PROOF. Let $M = A \oplus B$. Then $M' = f(A) + f(B)$. As M' is *qd*-continuous, it follows by Proposition 1.3 and Corollary 1.4 that $M' = A_1 \oplus B_1$ for some submodules $A_1 \subset f(A)$, $B_1 \subset f(B)$. Then $M = f^{-1}(A_1) + f^{-1}(B_1)$. However $f^{-1}(A_1) \subset f^{-1}(A_1) \cap A + \operatorname{Ker} f$ and $f^{-1}(B_1) \subset f^{-1}(B_1) \cap B + \operatorname{Ker} f$. Consequently $M = f^{-1}(A_1) \cap A + f^{-1}(B_1) \cap B$, as $\operatorname{Ker} f \underset{s}{\subsetneq} M$. We get

$$M = A \oplus B = f^{-1}(A_1) \cap A \oplus f^{-1}(B_1) \cap B.$$

Hence $f(A) = A_1$, $f(B) = B_1$, and $M = f(A) \oplus f(B)$. This shows that $\operatorname{Ker} f$ is invariant under every idempotent endomorphism of M.

We now prove two theorems analogous to [20, Proposition 2.2] and [6, Theorem 5.6] respectively.

THEOREM 3.7. *Let M be any qd-continuous module and f be an epimorphism of M onto a module M' with* $\operatorname{Ker} f \underset{s}{\subsetneq} M$. *Then M' is qd-continuous if and only if* $\operatorname{Ker} f$ *is invariant under every idempotent endomorphism of M.*

PROOF. Necessity follows from Lemma 3.6.

Conversely assume that $\operatorname{Ker} f$ is invariant under every idempotent endomorphism of M. Let A be a submodule of M'. Write $M = P \oplus Q$, with $P \subset f^{-1}(A)$ and $f^{-1}(A) \cap Q \underset{s}{\subsetneq} M$. Then the hypothesis on $\operatorname{Ker} f$ yields $M' = f(P) \oplus f(Q)$. Clearly $f(P) \subset A$. Further

$$f^{-1}(A \cap f(Q)) \subset f^{-1}(A) \cap Q + \operatorname{Ker} f \underset{s}{\subsetneq} M$$

yields $A \cap f(Q) \underset{s}{\subsetneq} M'$. Therefore M' satisfies condition (D_1). Now $A = f(P) \oplus A \cap f(Q)$, so if A is summand of M', we get $A \cap f(Q) = 0$ and hence $A = f(P)$.

Let B and C be summands of M' such that $M' = B + C$. As seen above there exist summands S, T of M such that $f(S) = B$, $f(T) = C$. Then $M = S + T + \operatorname{Ker} f = S + T$. As M is *qd*-continuous, by (D_3), $S \cap T$ is a summand of M. Consequently $M = S_1 \oplus S \cap T \oplus T_1$ with $S = S_1 \oplus S \cap T$, $T = T_1 \oplus S \cap T$. The hypothesis on $\operatorname{Ker} f$ yields $M' = f(S_1) \oplus f(S \cap T) \oplus f(T_1)$. This immediately yields $B \cap C = f(S \cap T)$. Hence M' is *qd*-continuous.

Theorem 3.8. *Let M be any module every summand of which admits a projective cover. Let $P \xrightarrow{f} M \to 0$ be a projective cover of M. Then M is qd-continuous if and only if* $\operatorname{Ker} f$ *is invariant under every idempotent endomorphism of P and P satisfies* (D_1).

Proof. Let P satisfy (D_1) and let $\operatorname{Ker} f$ be invariant under every idempotent endomorphism of P. As seen in the proof of [13, Theorem 2.3] any quasi-projective module satisfies (D_2). Consequently P is *qd*-continuous. So by Theorem 3.7, M is *qd*-continuous.

Conversely let M be *qd*-continuous. By Lemma 3.6, $\operatorname{Ker} f$ is invariant under every idempotent endomorphism of P. Let A be any submodule of P. Write $M = N_1 \oplus N_2$, such that $N_1 \subset f(A)$ and $N_2 \cap f(A) \subsetneq_s M$. This results in a decomposition $P = P_1 \oplus P_2$ such that

$$P_1 \xrightarrow{f|P_1} N_1 \to 0$$

$$P_2 \xrightarrow{f|P_2} N_2 \to 0$$

are projective covers of N_1 and N_2 respectively. Since $M = f(A) + f(P_2)$ we have

$$P = A + P_2 + \operatorname{Ker} f = A + P_2 = P_1 \oplus P_2.$$

By Lemma 2.10, $P = A_1 \oplus P_2$ for some $A_1 \subset A$. As $f(A \cap P_2) \subset f(A) \cap N_2 \subsetneq_s M$ and $\operatorname{Ker} f \subsetneq_s P$, we get $A \cap P_2 \subsetneq_s P$. Hence P satisfies (D_1). This proves the theorem.

We end the paper with the following

Remarks. (i) Consider any module M_R such that every homomorphic image of M has a projective cover. Let $P \xrightarrow{f} M \to 0$ be a projective cover of M. By [6, Theorem 5.6], P satisfies condition (D_1), and hence every homomorphic image of P has a projective cover. Let L be the sum of all those submodules K of $\operatorname{Ker} f$ which are invariant under idempotent endomorphisms of P. Let $\overline{M} = P/L$. Then $P \xrightarrow{\pi} P/L \to 0$ is the projective cover of $\overline{M}$, where π is the natural mapping, and we have the epimorphism $\bar{f}$: $\overline{M} \to M$ such that $\bar{f}\pi = f$. By Theorem 3.7, $\overline{M}$ is *qd*-continuous. It can be easily seen that given any *qd*-continuous module Q_R having a projective cover, and any epimorphism g: $Q \to M$, there exists an epimorphism $\bar{g}$: $Q \to \overline{M}$ such that $\bar{f}\bar{g} = g$. In this sense we can call $\overline{M}$ a *qd*-continuous cover of M.

(ii) The same proof as that of [13, Proposition 5.1] shows that given any module M and any two small submodules A and B of M, such that $M/A \oplus M/B$ is *qd*-continuous, then $M/A \cong M/B$. Thus in particular if two modules M and M' have isomorphic projective covers and $M \oplus M'$ is *qd*-continuous, then $M \cong M'$.

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