A NOTE ON WELL-DISTRIBUTED SEQUENCES

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A sequence $\{x_k\}_1^\infty$ is said to be *well distributed* (mod 1) **(3, 4, 5)** if the limit

$$\lim_{N\to\infty} N^{-1} \sum_{k=p+1}^{p+N} \chi_I((x_k)) = |I| \tag{1.1}$$

exists, uniformly in $p \geqslant 0$, for all intervals I in [0, 1], with length $|I|$, characteristic function $\chi_I(x)$, where (x) is the fractional part of x. If (1.1) is true for $p = 0$ and all I in [0, 1] we say that $\{x_k\}_1^\infty$ is *uniformly distributed* (mod 1).

In a paper of Dowidar and Petersen **(2)** it is proved that $\{r^k\theta\}_1^\infty$ is not well distributed (mod 1) for any real θ and integer r. For r rational Petersen and McGregor **(6)** have shown that $\{r^k\theta\}_1^\infty$ is not well distributed (mod 1) for almost all real θ. In this note we shall prove the generalization of this latter result for real r.

THEOREM. *Given a real number α, then $\{\alpha^k\theta\}_1^\infty$ is not well distributed* (mod 1) *for almost all real numbers θ.*

Proof. We first show that for $|\alpha| > 1$ the sequence $\{\alpha^k\theta\}_1^\infty$ is uniformly distributed (mod 1) for almost all θ. In a recent paper of Davenport, Erdös, and Le Veque **(1)** it is proved that if for integers $m \neq 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \int_a^b \left| \sum_{k=1}^{n} \exp[2\pi i m x_k(t)] \right|^2 dt < \infty, \tag{1.2}$$

then the sequence $\{x_k(t)\}_1^\infty$ is uniformly distributed (mod 1) for almost all t in $[a, b]$. Applying this to the case $x_k(t) = \alpha^k t$, with $|\alpha| > 1$, we get

$$\begin{aligned} \int_a^b \left| \sum_{k=1}^{n} \exp(2\pi i m \alpha^k t) \right|^2 dt &= \sum_{r,s=1}^{n} \int_a^b \cos 2\pi m(\alpha^r - \alpha^s) t\, dt \\ &< n(b-a) + \frac{1}{\pi|m|} \sum_{\substack{r,s=1 \\ r\neq s}}^{n} \frac{1}{|\alpha^r - \alpha^s|} \\ &< n(b-a) + \frac{|\alpha|}{|\alpha|^3 - 1} \end{aligned} \tag{1.3}$$

and hence $\{\alpha^k\theta\}_1^\infty$ is uniformly distributed (mod 1) for almost all θ. For $|\alpha| \leqslant 1$, $\{\alpha^k\theta\}_1^\infty$ is obviously not uniformly distributed or well distributed for any θ.

For $|\alpha| > 1$ we now consider separately the two cases: (i) α transcendental, (ii) α algebraic, not an integer.

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In case (i) we deduce that for arbitrary $v \geqslant 1$ and arbitrary integers $(m_1, \ldots, m_v)$, not all zero, the sequence

$$G_m^{\,v} = \left\{\alpha^k\theta \sum_{t=1}^{v} m_t \alpha^t\right\}_{k=1}^{\infty}$$

is uniformly distributed (mod 1) for almost all θ. Therefore all such sequences $G_m^{\,v}$ are simultaneously uniformly distributed for almost all θ. From this it follows, by means of the multidimensional form of Weyl's criterion for uniform distribution (7), that the sequence $\{\alpha^{k+1}\theta, \ldots, \alpha^{k+v}\theta\}_1^\infty$ is uniformly distributed (mod 1) in the v-dimensional unit cube C_v for all v simultaneously, for almost all θ. Thus, for any θ except in some set E of measure zero, any $N \geqslant 1$, there is an integer k such that

$$(1.4) \qquad 0 < (\alpha^{k+j}\theta) < \tfrac{1}{2} \qquad (1 \leqslant j \leqslant N)$$

and hence $\{\alpha^k\theta\}_1^\infty$ is not well distributed (mod 1) for θ not in E.

In case (ii), if α is algebraic of degree v, then for arbitrary $(m_1, \ldots, m_v)$ not all zero, and $q \geqslant 1$, the sequence

$$H_m^{\,q} = \left\{q^{-1}\alpha^k\theta \sum_{t=1}^{v} m_t \alpha^t\right\}_{k=1}^{\infty}$$

is uniformly distributed (mod 1) for almost all θ. Therefore all such sequences $H_m^{\,q}$ are simultaneously uniformly distributed for almost all θ. Hence, as before, the sequence

$$I_q^{\,v} = \left\{q^{-1}\alpha^{k+1}\theta, \ldots, q^{-1}\alpha^{k+v}\theta\right\}_{k=1}^{\infty}$$

is uniformly distributed (mod 1) in C_v for all q simultaneously, for almost all θ. If α satisfies the equation

$$\sum_{t=0}^{v} a_t \alpha^t = 0,$$

with integer coefficients a_i, $a_v > 0$, then there exist integers $A_t^{\,j}$ such that

$$(1.5) \qquad a_v^{\,j}\alpha^{v+j} = \sum_{t=1}^{v} A_t^{\,j}\alpha^t \qquad (j \geqslant 1).$$

Thus, for any θ except in a set F of measure zero and any $N \geqslant 1$, by the uniform distribution of $I_q^{\,v}$ with $q = a_v^{\,N-v}$, there exists an integer k such that

$$(1.6) \quad 0 \leqslant (q^{-1}\alpha^{k+t}\theta) < \left\{4 \max_{1 \leqslant j \leqslant N-v} \sum_{t=1}^{v} |A_t^{\,j}| a_v^{\,N-j}\right\}^{-1} \qquad (1 \leqslant t \leqslant v).$$

From (1.5) and (1.6) it follows that for $1 \leqslant j \leqslant N$

$$(1.7) \qquad 0 < \min\{(\alpha^{k+j}\theta), 1 - (\alpha^{k+j}\theta)\} < \tfrac{1}{4}$$

and so the sequence $\{\alpha^k\theta\}_1^\infty$ is not well distributed (mod 1) for θ not in F. This completes the proof of the theorem.

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Defining a *uniformly (well) distributed sequence* $\{x_n\}_1^\infty$ of degree v (mod 1) as one for which $\{x_{k+1}, \ldots, x_{k+v}\}$ is uniformly (well) distributed (mod 1) in C_v and a *normally distributed sequence* (mod 1) as one which is uniformly distributed of degree v for all $v \geqslant 1$ we derive from the above proof

COROLLARY 1. *If* $|\alpha| > 1$, *then* $\{\alpha^k\theta\}$ *is uniformly distributed* [*normally distributed*] *of degree* v (mod 1) *for almost all* θ *if* α *is algebraic* [*transcendental*] *of degree* v.

COROLLARY 2. *If* α *or* α^{-1} *is an algebraic integer of degree* v, *then* $\{\alpha^k\theta\}_1^\infty$ *is not well distributed of degree* v *for any* θ.

Proof. Corollary 1 has already been proved in the course of the proof of the Theorem.

For the proof of Corollary 2 we first note that if $\{\alpha^k\theta\}_1^\infty$ is well distributed of degree v it must be uniformly distributed of degree v; hence $|\alpha| > 1$. We consider the case when α is an algebraic integer of degree v, that is to say, α satisfies an equation

$$\sum_{t=0}^{v} a_t \alpha^t = 0$$

with $a_v = 1$. Applying the argument of case (ii) of the theorem with $q = 1$ to

$$I_1^v = \{\alpha^{k+1}\theta, \ldots, \alpha^{k+v}\theta\},$$

which is uniformly distributed, it follows that $\{\alpha^k\theta\}_1^\infty$ is not well distributed (mod 1) and hence is not well distributed of degree v. Thus we have a contradiction. Similarly, we obtain a contradiction if α^{-1} is an algebraic integer, so that $a_0 = 1$ instead of $a_v = 1$. In this case we express $(\alpha^{k-v-1}\theta, \ldots, \alpha^{k-N}\theta)$ in terms of $(\alpha^{k-v}\theta, \ldots, \alpha^{k-1}\theta)$ and obtain inequalities of type (1.7) with j replaced by $-j$.

Corollary 2, with $v = 1$, gives us the theorem of Dowidar and Petersen **(2)**, mentioned at the beginning of this note.

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