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A REPRESENTATION FOR THE REPRODUCING KERNEL FOR ELLIPTIC SYSTEMS

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1. Introduction

The kernel function method of Bergman and Schiffer (see (1) and (2)) has recently been used by Colton and Gilbert (see (3) and (4)) in connection with approximation theory and the numerical treatment of elliptic differential equations. In (5) Gilbert and Weinacht have successfully extended the kernel function method to elliptic systems of differential equations. Essential to their work is the concept of a matrix kernel satisfying the reproducing property. This reproducing kernel is defined initially as the difference of the Neumann matrix and the Dirichlet matrix. Thus actually to obtain the kernel matrix from this definition one has to solve both a Neumann problem and a Dirichlet problem. In view of this restriction, Gilbert and Weinacht derive an ingenious representation for the reproducing kernel in terms of purely geometric quantities which are obtained directly from the fundamental matrix for the differential system.

In this paper a new series representation for the reproducing kernel is given, the terms of which are more easily calculated than those appearing in the representation in (5). The terms of the series involve a certain parameter α which we introduce. For $\alpha = 0$ the representation is shown to reduce to the one given in (5). We also obtain the value of α leading to the fastest rate of convergence and make a comparison with the $\alpha = 0$ case.

2. Notation and Statement of the Problem

Here we state the problem under consideration and give our notation. Consider the elliptic system of differential equations

$$LU \equiv \Delta U - C(x, y)U = 0 \tag{2.1}$$

where U and C are complex $n \times n$-matrices and Δ is the Laplacian operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.2}$$

The domain $\dot{D}$ for (2.1) is a bounded simply connected region with analytic boundary D. C is a positive definite Hermitian matrix belonging to class $C^1(\bar{D})$. To avoid confusion, we adopt the notation of (5). $S(Q, P)$ is the fundamental matrix of L with a logarithmic singularity for $P = Q$ and is considered a known quantity throughout.

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$I(Q, P)$ denotes the geometric quantity

$$I(Q, P) \equiv \int_{\dot{D}} \frac{\partial S^*}{\partial \nu_T}(T, Q) S(T, P)\, ds_T \tag{2.3}$$

where $*$ is the conjugate-transpose and ν is the inner unit normal vector. $K(Q, P)$ is the reproducing kernel for (2.1), whose representation we seek, and is defined formally in (5) (see Eqs. (3.1) and (3.2) there). There it is shown that the kernel matrix

$$\mathcal{K}(Q, T) \equiv \frac{\partial}{\partial \nu_T}[K(T, Q) - 4I(T, Q)] \tag{2.4}$$

is continuous in $\dot{D} \times \dot{D}$ and furthermore that the homogeneous equation

$$\Phi(Q) = -\lambda \int_{\dot{D}} \mathcal{K}^*(Q, T)\Phi(T)\, ds_T \tag{2.5}$$

has real positive characteristic values λ_j greater than unity. This particular result is crucial in what follows.

From (5), the reproducing kernel satisfies the following integral equation

$$K(Q, P) = 4I(Q, P) + \mathcal{A}K(Q, P) \tag{2.6}$$

where

$$\mathcal{A}K(Q, P) \equiv -\int_{\dot{D}} \mathcal{K}^*(Q, T)K(T, P)\, ds_T. \tag{2.7}$$

From (2.4), we note that the unknown kernel K appears in the kernel matrix $\mathcal{K}^*$. Finally let $\mathcal{I}$ denote the identity operator and if V and W are $n \times n$ matrices whose entries are complex valued functions of class $C^2(\bar{D})$, let

$$E\{V, W\} \equiv \iint_D [V_x^* W_x + V_y^* W_y + V^* CW]\, dx\, dy. \tag{2.8}$$

3. A Representation for the Reproducing Kernel

In this section we obtain a series representation for the reproducing kernel. Adding $\alpha K(Q, P)$ to both sides of (2.6) and dividing by $1 + \alpha$ we obtain

$$K(Q, P) = \frac{4}{1+\alpha} I(Q, P) + \frac{1}{1+\alpha}[\alpha K(Q, P) + \mathcal{A}K(Q, P)]. \tag{3.1}$$

We now establish that (3.1) can be solved by iteration for appropriate values of α and determine the value of α giving the fastest convergence. This result is analogous to one for matrices given by Isaacson and Keller (6, pp. 75–77).

Theorem 3.1. *The iteration scheme*

$$K^{(n+1)}(Q, P) = \frac{4}{1+\alpha} I(Q, P) + \frac{1}{1+\alpha}[\alpha K^{(n)}(Q, P) + \mathcal{A}K^{(n)}(Q, P)] \tag{3.2}$$

$$K^{(0)}(Q, P) = \frac{4}{1+\alpha} I(Q, P)$$

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converges for $\alpha > -\frac{1}{2}$. *The fastest rate of convergence occurs when* $\alpha = -\frac{1}{2}\lambda_0$ *where* λ_0 *denotes the smallest eigenvalue of* (2.5).

Proof. Let

$$B(\alpha) = \frac{1}{1+\alpha}(\alpha\mathcal{I} + \mathcal{A}). \tag{3.3}$$

Letting $n_j = \lambda_j^{-1}$, the eigenvalues for $B(\alpha)$ satisfy

$$\mu_j(\alpha) = \frac{\alpha + n_j}{1+\alpha}. \tag{3.4}$$

Let $\beta = \dfrac{1}{1+\alpha}$ and $m_j = n_j - 1$. Then

$$\mu_j = 1 + m_j\beta. \tag{3.5}$$

From (5) we have that

$$1 > \eta_0 \geqslant \eta_1 \geqslant \eta_2 \geqslant \cdots \geqslant 0 \tag{3.6}$$

and since $\mathcal{A}$ is a compact operator, $\lim_{j\to\infty} \eta_j = 0$. For $-\frac{1}{2} < \alpha < \infty$, $0 < \beta < 2$ and so

$$\mu_0 = (\eta_0 - 1)\beta + 1 \geqslant \mu_j = (\eta_j - 1)\beta + 1 \geqslant 1 - \beta. \tag{3.7}$$

Therefore the spectral radius $\rho(\alpha)$ of $B(\alpha)$ is

$$\rho(\alpha) = \max\{|1 + (\eta_0 - 1)\beta|, |1 - \beta|\}. \tag{3.8}$$

Since $0 < \beta < 2$ and $0 \leqslant \eta_0 < 1$, $\rho(\alpha) < 1$. Hence the iteration scheme (3.2) converges for these values of α.

The fastest rate of convergence occurs for $\min \rho(\alpha) \equiv \rho_*$. ρ_* occurs when

$$1 + (\eta_0 - 1)\beta = \beta - 1 \tag{3.9}$$

(see Figure 1). Solving for β we obtain $\beta = 2/(2-\eta_0)$. Thus the value α_* for which $B(\alpha)$ has smallest spectral radius is $\alpha = -\eta_0/2$.

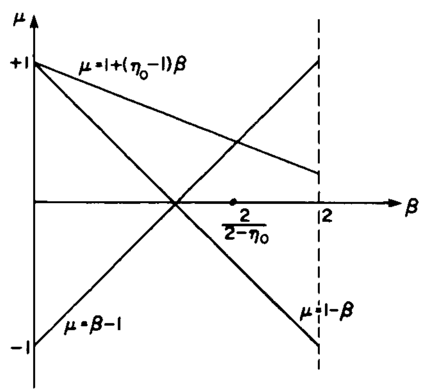

Fig. 1.

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In (5), $\alpha = 0$ is the basis of the representation for the reproducing kernel. Substituting this value into (3.8), it is seen that the corresponding spectral radius is

$$\rho(0) = \eta_0 = \frac{1}{\lambda_0}. \tag{3.10}$$

For $\alpha = -\eta_0/2$ we get

$$\rho_* = \rho(-\eta_0/2) = \frac{1}{2\lambda_0 - 1} < \frac{1}{\lambda_0}. \tag{3.11}$$

Obtaining estimates for λ_0 is in general hard and thus actually obtaining α_* for each matrix $C(x, y)$ is difficult. Certain approximations for the optimal choice of α, however, can be made. If $\lambda_0 \simeq 1$, $\alpha_* \simeq -\frac{1}{2}$ and if $\lambda_0 \gg 1$, $\alpha_* \simeq 0$. In any event

$$-\tfrac{1}{2} < \alpha_* < 0. \tag{3.12}$$

In what follows it is assumed that $\alpha > -\frac{1}{2}$. From (3.2) and using the Binomial Theorem, we have that

$$K(Q, P) = \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)^{n+1}} (\alpha \mathcal{I} + \mathcal{A})^n 4I(Q, P) \tag{3.13}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} \mathcal{A}^j 4I(Q, P). \tag{3.14}$$

The j-th iterated kernel $\mathcal{K}_j$ is defined by

$$\mathcal{K}_j^*(Q, P) \equiv \int_{\dot{D}} \mathcal{K}^*(Q, P) \mathcal{K}_{j-1}^*(T, P)\, ds_T \tag{3.15}$$

for $j \geq 2$ with $\mathcal{K}_1(T, P) \equiv \mathcal{K}(T, P)$. It can be shown that

$$\mathcal{A}^n I(Q, P) = (-1)^n \int_{\dot{D}} \mathcal{K}_n^*(Q, T) I(T, P)\, ds_T. \tag{3.16}$$

Hence

$$K(Q, P) = \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} (-1)^j \int_{\dot{D}} \mathcal{K}_j^*(Q, T) 4I(T, P)\, ds_T \tag{3.17}$$

and using Green's identities and (2.8) we have

$$K(Q, P) = \sum_{n=0}^{\infty} \frac{1}{(1+\alpha)^{n+1}} \sum_{j=0}^{n} \binom{n}{j} \alpha^{n-j} E\{M_j^*(T, Q), 4I(T, P)\} \tag{3.18}$$

where

$$\begin{aligned} M_j(T, Q) &= E\{M(0, T), M_{j-1}(0, Q)\} \\ M_1(T, Q) &\equiv K(T, Q) - 4I(T, Q) \\ M_0(T, Q) &\equiv K(T, Q) \end{aligned} \tag{3.19}$$

Following (5) we introduce the pure geometric quantities

$$\begin{aligned} i_1(Q, P) &\equiv 4I(Q, P) \\ i_k(Q, P) &\equiv E\{i_{k-1}^*(Q, T), 4I(T, P\} \quad k \geq 2 \end{aligned} \tag{3.20}$$

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and satisfy

$$M_j(Q,P)=\sum_{k=0}^{j}(-1)^k\binom{j}{k}i_j(Q,P). \tag{3.21}$$

From (3.18), (3.20) and (3.21) we have

$$K(Q,P)=\sum_{n=0}^{\infty}\frac{1}{(1+\alpha)^{n+1}}\sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}\sum_{k=0}^{j}(-1)^k\binom{j}{k}i_{k+1}(Q,P) \tag{3.22}$$

which involves only integrals of the fundamental matrix S. In the limiting case $\alpha=0$ this coincides with the representation obtained by Gilbert and Weinacht (5, Eq. (4.8)).

We now put (3.22) in a more usable form. From Green's identities and the fact that $LI\equiv 0$, it can be shown that

$$E\{i_k^*(Q,T),4I(T,P)\}=-\int_{\dot D}i_k(Q,T)\frac{\partial}{\partial\nu_T}4I(T,P)\,ds_T. \tag{3.23}$$

Let $\mathscr{B}$ denote the following integral operator

$$\mathscr{B}g(Q,P)\equiv-\int_{\dot D}g(Q,T)\frac{\partial}{\partial\nu_T}4I(T,P)\,ds_T. \tag{3.24}$$

By induction

$$i_{k+1}(Q,P)=\mathscr{B}^k 4I(Q,P). \tag{3.25}$$

Substituting (3.25) into (3.22) and applying the Binomial Theorem, we have

$$K(Q,P)=\sum_{n=0}^{\infty}\frac{1}{(1+\alpha)^{n+1}}\sum_{j=0}^{n}\binom{n}{j}\alpha^{n-j}(\mathscr{I}-\mathscr{B})^j 4I(Q,P) \tag{3.26}$$

and again applying the Binomial Theorem we obtain the following important representation

$$K(Q,P)=\frac{4}{1+\alpha}\sum_{n=0}^{\infty}\left(\mathscr{I}-\frac{1}{1+\alpha}\mathscr{B}\right)^n I(Q,P)\quad \alpha>-\tfrac{1}{2} \tag{3.27}$$

which involves only integrals of the fundamental matrix S.

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