ASYMPTOTIC FORMULAE FOR LINEAR OSCILLATIONS

by F. V. ATKINSON

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1. A number of formulae are known which exhibit the asymptotic behaviour as $t \to \infty$ of the solutions of

$$\ddot{x} + F(t)x = 0, \qquad (t \geqslant t_0, \ \cdot = d/dt). \quad \text{.......(1.1)}$$

The aim of this note is to unify a group of such formulae, relating to the case in which $F(t)$ is on the whole positive, and suitably continuous though not necessarily analytic.

The two special cases in which $F(t)$ approximates closely to a positive constant as $t \to \infty$, and in which $\log F(t)$ is of bounded variation over (t_0, ∞), are summed up in a result due to Ascoli [**1**, **2**]. Writing $F(t) = f(t) + g(t)$, where $\log f$ is of bounded variation over (t_0, ∞) and g absolutely integrable over (t_0, ∞), f being positive and f and g suitably continuous, we have as $t \to \infty$ the asymptotic formula

$$x = A \cos\left\{\int_{t_0}^{t} f^{\frac{1}{2}}\, dt + B + o(1)\right\}, \quad \text{.......(1.2)}$$

where A and B are constants of integration, A being positive for a non-trivial solution. An extension to the complex field is due to Levinson [**3**].

The validity of (1.2) admits of some extension beyond the above conditions (see for example [**4**]), but modifications seem essential for cases in which $F(t) \to \infty$ or $F(t) \to 0$ as $t \to \infty$. Under various conditions there hold formulae of the type

$$x = AF^{-\frac{1}{4}} \cos\left\{\int_{t_0}^{t} k(t)\, dt + B + o(1)\right\}, \quad \text{.......(1.3)}$$

where we may have, as previously, as for example in Wintner [**5**],

$$k(t) = F^{\frac{1}{2}}, \quad \text{.......(1.4)}$$

or (Atkinson [**6**, p. 84], extended by Hartman and Wintner [**7**]†).

$$k(t) = \{F - \dot{F}^2/(16F^2)\}^{\frac{1}{2}}, \quad \text{.......(1.5)}$$

or again ([**6**, p. 86]),

$$k(t) = (F - \tfrac{1}{4}t^{-2})^{\frac{1}{2}}. \quad \text{.......(1.6)}$$

In Theorem 1 I give a result which includes and extends these formulae.

Finally, I show that certain boundedness criteria are also included.

2. In order to formulate the main result, we split $F(t)$ into a " smooth " part $f(t)$ and a " small " part $g(t)$, so that (1.1) takes the form

$$\ddot{x} + \{f(t) + g(t)\}\, x = 0 \qquad (t \geqslant t_0). \quad \text{.......(2.1)}$$

We have then

THEOREM 1. *Let $f(t)$ be positive and continuously differentiable, $g(t)$ continuous and such that*

$$\int_{t_0}^{\infty} |\, g f^{-\frac{1}{2}} \,|\, dt < \infty. \quad \text{.......(2.2)}$$

† An error in sign in [7, p. 83] is corrected on p. 932.

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Let

$$ff^{-\frac{3}{2}} = h(t) + h_1(t), \quad \text{...(2.3)}$$

where $h(t)$ *is continuously differentiable and such that*

$$\int_{t_0}^{\infty} |\dot{h}|\, dt < \infty, \quad -4 < h(\infty) < 4, \quad \text{................................(2.4–5)}$$

and $h_1(t)$ *satisfies*

$$\int_{t_0}^{\infty} |h_1 f^{\frac{1}{2}}|\, dt < \infty. \quad \text{..(2.6)}$$

Then the general solution of (2.1) *admits as* $t \to \infty$ *the asymptotic representation*

$$x = Af^{-\frac{1}{4}} \cos\left\{\int_{t_0}^{t} f^{\frac{1}{2}}(1 - h^2/16)^{\frac{1}{2}}\, dt + B + o(1)\right\}, \quad \text{.....................(2.7)}$$

where $A > 0$ *for a non-trivial solution.*

For the proof we introduce a new independent variable by putting $u = \int_{t_0}^{t} f^{\frac{1}{2}}\, dt$, so that (2.1) transforms to

$$\frac{d^2x}{du^2} + \tfrac{1}{2}\dot{f}f^{-\frac{3}{2}}\frac{dx}{du} + (1 + g/f)x = 0.$$

On putting $y_1 = x$, $y_2 = dx/du$, this may be written

$$\frac{dy_1}{du} = y_2, \quad \frac{dy_2}{du} = -(1 + g/f)y_1 - \tfrac{1}{2}\dot{f}f^{-\frac{3}{2}}y_2,$$

or, in vector-matrix notation, in the form

$$\frac{d\mathbf{y}}{du} = (\mathbf{A} + \mathbf{B} + \mathbf{C})\mathbf{y}, \quad \text{.................................. ......(2.8)}$$

where $\mathbf{y} = (y_1, y_2)$ and

$$\mathbf{A} = \begin{Vmatrix} 0 & 1 \\ -1 & -\tfrac{1}{2}h(\infty) \end{Vmatrix}, \qquad \mathbf{B} = \begin{Vmatrix} 0 & 0 \\ 0 & \tfrac{1}{2}(h(\infty) - h(t)) \end{Vmatrix}$$

and

$$\mathbf{C} = \begin{Vmatrix} 0 & 0 \\ -g/f & \tfrac{1}{2}\{h(t) - \dot{f}f^{-\frac{3}{2}}\}, \end{Vmatrix},$$

in which form we may apply a powerful result due to Levinson [**3**].†

In verifying that Levinson's hypotheses are satisfied, we have first to show that $u \to \infty$ as $t \to \infty$. In the contrary event, if

$$\int_{t_0}^{\infty} f^{\frac{1}{2}}\, dt < \infty, \quad \text{..(2.9)}$$

it would follow from (2.3), (2.4) and (2.6) that $\int_{t_0}^{\infty} |\dot{f}f^{-1}|\, dt < \infty$, so that f would have a positive limit as $t \to \infty$, in contradiction to (2.9).

† This result is also accessible in the text of Bellman [**8**, p. 50]. Actually it would be possible, at the cost of further calculations, to deduce Theorem 1 from a more primitive result than that of Levinson, for the case of a system with absolutely integrable coefficients.

We next observe that **A** is a constant matrix, that **B** is continuously differentiable and of bounded variation near $u=\infty$, with $\mathbf{B}(\infty)=0$, and that **C** is absolutely integrable over $(0, \infty)$ with respect to u. These statements all follow from the hypotheses of Theorem 1. Finally, the characteristic roots of $\mathbf{A}+\mathbf{B}$ are

$$-\tfrac{1}{4}h \pm i(1-h^2/16)^{\frac{1}{2}}, \qquad \text{.....(2.10)}$$

which have the same real part (cf. [**3**, (1.3)]).

In applying Levinson's result, we need the characteristic vectors of **A**, which we may take to be

$$1, -\tfrac{1}{4}h(\infty) \pm i\{1-h^2(\infty)/16\}^{\frac{1}{2}}.$$

If we seek an approximation to x only, and not to $\dot{x}$, we may disregard the second element of these vectors. From Levinson's result we deduce that (2.8) has solutions such that, as $t\to\infty$,

$$y_1 \sim \exp\left(\int_0^u \{-\tfrac{1}{4}h \pm i(1-h^2/16^{\frac{1}{2}}\}\, du\right) = \exp\left(\int_{t_0}^t \{-\tfrac{1}{4}f^{\frac{1}{2}}h \pm i\,(f-fh^2/16)^{\frac{1}{2}}\}\, dt\right) \quad \text{....(2.11)}$$

But

$$\int_{t_0}^t f^{\frac{1}{2}}h\, dt = \int_{t_0}^t \dot{f}f^{-1}\, dt - \int_{t_0}^t f^{\frac{1}{2}}h_1\, dt,$$

and the latter integral on the right converges as $t\to\infty$, by (2.6). Hence for some positive constant D the pair of solutions satisfying (2.11) are such that

$$y_1 \sim Df^{-\frac{1}{4}} \exp\left\{\pm\int_{t_0}^t i(f-fh^2/16)^{\frac{1}{2}}\, dt\right\},$$

so that (2.1) has a pair of solutions satisfying

$$x \sim f^{-\frac{1}{4}} \exp\left\{\pm\int_{t_0}^t i(f-fh^2/16)^{\frac{1}{2}}\, dt\right\},$$

which is equivalent to the assertion of Theorem 1.

3. Three of the asymptotic integrations referred to in § 1 are immediately derivable from Theorem 1 as special cases.

To obtain Ascoli's result (1.2) we take $h(t)=0$.

To obtain (1.3) with (1.5) we take $f(t)=F(t)$, $g(t)=0$, and $h_1(t)=0$. The conditions of the theorem require that F should be positive and $\ddot{F}$ continuous, that $\dot{F}F^{-\frac{3}{2}}$ should be of bounded variation over (t_0, ∞) and such that $\alpha = \lim_{t\to\infty} \dot{F}F^{-\frac{3}{2}}$ should satisfy $-4<\alpha<4$.

Actually, if $F^{-\frac{1}{2}}$ is to have its positive value, the last restriction can be replaced by $-4<\alpha\leqslant 0$. For if $\dot{F}F^{-\frac{3}{2}}\to\alpha\neq 0$ an integration shows that $F^{-\frac{1}{2}}\sim -\tfrac{1}{2}\alpha t$ as $t\to\infty$, so that α must be negative. So far as real F are concerned, Hartman and Wintner's conditions ([**7**, (171) and (172)] with due correction as regards sign) also confine us to such cases. The case $\alpha<-4$ could also be included in the above investigation, using a different case of Levinson's result, though an exponential form would be more appropriate than (2.7). The case $\alpha=-4$ is more delicate.

To obtain the case of (1.3) with (1.6) we take $f(t)=F(t)$, $g(t)=0$, and $h(t)=-2t^{-1}F^{-\frac{1}{2}}$. The conditions require that F should be positive with $\ddot{F}$ continuous, that $tF^{\frac{1}{2}}$ should be of bounded variation over (t_0, ∞), tending as $t\to\infty$ to a limit greater than $\tfrac{1}{2}$.

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4. To show that Wintner's result of (1.3) with (1.4) is included, and to obtain a comparison with other results, I require the

LEMMA. *Let $f(t)$ be positive for $t \geqslant t_0$, with f continuous, and, for some constant β with $\beta < \frac{3}{2}$, let*

$$\int_{t_0}^{\infty} | \ddot{f} f^{-\frac{3}{2}} - \beta \dot{f}^2 f^{-\frac{5}{2}} | \, dt < \infty. \quad \text{......(4.1)}$$

If $\beta < 1$, it follows that

$$\int_{t_0}^{\infty} \dot{f}^2 f^{-\frac{5}{2}} \, dt < \infty, \quad \text{......(4.2)}$$

and further that $\dot{f} f^{-\frac{3}{2}}$ is of bounded variation over (t_0, ∞), tending to 0 as $t \to \infty$.

If $1 \leqslant \beta < \frac{3}{2}$, then either the previous conclusions hold, or else

$$\int_{t_0}^{\infty} t f \, dt < \infty. \quad \text{......(4.3)}$$

Assume first that (4.2) holds. Since

$$\frac{d}{dt} (\dot{f} f^{-\frac{3}{2}}) = (\ddot{f} f^{-\frac{3}{2}} - \beta \dot{f}^2 f^{-\frac{5}{2}}) - (\tfrac{3}{2} - \beta) \dot{f}^2 f^{-\frac{5}{2}}, \quad \text{......(4.4)}$$

it follows from (4.1–2) that $\dot{f} f^{-\frac{3}{2}}$ is of bounded variation over (t_0, ∞). Denoting $\lim_{t \to \infty} \dot{f} f^{-\frac{3}{2}}$ by α, we must have $\alpha = 0$. For if $\dot{f} f^{-\frac{3}{2}} \to \alpha \neq 0$, we must have $f^{-\frac{1}{2}} \sim -\frac{1}{2} \alpha t$, whence $\dot{f}^2 f^{-\frac{5}{2}} \sim -2\alpha/t$, in contradiction to (4.2).

Assume next that (4.2) does not hold, so that

$$\int_{t_0}^{\infty} \dot{f}^2 f^{-\frac{5}{2}} \, dt = \infty \; ; \quad \text{......(4.5)}$$

we show that this is impossible if $\beta < 1$, which will prove the first part of the Lemma. We also show that (4.5), together with the other assumptions of the Lemma, implies (4.3), thus completing the proof of the Lemma.

Integrating (4.4) over (t_0, t) and making $t \to \infty$, we have, in view of (4.1) and (4.5),

$$\dot{f} f^{-\frac{3}{2}} \to -\infty \quad \text{as } t \to \infty, \quad \text{......(4.6)}$$

so that ultimately

$$\dot{f} < 0 \quad \text{......(4.7)}$$

and further

$$f \to 0 \quad \text{as } t \to \infty, \quad \text{......(4.8)}$$

since f is non-negative and cannot, in view of (4.6), tend to a positive constant as $t \to \infty$.

Write now

$$\ddot{f} f^{-\frac{3}{2}} - \beta \dot{f}^2 f^{-\frac{5}{2}} = \gamma(t), \quad \text{......(4.9)}$$

so that by (4.1)

$$\int_{t_0}^{\infty} | \gamma(t) | \, dt < \infty. \quad \text{......(4.10)}$$

We have then

$$\frac{d}{dt} (\dot{f} f^{-\beta}) = \gamma(t) f^{\frac{3}{2} - \beta}. \quad \text{......(4.11)}$$

From (4.8) and (4.10) it follows that $\dot{f} f^{-\beta}$ tends to a limit, δ say, as $t \to \infty$. Furthermore, integrating (4.11) over (t, ∞), we have, in view of (4.7) and (4.10), for large t,

$$\dot{f} f^{-\beta} = \delta + o(f^{\frac{3}{2} - \beta}). \quad \text{......(4.12)}$$

Here δ cannot vanish, since that would imply $\dot{f}f^{-\frac{3}{2}} = o(1)$, in contradiction to (4.6). Moreover δ cannot be positive, in view of (4.7) and (4.8). Hence

$$\dot{f}f^{-\beta} \to \delta < 0 \quad \text{.......................................(4.13)}$$

as $t \to \infty$.

Now if $\beta \neq 1$, an integration of (4.13) gives

$$f^{1-\beta} \sim (1-\beta)\,\delta t, \quad \text{.......................................(4.14)}$$

as $t \to \infty$. This is impossible if $\beta < 1$, since it would imply that f is ultimately negative, contrary to hypothesis; we have therefore shown that (4.5) does not hold if $\beta < 1$, and thereby proved the first half of the lemma.

If $1 < \beta < \frac{3}{2}$, then (4.14) can occur, and implies that

$$f \sim \text{const. } t^{-1/(1-\beta)},$$

which establishes (4.3).

If $\beta = 1$, an integration of (4.13) gives

$$\log f \sim \delta t,$$

and since $\delta < 0$ this again ensures (4.3). This completes the proof of the lemma.

As a special case of Theorem 1 we have now

THEOREM 2. *Let F be positive, $\ddot{F}$ continuous, and for some constant β with $\beta < \frac{3}{2}$ let*

$$\int_{t_0}^{\infty} |\,\ddot{F}F^{-\frac{3}{2}} - \beta \dot{F}^2 F^{-\frac{5}{2}}\,|\, dt < \infty. \quad \text{.................................(4.15)}$$

If $1 \leqslant \beta \leqslant \frac{3}{2}$ let also

$$\int_{t_0}^{\infty} tF\, dt = \infty. \quad \text{...(4.16)}$$

Then the general solution of (1.1) *admits as $t \to \infty$ the asymptotic representation*

$$x = AF^{-\frac{1}{4}} \cos\left\{\int_{t_0}^{t} F^{\frac{1}{2}}\, dt + B + o(1)\right\}. \quad \text{..........................(4.17)}$$

In view of the lemma, (4.15) and (4.16) ensure the validity of the asymptotic integration (1.3) with (1.5), sufficient conditions for which were enumerated in § 3. However, here (1.5) is equivalent to (1.4), since the extra term in (1.5) contributes to the phase an amount

$$\int_{t_0}^{\infty} O(\dot{F}^2 F^{-\frac{5}{2}})\, dt,$$

which is asymptotic to a constant as $t \to \infty$, by (4.2).

Wintner's result [5] is the special case $\beta = \frac{5}{4}$, the actual value of β in the range $1 \leqslant \beta < \frac{3}{2}$ being immaterial if (4.16) is postulated. The effect of the restriction (4.16) is to exclude cases in which $F \to 0$ as $t \to \infty$ so rapidly that the solutions of (1.1) are asymptotically constant.

5. The question of the boundedness of the solutions of (1.1) is often discussed under assumptions which ensure, not merely the boundedness of solutions, but the existence of an asymptotic expression for them. Here I note sufficient conditions for boundedness which follow from Theorem 2, and give its relation to two other such sets of conditions. Naturally, more general boundedness conditions could be deduced from Theorem 1. We have

THEOREM 3. *Let F have a positive lower bound, and let $\ddot{F}$ be continuous. For some β with $\beta<\frac{3}{2}$ let*

$$\int_{t_0}^{\infty} | \ddot{F}F^{-\frac{3}{2}} - \beta\dot{F}^2F^{-\frac{5}{2}} | \, dt < \infty. \qquad (5.1)$$

Then all solutions of (1.1) *are bounded as $t\to\infty$.*

This follows directly from Theorem 2, the condition (4.16) being superfluous since F has a positive lower bound. The conditions (5.1) for varying $\beta<\frac{3}{2}$ are mutually equivalent, again since F has a positive lower bound.

Bellman [**9**] has recently given a fresh proof of a set of boundedness conditions, which he attributes to Gusarov [**10**], namely

$$F\geqslant \text{const.}>0, \quad \text{and} \quad \int_{t_0}^{\infty} | \ddot{F} | \, dt < \infty. \qquad (5.2,\ 3)$$

Sobol′ [**11**, p. 710], requires in addition to (5.2) that $F^{\frac{1}{2}}$ should have a derivative of bounded variation over (t_0, ∞), i.e. that

$$\int_{t_0}^{\infty} | \ddot{F}F^{-\frac{1}{2}} - \tfrac{1}{2}\dot{F}^2F^{-\frac{3}{2}} | \, dt < \infty, \qquad (5.4)$$

where we have added to Sobol′'s assumptions the continuity of $\ddot{F}$.

Apart from the latter point, I show that Theorem 3 includes the conditions of Sobol′, which in turn include those of Gusarov.

As to the first assertion, it follows from (5.2) and (5.4) that

$$\int_{t_0}^{\infty} | \ddot{F}F^{-\frac{3}{2}} - \tfrac{1}{2}\dot{F}^2F^{-\frac{5}{2}} | \, dt < \infty,$$

and this is one of the (mutually equivalent) conditions (5.1).

We have next to show that (5.2, 3) imply (5.4). It is clear that they imply

$$\int_{t_0}^{\infty} | \ddot{F}F^{-\frac{1}{2}} | \, dt < \infty. \qquad (5.5)$$

We show that this implies, subject to (5.2),

$$\int_{t_0}^{\infty} \dot{F}^2F^{-\frac{3}{2}} \, dt < \infty. \qquad (5.6)$$

Suppose that we have (5.5), and the contrary to (5.6). From the result

$$\left[\dot{F}F^{-\frac{1}{2}}\right]_{t_0}^{t} = \int_{t_0}^{t} \ddot{F}F^{-\frac{1}{2}} \, dt - \tfrac{1}{2}\int_{t_0}^{t} \dot{F}^2F^{-\frac{3}{2}} \, dt,$$

it then follows that

$$\dot{F}F^{-\frac{1}{2}} \to -\infty \qquad (5.7)$$

as $t\to\infty$. This implies that F is ultimately monotonically decreasing, and this by (5.2) implies that F tends to a positive constant. By (5.7) this implies that $\dot{F}\to-\infty$, which is clearly contradictory. Hence (5.5) and (5.2) imply (5.6), which imply (5.4), as asserted.

As an example, we cite the case of $\ddot{x}+t^n x=0$, the solutions of which are easily seen to be bounded for all $n\geqslant 0$. The conditions of Gusarov work here for $0\leqslant n\leqslant 1$, while those of Sobol′ work for $0\leqslant n\leqslant 2$. The conditions of Theorem 3, e.g. with $\beta=0$, work for all $n\geqslant 0$.

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INFINITE INTEGRALS INVOLVING PRODUCTS OF LEGENDRE FUNCTIONS

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1. In this paper we evaluate a few infinite integrals involving products of Legendre functions. The results obtained herein are quite general and include, as particular cases, some known results.

We shall evaluate these integrals with the help of a theorem in operational calculus proved in § 2.

We write

$$\psi(p) \doteqdot f(x),$$

when

$$\psi(p) = p \int_0^\infty e^{-px} f(x)\, dx, \qquad (1)$$

and

$$\phi(p) \underset{K}{\overset{K}{=}} f(x),$$

when

$$\phi(p) = (2/\pi)^{\frac{1}{2}}\, p \int_0^\infty (px)^{\frac{1}{2}} K_\nu(px) f(x)\, dx. \qquad (2)$$

Formula (2) is a generalisation of (1) as given by Meijer [**7**] and it reduces to (1) when $\nu = \pm\frac{1}{2}$, since

$$K_{\pm\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x}.$$

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