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Divergence of Fourier series

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Carleson has proved that the Fourier series of functions belonging to the class L^2 converge almost everywhere. Improving his method, Hunt proved that the Fourier series of functions belonging to the class L^p $(p > 1)$ converge almost everywhere. On the other hand, Kolmogoroff proved that there is an integrable function whose Fourier series diverges almost everywhere. We shall generalise Kolmogoroff's Theorem as follows: There is a function belonging to the class $L(\log L)^p$ $(p > 0)$ whose Fourier series diverges almost everywhere. The following problem is still open: whether "almost everywhere" in the last theorem can be replaced by "everywhere" or not. This problem is affirmatively answered for the class L by Kolmogoroff and for the class $L(\log \log L)^p$ $(0 < p < 1)$ by Tandori.

1. Introduction

See [1], [3], [6], and [5] for the statements in the above abstract. Generalising Kolmogoroff's Theorem [6], Chen [2] and Prohorenko [4] proved that there is a function belonging to the class $L(\log \log L)^p$ $(0 < p < 1)$ whose Fourier series diverges almost everywhere. We shall prove the following:

THEOREM. *There is a function belonging to the class* $L(\log L)^p$ $(p > 0)$ *whose Fourier series diverges almost everywhere.*

For the proof of the theorem, we can suppose that p is a positive

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integer since $L(\log L)^p \supset L(\log L)^q$ for $p < q$.

2.

We shall first prove the following

PROPOSITION. *There are a sequence of trigonometric polynomials* (f_n) *and a sequence of sets* (E_n) *in the interval* $(0, 2\pi)$ *, satisfying the following conditions:*

$1°$ *the measure of* E_n *tends to* 2π *as* $n \to \infty$ *,*

$2°$ *for each* $t \in E_n$ *, there corresponds a partial sum of* f_n *whose absolute value is not less than* $A\log n/\log\log n$.

We denote the mth Dirichlet kernel by $D_m(t)$ and the mth Fejér kernel by $F_m(t)$; then

$$D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} \cos kt = \frac{\sin(m+1/2)t}{2\sin t/2}$$

and

$$F_m(t) = \frac{1}{m+1} \sum_{k=0}^{m} D_k(t) = \frac{\sin^2(m+1)t}{(m+1)(2\sin t/2)^2}$$

$$= \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{m+1}\right) \cos kt .$$

It is known that

$$0 \le F_m(t) \le A\min(m, 1/mt^2) .$$

By $F_m^*(t)$ we denote $F_m(t)$ without the constant term. Let n be a large positive integer and take n points (c_i) on the interval $(0, 2\pi)$, defined by

$$c_i = 4i\pi/(2n+1) \quad (i = 1, 2, \ldots, n)$$

and

$$c_i' = (c_i+c_{i+1})/2 \quad (i = 1, 2, \ldots, n) .$$

Let (m_i) be an increasing sequence of integers defined by

$$m_i = \tfrac{1}{2}\left((2n+1)^q 3^i - 1\right) \quad (i = 1, 2, \ldots, n)$$

where q is a positive integer $\geq p + 2$. Then

$$(2n+1) \mid (2m_i+1) \quad (i = 1, 2, \ldots, n) .$$

Let h be a function periodic with period 2π, defined by

$$\begin{aligned} h(t) &= 0 \quad \text{for} \quad c_i - 1/n \log\log n < t < c_i + 1/n \log\log n \quad (i = 1, 2, \ldots, n) \\ &= 1 \quad \text{otherwise in } (0, 2\pi) . \end{aligned}$$

We denote by $s_m(t, h)$ the mth partial sum of the Fourier series of the function h at the point t and consider the trigonometric polynomial

$$f_n(t) = \frac{1}{n} \sum_{i=1}^{n} F^*_{m_i}(t-c_i) s_N(t; h)$$

where $N = (2n+1)^{p+2}$. The (m_j+N)th partial sum of the Fourier series of f_n is

$$\begin{aligned} s_{m_j+N}(t, f_n) &= \frac{1}{n} \sum_{i=1}^{j} F^*_{m_i}(t-c_i) s_N(t; h) \\ &\quad + \frac{1}{n} \sum_{i=j+1}^{n} \sum_{k=1}^{m_j} \left(1 - \frac{k}{m_i+1}\right) \cos k(t-c_i) s_N(t; h) \\ &\quad + \frac{1}{n} \sum_{i=j+1}^{n} \sum_{k=m_j+1}^{m_j+N} \left(1 - \frac{k}{m_i+1}\right) \cos k(t-c_i) s_{N+m_j-k}(t; h) \\ &= P_1 + P_2 + P_3 . \end{aligned}$$

We suppose that

$$\log n < j < n - \sqrt{n}$$

and

$$c_j + 2/n \log\log n < t < c_j' - 1/n \log\log n \tag{1}$$

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or

$$c_j'+1/n \log\log n < t < c_{j+1}-2/n \log\log n .$$

Then we get

$$|P_1| \le \frac{A}{n}\sum_{i=1}^{j}\frac{1}{m_i}\frac{\sin^2(m_i+1)(t-c_i)}{(2\sin(t-c_i)/2)^2}+A$$
$$\le \frac{A}{n}\sum_{i=1}^{j-1}\frac{n^2}{m_i(i-j)^2}+\frac{A}{nm_j}(n\log\log n)^2+A \le A$$

since $t-c_i > A(i-j)/n$ for $1 \le i \le j-1$, and similarly

$$P_2 = \frac{1}{n}\sum_{i=j+1}^{n}\frac{m_j+1}{m_i+1}\sum_{k=1}^{m_j}\left(1-\frac{k}{m_j+1}\right)\cos k(t-c_i)s_N(t;\ h)$$
$$+\frac{1}{n}\sum_{i=j+1}^{n}\frac{m_i-m_j}{m_i+1}\sum_{k=1}^{m_j}\cos k(t-c_i)s_N(t;\ h)$$
$$=\frac{1}{n}\sum_{i=j+1}^{n}\frac{m_j+1}{m_i+1}F_{m_j}(t-c_i)s_N(t;\ h)$$
$$+\frac{1}{n}\sum_{i=j+1}^{n}\frac{m_i-m_j}{m_i+1}D_{m_j}(t-c_i)s_N(t;\ h)+O(1)$$
$$=\frac{\sin(m_j+1/2)t_j}{n}\sum_{i=j+1}^{n}\frac{m_i-m_j}{m_i+1}\frac{s_N(t;h)}{2\sin(t-c_i)/2}+O(1)$$
$$=P_2'+O(1)$$

where $t_j = t - c_j$. On the other hand, using Abel's transformation for the inner summation of P_3 , we write

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$$P_3 = \frac{1}{\pi n} \sum_{i=j+1}^{n} \sum_{k=m_j+1}^{m_j+N} \left(1 - \frac{k}{m_i+1}\right) \cos k\left(t-c_i\right) \int_0^{2\pi} h(u) \frac{\sin\left(N+m_j-k+1/2\right)(t-u)}{2\sin(t-u)/2}\, du$$

$$= \frac{1}{\pi n} \left[\sum_{i=j+1}^{n} \frac{1}{2} \left(1 - \frac{m_j+N}{m_i+1}\right) \frac{\sin\left(m_j+N+1/2\right)\left(t-c_i\right)}{2\sin\left(t-c_i\right)/2} \int_0^{2\pi} h(u)du \right.$$

$$- \left(1 - \frac{m_j+1}{m_i+1}\right) \frac{\sin\left(m_j+1/2\right)\left(t-c_i\right)}{2\sin\left(t-c_i\right)/2} \int_0^{2\pi} h(u) \frac{\sin(N-1/2)(t-u)}{2\sin(t-u)/2}\, du$$

$$+ \sum_{k=m_j+1}^{m_j+N-1} \frac{\sin(k+1/2)\left(t-c_i\right)}{2\sin\left(t-c_i\right)/2} \left\{ \frac{1}{m_i+1} \int_0^{2\pi} h(u) \frac{\sin\left(N+m_j-1/2\right)(t-u)}{2\sin(t-u)/2}\, du \right.$$

$$\left.\left. - \left(1 - \frac{k}{m_i+1}\right) \int_0^{2\pi} h(u)\cos\left(m_j+N-k\right)(t-u)du \right\}\right]$$

$$= Q_1 - Q_2 + \left(Q_3 - Q_4 + Q_5\right) ;$$

then we can easily see that

$$P_2' - Q_2 = O\left(\log n/n^{p+1}\right)$$

and

$$Q_3 = O(1) .$$

Now we shall estimate Q_4 .

$$Q_4 = \frac{1}{\pi n} \sum_{i=j+1}^{n} \frac{1}{2\sin\left(t-c_i\right)/2} \sum_{k=1}^{N-1} \sin\left(m_j+k+1/2\right)\left(t-c_i\right) \int_0^{2\pi} h(u)\cos(N-k)(t-u)du$$

$$= \frac{1}{\pi n} \sum_{k=1}^{N-1} \sum_{i=j+1}^{n} \frac{\sin\left(m_j+k+1/2\right)\left(t-c_i\right)}{2\sin\left(t-c_i\right)/2} \int_0^{2} h(u)\cos(N-k)(t-u)du ,$$

and we write the inner sum of Q_4 as follows:

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$$R_k = \sum_{i=j+1}^{n} \frac{\sin(m_j+k+1/2)(t-c_i)}{2\sin(t-c_i)/2}$$
$$= \sum_{i=j+1}^{n} \frac{\sin(m_j+k+1/2)(t_j-4(i-j)\pi/(2n+1))}{2\sin(t_j-4(i-j)\pi/(2n+1))/2}$$
$$= \int_{j+1/2}^{n+1/2} \frac{\sin(m_j+k+1/2)(t_j-4(v-j)\pi/(2n+1))}{2\sin(t_j-4(v-j)\pi/(2n+1))/2}(dv+dJ(v))$$
$$= R_k' + R_k'' ,$$

where

$$J(v) = [v] - v + 1/2 = \sum_{l=1}^{\infty} \frac{\sin 2\pi l v}{2\pi l}$$

and further

$$R_k' = \frac{2n+1}{4\pi} \int_{-2+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)} \sin(m_j+k+1/2)w\left(\frac{1}{w} - \frac{2\sin w/2-w}{2w\sin w/2}\right)dw$$
$$= S_k' + S_k'' .$$

If t belongs to the interval (1), then

$$t_j - 2\pi/(2n+1) \leq -1/n \log\log n$$

and then $S_k' = O\left(n^2 \log\log n/m_j\right)$. Similarly S_k'' is also of order not greater than $n^3 \log\log n/m_j \log n$, by using the second mean value theorem and the fact that $(w/2\sin w/2) - 1 > 0$, and decreasing in the interval of integration. Therefore

$$Q_4' = \frac{1}{\pi n} \sum_{k=1}^{N-1} R_k' \int_0^{2\pi} h(u)\cos(N-k)(t-u)du = O(1) .$$

On the other hand, if t belongs to the interval (2), then

$$t_j - 2\pi/(2n+1) \geq 1/n \log\log n$$

and we can estimate

$$S'_k = \frac{2n+1}{4\pi}\left\{\int_{-\infty}^{\infty}\frac{\sin w}{w}\,dw - \right.$$

$$\left. - \left(\int_{t_j-2\pi/(2n+1)}^{\infty} + \int_{-\infty}^{-2\pi+4j\pi/(2n+1)+t_j}\right)\frac{\sin(m_j+k+1/2)w}{w}\,dw\right\}$$

$$= A(2n+1) + O\left(n^2 \log\log n/m_j\right) ,$$

$$S''_k = A\,\frac{2n+1}{4\pi}\left\{\frac{n}{\log n}\int_{-2\pi+4j\pi/(2n+1)+t_j}^{-1/n}\frac{\sin(m_j+k+1/2)w}{w}\,dw\right.$$

$$\left. + \int_{-1/n}^{t_j-2\pi/(2n+1)} \sin(m_j+k+1/2)w\left(\frac{w-2\sin w/2}{2\sin w/2}\right)dw\right\}$$

$$= O\left(n^3 \log\log n/m_j \log n\right) + O\left(1/n^2\right) ,$$

similarly as before; so that Q'_4 is also bounded for t in the interval (2). Hence we have to estimate the rest of Q_4 :

$$Q''_4 = Q_4 - Q'_4$$

$$= \frac{1}{\pi n}\sum_{k=1}^{N-1} R''_k \int_0^2 h(u)\cos(N-k)(t-u)du$$

$$= \frac{1}{\pi n}\sum_{k=1}^{N-1} R''_n\left(s_{N-k}(t;\ h)-s_{N-k-1}(t;\ h)\right)$$

$$= \frac{1}{\pi n}\left\{-s_0(t;\ h)R''_{N-1}+s_{N-1}(t;\ h)R''_1 + \sum_{k=1}^{N-2} s_k(t;\ h)\left(R''_{N-k}-R''_{N-k-1}\right)\right\}$$

$$= -T_1 + T_2 + T_3 ,$$

where

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$$R''_k = -\int_{j+1/2}^{n+1/2} J(v)\,\frac{d}{dv}\left[\frac{\sin(m_j+k+1/2)(t_j-4(v-j)\pi/(2n+1))}{2\sin(t_j-4(v-j)\pi/(2n+1))/2}\right]dv$$

$$= \frac{-1}{2\pi}\sum_{l=1}^{\infty}\frac{1}{l}\int_{j+1/2}^{n+1/2}\sin 2\pi l v\,\frac{d}{dv}\left[\frac{\sin(m_j+k+1/2)(t_j-4(v-j)\pi/(2n+1))}{2\sin(t_j-4(v-j)\pi/(2n+1))/2}\right]dv$$

$$= \sum_{l=1}^{\infty}\int_{j+1/2}^{n+1/2}\frac{\cos 2\pi l v\cdot\sin(m_j+k+1/2)(t_j-4(v-j)\pi/(2n+1))}{2\sin(t_j-4(v-j)\pi/(2n+1))/2}\,dv$$

$$= \frac{n+1/2}{2\pi}\sum_{l=1}^{\infty}\int_{-2+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)}\frac{\cos l(n+1/2)(w-t_j)\sin(m_j+k+1/2)w}{2\sin w/2}\,dw$$

$$= \frac{n+1/2}{2\pi}\int_{1/2}^{\infty}(dl+dJ(l))\int_{-2+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)} \cdot\frac{\cos l(m+1/2)(w-t_j)\sin(m_j+k+1/2)w}{2\sin w/2}\,dw$$

$$= U'_k + U''_k\ .$$

We shall first estimate T_1. Changing the order of integration and using the second mean value theorem, the first part of R''_{N-1} becomes

$$\frac{1}{n}U'_{n-1} = \frac{n+1/2}{n}\int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)}\frac{\sin(m_j+N-1/2)w}{2\sin w/2}\,dw\ \cdot$$

$$\cdot\int_{1/2}^{\infty}\cos l(n+1/2)(w-t_j)\,dl$$

$$= \frac{1}{n}\lim_{M\to\infty}\int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)}\frac{\sin M(n+1/2)(w-t_j)\cdot\sin(m_j+N-1/2)w}{(w-t_j)\cdot 2\sin w/2}\,dw$$

$$-\frac{1}{n}\int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)}\frac{\sin(n+1/2)(w-t_j)/2\cdot\sin(m_j+N-1/2)w}{(w-t_j)\cdot 2\sin w/2}\,dw$$

$$= O(1)\ ,$$

and the second part of R''_{N-1} is

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$$\frac{1}{n} U''_{N-1} = \frac{n+1/2}{n} \int_{1/2}^{\infty} dJ(l) \int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)} \cdot \frac{\cos l(n+1/2)(w-t_j)\cdot\sin(m_j+N-1/2)w}{2\sin w/2}\, dw$$

$$= \frac{(n+1/2)^2}{n} \int_{1/2} J(l)dl \int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)} \cdot \frac{(w-t_j)\sin l(n+1/2)(w-t_j)\cdot\sin(m_j+N-1/2)w}{2\sin w/2}\, dw$$

$$= \frac{(n+1/2)^2}{n} \lim_{M\to\infty} \int_{-2\pi+4j\pi/(2n+1)+t_j}^{t_j-2\pi/(2n+1)} \frac{(w-t_j)\sin(m_j+N-1/2)w}{2\sin w/2}\, dw \cdot \int_{1/2}^{M} J(l)\sin l(n+1/2)(w-t_j)dl ,$$

where we can suppose that M is an even integer. The last inner integral is

$$\int_{1/2}^{M} J(l)\sin(n+1/2)(w-t_j)dl$$

$$= \sum_{k=1}^{\infty} \frac{1}{2\pi k} \int_{1/2}^{M} \sin 2lk\pi\cdot\sin(n+1/2)(w-t_j)dl$$

$$= \sum_{k=1}^{\infty} \frac{1}{4\pi k} \int_{1/2}^{M} \{\cos(n+1/2)(w-t_j-2\pi k)l - \cos((n+1/2)(w-t_j)+2\pi k)l\}dl$$

$$= -\frac{1}{4\pi} \sum_{k=1} \frac{1}{k} \left\{\frac{\sin((n+1/2)(w-t_j)/2-k\pi)}{(n+1/2)(w-t_j)-2k\pi} - \frac{\sin((n+1/2)(w-t_j)/2+k\pi)}{(n+1/2)(w-t_j)+2k\pi}\right\}$$

$$+ \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{1}{k} \left\{\frac{\sin((n+1/2)(w-t_j)-2k\pi)M}{(n+1/2)(w-t_j)-2k\pi} - \frac{\sin((n+1/2)(w-t_j)+2k\pi)M}{(n+1/2)(w-t_j)+2k\pi}\right\}$$

$$= -\sin(n+1/2)(w-t_j)/2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1/2)^2(w-t_j)^2-(2k\pi)^2}$$

$$+ \sum_{k=1}^{\infty} \frac{\sin((n+1/2)(w-t_j)-2k\pi)M}{\{(n+1/2)(w-t_j)-2k\pi\}\{(n+1/2)(w-t_j)+2k\pi\}} ,$$

where we use the convention that $\sin x/x = 1$ for $x = 0$. Substituting

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the first term on the right side into U''_{n-1}/n, we get

$$-\frac{(n+1/2)^2}{n}\int_{t_j-2\pi+4j\pi/(2n+1)}^{t_j-2\pi/(2n+1)} \frac{(w-t_j)\sin(m_j+N-1/2)w}{2\sin w/2}\cdot$$

$$\cdot \sin(n+1/2)(w-t_j)/2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1/2)^2(w-t_j)^2-(2k\pi)^2}\,dw$$

$$= -\frac{(n+1/2)^2}{n}\sum_{m=1}^{n-j}\int_{t_j-2(2m+1)\pi/(2n+1)}^{t_j-2(2m-1)\pi/(2n+1)} D_{m_j+N-1}(w)\cdot$$

$$\cdot (w-t_j)\sum_{k=1}^{\infty}\frac{(-1)^k \sin(n+1/2)(w-t_j)/2}{(n+1/2)^2(w-t_j)^2-(2k\pi)^2}\,dw$$

$$= O(1)$$

by the second mean value theorem, since the last sum is expressed as the difference of bounded monotone increasing functions in each interval of integration. Therefore

$$\frac{1}{n}U''_{N-1} = \frac{1}{n}\lim_{M\to\infty}\sum_{k=1}^{\infty}\int_{t_j-2\pi+4j\pi/(2n+1)}^{t_j-2\pi/(2n+1)} \frac{(w-t_j)\sin(m_j+N-1/2)w}{2\sin w/2\,\{w-(t_j+2k\pi/(n+1/2))\}/2}\cdot$$

$$\cdot\frac{\sin\{w-(t_j-2k\pi/(n+1/2))(n+1/2)\}M}{w-(t_j-2k\pi/(n+1/2))}\,dw + O(1)$$

$$= \frac{\pi}{2n}\sum_{k=1}^{n-j}\frac{\sin(m_j+N-1/2)(t_j-2k\pi/(n+1/2))}{2\sin(t_j-2k\pi/(n+1/2))/2} + O(1)$$

$$= \frac{\pi}{2n}\sum_{i=j+1}^{n}\frac{\sin(m_j+N+1/2)(t-c_i)}{2\sin(t-c_i)/2} + O(1)$$

by the Dirichlet formula and then we get

$$T_1 = \frac{1}{2n}s_0(t;\ h)\sum_{i=j+1}^{n}\frac{\sin(m_j+N+1/2)(t-c_i)}{2\sin(t-c_i)/2} + O(1)$$

$$= Q_1 + O(\log\log n)\ .$$

By a similar estimation, we can see that

$$T_2 = \frac{1}{2n} s_{N-1}(t;\ h) \sum_{i=j+1}^{n} \frac{\sin\left(m_j+3/2\right)\left(t-c_i\right)}{2\sin\left(t-c_i\right)/2} + O(1)\ .$$

Now

$$T_3 = \frac{1}{\pi n} \sum_{k=1}^{N-2} s_k(t;\ h) \int_{j+1/2}^{n+1/2} J(v) \frac{d}{dv} \sin\left(m_j+k\right)\left(t-4v\pi/(2n+1)\right)dv$$

where the above integral equals

$$T_3' = \frac{4\pi\left(m_j+k\right)}{2n+1} \int_{j+1/2}^{n+1/2} J(v)\cos\left(m_j+k\right)\left(t-4v\pi/(2n+1)\right)dv\ .$$

By the transformation $4v\pi/(2n+1) = w$,

$$\begin{aligned}
T_3' &= \left(m_j+k\right) \int_{2(2j+1)\pi/(2n+1)}^{2\pi} J\left(\frac{(2n+1)w}{4}\right)\cos\left(m_j+k\right)(t-w)dw \\
&= \frac{m_j+k}{2} \sum_{l=1}^{\infty} \frac{1}{l} \int_{2(2j+1)\pi/(2n+1)}^{2\pi} \sin(n+1/2)lw\cdot\cos\left(m_j+k\right)(t-w)dw \\
&= -\frac{m_j+k}{4\pi} \sum_{l=1}^{\infty}{}^{*} \frac{1}{l} \left\{\left[\frac{\cos\left(\left\{(n+1/2)l-\left(m_j+k\right)\right\}w+\left(m_j+k\right)t\right)}{(n+1/2)l-\left(m_j+k\right)}\right.\right. \\
&\qquad \left.\left. + \frac{\cos\left(\left\{(n+1/2)l+\left(m_j+k\right)\right\}w-\left(m_j+k\right)t\right)}{(n+1/2)l+\left(m_j+k\right)}\right]_{2(2j+1)\pi/(2n+1)}^{2\pi}\right\} \\
&= -\frac{m_j+k}{4\pi} \sum_{l=1}^{\infty}{}^{*} \frac{1}{l} \left\{\frac{(-1)^l\cos\left(m_j+k\right)t}{(n+1/2)l-\left(m_j+k\right)} - \frac{(-1)^l\cos\left(\left(m_j+k\right)(2j+1)\pi/(n+1/2)+\left(m_j+k\right)t\right)}{(n+1/2)l-\left(m_j+k\right)}\right. \\
&\qquad \left. + \frac{(-1)^l\cos\left(m_j+k\right)t}{(n+1/2)l+\left(m_j+k\right)} - \frac{(-1)^l\cos\left(\left(m_j+k\right)(2j+1)\pi/(n+1/2)-\left(m_j+k\right)t\right)}{(n+1/2)l+\left(m_j+k\right)}\right\} \\
&= -\frac{m_j+k}{4\pi}\left[\left\{\cos\left(m_j+k\right)t-\cos\left(\left(m_j+k\right)\left((2j+1)\pi/(n+1/2)+t\right)\right)\right\} \sum_{l=1}^{\infty}{}^{*} \frac{(-1)^l}{l\left\{(n+1/2)l-\left(m_j+k\right)\right\}}\right. \\
&\qquad \left. +\left\{\cos\left(m_j+k\right)t-\cos\left(\left(m_j+k\right)\left((2j+1)\pi/(n+1/2)-t\right)\right)\right\} \sum_{l=1}^{\infty} \frac{(-1)^l}{l\left\{(n+1/2)l+\left(m_j+k\right)\right\}}\right] \\
&= A\left\{\cos\left(m_j+k\right)t-\cos\left(m_j+k\right)\left(t+(2j+1)\pi/(n+1/2)\right)\right\} \\
&\qquad + A\left\{\cos\left(m_j+k\right)t-\cos\left(m_j+k\right)\left(t-(2j+1)\pi/(n+1/2)\right)\right\} + O\left(n^2/m_j\right)\ ,
\end{aligned}$$

where $*$ means the sum omitting the term with vanishing denominator.

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Therefore

$$T_3 = \frac{1}{n}\sum_{k=1}^{N-2} s_k(t;\ h)\{A\cos(m_j+k)t - A(m_j+k)(t+(2j+1)\pi/(n+2))$$
$$- A(m_j+k)(t-(2j+1)\pi/(n+2))\} + O(1)$$
$$= V_1 - V_2 - V_3 + O(1)\ ,$$

where

$$V_1 = \frac{A}{n}\int_0^{2\pi} h(t-u)\sum_{k=1}^{N-2} D_k(u)\cos(m_j+k)t\,du$$
$$= \frac{A}{n}\sum_{k=1}^{N-2}\cos(m_j+k)t$$
$$- \frac{A}{n}\int_0^{2\pi}(1-h(t-u))\left\{\frac{\cos((u-t)/2+u/2-m_jt)-\cos((N-3/2)(u-t)+u/2-m_jt)}{2\sin u/2\cdot 2\sin(u-t)/2}\right.$$
$$\left. + \frac{\cos((u+t)/2+u/2+m_jt)-\cos((N-3/2)(u+t)+u/2+m_jt)}{2\sin u/2\cdot 2\sin(u+t)/2}\right\}du$$
$$= O\left(\frac{1}{j}\right) + O\left[\frac{1}{n}\sum_{i=1}^{n}\int_{t-c_i-1/n\log\log n}^{t-c_i+1/n\log\log n}\frac{du}{\sin u/2\cdot\sin(u-t)/2}\right]$$
$$= O(1) + O\left[\frac{1}{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{n^2}{|i-j|\,i n\log\log n} + \frac{1}{n}\frac{(n\log\log n)^2}{n\log\log n}\right]$$
$$= O(\log\log n)$$

and similarly V_2 and V_3 are also $O(\log\log n)$. Collecting the above estimates, we get

$$s_{m_j+N}(t;\ f_n) = Q_1 - Q_4 + O(\log\log n) = T_2 + O(\log\log n)$$
$$= \frac{\sin(m_j+1/2)t_j}{n}\sum_{i=j+1}^{n}\frac{1}{2\sin(t-c_i)/2} + O(\log\log n)$$

for all t belonging to the intervals (1) and (2).

We denote by $E_{n,j}$ the set of t in the intervals (1) and (2), satisfying the condition

$$|\sin(m_j+1/2)t_j| > 1/\log \log n$$

then

$$\left|s_{m_j+N}(t;\ f_n)\right| \geq A \log n/\log \log n \quad \text{for all} \quad t \quad \text{in} \quad E_{n,j} .$$

We write

$$E_n = \bigvee (E_{n,j}\ ;\ \log n < j < n - \sqrt{n}) \ ;$$

then (f_n) and (E_n) satisfy the conditions of the proposition stated at the beginning of this section.

3.

Let $(r_k(x))$ be the Rademacher system on the interval $(0, 1)$ and we shall consider the function

$$f(t,\ x) = \sum_{k=1}^{\infty} f_{p_k}(q_k t) r_k(x)/\log \log p_k$$

where f_{p_k} is defined in Section 2 and (q_k) is the sequence of integers defined such that $f_{p_k}(q_{k+1}t)$ have no overlapping terms for each k and (p_k) will be determined such that $f \in L(\log L)^p$. We write

$$Y(t) = t(\log(t+e))^p\ , \quad Z(t) = Y(\sqrt{t}) \quad \text{for} \quad t > 0\ ;$$

then, by the Jensen inequalities, we get (*cf.* [5]),

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$$\int_0^1 dx \int_0^{2\pi} Y(|f(t, x)|)dt = \int_0^{2\pi} dt \int_0^1 Y(|f(t; x)|)dx$$
$$= \int_0^{2\pi} dt \int_0^1 Z(|f(t, x)|^2)dx$$
$$\le \int_0^{2\pi} Z\left[\int_0^1 (f(t, x))^2 dx\right]dt$$
$$= \int_0^{2\pi} Z\left[\sum_{k=1}^{\infty} \left(f_{p_k}(q_k t)/\log\log p_k\right)^2\right]dt$$
$$\le \int_0^{2\pi} \sum_{k=1}^{\infty} Z\left[\left(f_{p_k}(q_k t)/\log\log p_k\right)^2\right]dt$$
$$= \int_0^{2\pi} \sum_{k=1}^{\infty} Y\left[f_{p_k}(q_k t)/\log\log p_k\right]dt$$
$$= \sum_{k=1}^{\infty} \int_0^{2\pi} Y\left[f_{p_k}(q_k t)/\log\log p_k\right]dt .$$

If the last series is convergent, then there is an x_0 such that $f(t, x_0) \in L(\log L)^p$. Therefore it is sufficient to prove that

$$\int_0^{2\pi} Y(|f_n(t)|/\log\log n)dt \to 0 \quad \text{as} \quad n \to \infty$$

which follows from

$$\int_0^{2\pi} Y(|f_n(t)|)dt \le A \quad \text{for all} \quad n . \tag{3}$$

The left side integral is

$$\frac{1}{n}\left[\int_0^{2\pi/(2n+1)} + \sum_{k=1}^{n-1} \int_{c_k-2\pi/(2n+1)}^{c_k+2\pi/(2n+1)} + \int_{c_{n-1}+2\pi/(2n+1)}^{2\pi}\right] \left|\sum_{i=1}^{n} F^*_{m_i}(t-c_i)s_N(t; h)\right| \left|\log(|f_n(t)|+e)\right|^p dt$$
$$= \frac{1}{n}\left[W_0 + \sum_{k=1}^{n-1} W_k + W_n\right] .$$

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We shall prove that each W_k is bounded. For $1 \le k \le n-1$, we write

$$W_k = \int_{c_k-2\pi/(2n+1)}^{c_k+2\pi/(2n+1)} = \int_{c_k-2\pi/(2n+1)}^{c_k} + \int_{c_k}^{c_k+2\pi/(2n+1)}$$

$$= W_k' + W_k''$$

and

$$W_k'' = \int_{c_k}^{c_k+1/m_k} + \int_{c_k+1/m_k}^{c_k+1/2n\log\log n} + \int_{c_k+1/2n\log\log n}^{c_k+2\pi/(2n+1)}$$

$$= X_1 + X_2 + X_3 .$$

Since

$$\log\left(|f_n(t)|+e\right) \le An$$

and

$$s_N(t;\ h) = O(n\log\log n/N) \quad \text{for} \quad c_k < t < c_k+1/2n\log\log n ,$$

we have

$$|X_1| \le \frac{An^{p+1}\log\log n}{n} \int_{c_k}^{c_k+1/m_k} \left[F_{m_k}(t-c_k) + \sum_{\substack{i=1\\ i\ne k}}^{n} F_{m_i}(t-c_i)\right] dt + A$$

$$\le \frac{An^{p+1}\log\log n}{n} \left\{A + \sum_{\substack{i=1\\ i\ne k}}^{n} \frac{An^2}{m_i(i-k)^2}\,\frac{1}{m_k}\right\} + A$$

$$\le A ,$$

$$|X_2| \le \frac{An^{p+1}\log\log n}{n} \left\{\int_{1/m_k}^{1/2n\log\log n} \frac{dt}{m_k t^2} + A\right\}$$

$$\le A$$

and

$$X_3 \le An^p \left\{\int_{1/2n\log\log n}^{2\pi/(2n+1)} \left[\frac{1}{m_k t^2} + \sum_{\substack{i=1\\ i\ne k}}^{n} \frac{n^2}{m_i(i-k)^2}\right] dt\right\}$$

$$\le A .$$

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Therefore we have proved that W_k'' is bounded. Similarly W_k' is bounded. Since W_0 and W_n are also bounded, all W_k are bounded, which proves (3).

Thus we have proved that $f(t, x_0) \in L(\log L)^p$. The theorem is now completely proved, since the Fourier series of $f(t, x_0)$ diverges almost everywhere which follows from Section 2.

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