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Lagrangian configurations and Hamiltonian maps

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Compositio Math. **159** (2023), 2483–2520.

[doi:10.1112/S0010437X23007455](https://doi.org/10.1112/S0010437X23007455)



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ABSTRACT

We study configurations of disjoint Lagrangian submanifolds in certain low-dimensional symplectic manifolds from the perspective of the geometry of Hamiltonian maps. We detect infinite-dimensional flats in the Hamiltonian group of the two-sphere equipped with Hofer's metric, prove constraints on Lagrangian packing, find instances of Lagrangian Poincaré recurrence, and present a new hierarchy of normal subgroups of area-preserving homeomorphisms of the two-sphere. The technology involves Lagrangian spectral invariants with Hamiltonian term in symmetric product orbifolds.

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1. Introduction and main results

1.1 Overview

A symplectic structure ω on an even-dimensional manifold M^{2n} can be viewed as a far-reaching generalization of the two-dimensional area on surfaces: by definition, ω is a closed differential 2-form whose top wedge power ω^n is a volume form. The group $\text{Symp}(M, \omega)$ of all symmetries of a symplectic manifold, i.e. of diffeomorphisms preserving the symplectic structure, contains a remarkable subgroup of *Hamiltonian diffeomorphisms* $\text{Ham}(M, \omega)$. When M is closed and its first cohomology vanishes, Ham coincides with the identity component of Symp . At the same time, in classical mechanics, where M models the phase space, Ham arises as the group of all admissible motions. This group became a central object of interest in symplectic

Received 15 April 2021, accepted in final form 17 May 2023, published online 18 September 2023.

2020 Mathematics Subject Classification 53Dxx, 37Jxx (primary).

Keywords: symplectic manifold, Hamiltonian diffeomorphism, Lagrangian submanifold, Poincaré recurrence, Hofer's metric, Floer theory, orbifold, symmetric product.

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topology in the past three decades. In spite of that, some very basic questions about the algebra, geometry and topology of $\text{Ham}(M, \omega)$ are far from being understood even when M is a surface.

Let us briefly outline the contents of the paper. First, we obtain new results on Hofer's bi-invariant geometry of $\text{Ham}(M, \omega)$ in the case where $M = S^2$ is the two-dimensional sphere. We establish, roughly speaking, that $\text{Ham}(S^2)$ contains flats of arbitrary dimension, thus solving a question open since 2006. A similar result was obtained simultaneously and independently, by using a different technique, in a recent paper by Cristofaro-Gardiner, Humilière, and Seyfaddini [CHS21]. Furthermore, our method yields an infinite hierarchy of normal subgroups of area-preserving homeomorphisms of the two-sphere, all of which contain the normal subgroup of homeomorphisms of finite energy discovered in [CHS21]. In addition, we find a new constraint on rotationally symmetric homeomorphisms of finite energy.

Second, we extend a number of elementary two-dimensional phenomena taking place on S^2 to the *stabilized* space $S^2 \times S^2$, where the area of the second factor is 'much smaller' than that of the first. These phenomena include constraints on packing by circles, which correspond to packings by two-dimensional tori after stabilization, and a version of Poincaré recurrence theorem for sets of zero measure in the context of Hamiltonian diffeomorphisms. Furthermore, we prove a stabilized version of our results on Hofer's geometry presented above.

To illustrate the stabilization paradigm, an elementary area count shows that one cannot fit into the sphere of unit area k pairwise disjoint Hamiltonian images of a circle L bounding a disk of area $> 1/k$. We shall show that the same is true for $L \times$ equator in the stabilized space. Here the area/volume control miserably fails: a two-dimensional torus does not bound any volume in a four-manifold.

The first instance of such a stabilization was discovered in a recent paper by Mak and Smith [MS21], who studied constraints on the displaceability of collections of curves on S^2 . Recall, that a set $X \subset M$ is called displaceable if there exists a Hamiltonian diffeomorphism ϕ with $\phi(X) \cap X = \emptyset$. This notion, introduced by Hofer in [Hof90], gives rise to a natural 'small scale' on symplectic manifolds. Mak and Smith noticed that certain 'small sets' on M become more rigid when one looks at them in the symmetric product, a *symplectic orbifold* $(M)^k / \text{Sym}_k$ where Sym_k stands for the permutation group. On the technical side, our main observation is that a powerful Floer-theoretical tool, Lagrangian spectral invariants with Hamiltonian term, as developed by Leclercq–Zapolsky and Fukaya–Oh–Ohta–Ono, extends to Lagrangian tori in symmetric product orbifolds and can be applied to the study of the above-mentioned questions on Hamiltonian maps. With this language, our paper provides a toolbox for *measurements of large energy symplectic effects on small geometric scales* by using Floer theory in symmetric products.

Let us note that the idea of looking at configuration spaces of points on a surface, the two-sphere in particular, in order to construct invariants of Hamiltonian diffeomorphisms goes back to Gambaudo and Ghys [GG04]. Technically, it turns out that for this paper it is beneficial to work with symmetric products, which are certain compactifications of unordered configuration spaces, and instead of the sphere to look at a certain stabilization of it to a four-manifold. Thus, our approach can be considered as a 'symplectization' of that by Gambaudo and Ghys. Furthermore, the central objects of interest to us are certain collections of pairwise disjoint Lagrangian submanifolds which in some sense govern Hamiltonian dynamics. These are *the Lagrangian configurations* appearing in the title of the paper.

1.2 Flats in Hofer's geometry

In [Hof90] Hofer has introduced a remarkable bi-invariant metric on the group $\text{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of a symplectic manifold (M, ω) . It can intuitively be thought of as the minimal L^∞ energy required to generate a given Hamiltonian diffeomorphism. For a time-dependent Hamiltonian H in $C^\infty([0, 1] \times M, \mathbb{R})$ we denote by $\{\phi_H^t\}_{t \in [0, 1]}$ the Hamiltonian isotopy generated by the vector field X_H^t given by

$$\omega(X_H^t, \cdot) = -dH_t(\cdot),$$

where $H_t(\cdot) = H(t, \cdot)$ for all $t \in [0, 1]$. We say that H has zero mean if $\int H_t \omega^n = 0$ for all $t \in [0, 1]$. When the symplectic manifold is closed, the Hofer distance of $\phi \in \text{Ham}(M, \omega)$ to the identity is defined as

$$d_{\text{Hofer}}(\phi, \text{id}) = \inf_{\phi_H^1 = \phi} \int_0^1 \max_M |H(t, -)| dt,$$

where the infimum is taken over all zero mean Hamiltonians generating ϕ . It is extended to arbitrary pairs of diffeomorphisms by the bi-invariance,

$$d_{\text{Hofer}}(\psi\phi, \psi\phi') = d_{\text{Hofer}}(\phi\psi, \phi'\psi) = d_{\text{Hofer}}(\phi, \phi'),$$

for all $\phi, \phi', \psi \in \text{Ham}(M, \omega)$.

The non-degeneracy of d_{Hofer} was studied in many further publications such as [Vit92, Pol93] and was proven to hold for arbitrary symplectic manifolds in [LM95]. Let us mention also that Hofer's metric naturally lifts to a (pseudo-)metric on the universal cover $\widetilde{\text{Ham}}(M, \omega)$, where the question about its non-degeneracy is still open in full generality. We refer to [Pol01] for a detailed introduction to the Hofer metric and many of its aspects and properties.

The main question related to the Hofer metric, Problem 20 in [MS17], is whether its diameter is infinite for all symplectic manifolds, and when it is, which unbounded groups can be quasi-isometrically embedded into $(\text{Ham}(M, \omega), d_{\text{Hofer}})$.

THEOREM A. *The additive group $\mathcal{G} = C_c^\infty(I)$ of compactly supported smooth functions on an open interval I with the C^0 distance embeds isometrically into $\text{Ham}(S^2)$ endowed with d_{Hofer} .*

This settles a question of the first author and Kapovich from 2006, cf. Problem 21 [MS17]. A similar result was obtained simultaneously and independently in [CHS21] by completely different methods based on periodic Floer homology. As, by a classical theorem [Ban93, Théorème 10, p. 187] from a book of Banach, every separable metric space admits an isometric embedding into $C_c^0(I)$, and by smooth approximation, the latter is quasi-isometric to $C_c^\infty(I)$, Theorem A implies the following.

COROLLARY 1. *Every separable metric space admits a quasi-isometric embedding into $\text{Ham}(S^2)$ endowed with d_{Hofer} .*

For closed surfaces of higher genus, flats of arbitrary dimension were found in [Py08]. The proof is based on the fact that such a surface admits an incompressible annulus foliated by non-displaceable closed curves. Symplectic rigidity of these curves yields the result. The lack of such an annulus in the case of S^2 requires a new tool, *Lagrangian estimators*, which we develop by using Lagrangian Floer theory in orbifolds, see §§ 2 and 6.

In fact, our method of proof yields the following statement about Hofer's geometry in dimension four. Throughout the paper $S^2(b)$ stands for the sphere equipped with the standard area form normalized in such a way that the total area equals b . We abbreviate $S^2 = S^2(1)$.

For $a > 0$, consider a symplectic manifold $M_a = S^2 \times S^2(2a)$. The natural monomorphism

$$\iota : \text{Ham}(S^2) \rightarrow \text{Ham}(M_a), \quad \phi \mapsto \phi \times \text{id},$$

satisfies $d_{\text{Hofer}}(\iota(\phi), \iota(\psi)) \leq d_{\text{Hofer}}(\phi, \psi)$.

THEOREM B. *Assume that $a > 0$ is small enough. The isometric monomorphism $\Phi : \mathcal{G} \rightarrow \text{Ham}(S^2)$ from Theorem A can be chosen in such a way that*

$$\Psi = \iota \circ \Phi : (\mathcal{G}, d_{C^0}) \hookrightarrow (\text{Ham}(M_a), d_{\text{Hofer}})$$

is a bi-Lipschitz embedding. Furthermore, the monomorphism of the universal covers $\tilde{\Psi} : \mathcal{G} \rightarrow \text{Ham}(M_a)$ covering Ψ is an isometric embedding.

1.3 Lagrangian packing

Let K_r be a simple closed curve on the sphere $S^2 = S^2(1)$ bounding a disk of area $1/k > r > 1/(k + 1)$, $k \in \mathbb{N}$, $k \geq 2$. Let $S \subset S^2(2a)$ be the equator. Note that the Lagrangian torus $\Lambda_r = K_r \times S$ can be Hamiltonianly k -packed into M_a , that is, there are k Hamiltonian diffeomorphisms $\phi_1 = \text{id}, \phi_2, \dots, \phi_k$ of M_a such that $\{\phi_j(\Lambda_r)\}_{1 \leq j \leq k}$ are pairwise disjoint. Indeed, such a packing exists for $K_r \subset S^2$.

THEOREM C (Lagrangian packing). *One cannot pack $M_a = S^2 \times S^2(2a)$ by $k + 1$ Hamiltonian images of Λ_r if a is sufficiently small.*

We present an argument, based on asymptotic Hofer’s geometry, in § 3.6.

1.4 Lagrangian Poincaré recurrence

Using Lagrangian packing obstructions, we are able to make the following progress on the well-known Lagrangian Poincaré recurrence conjecture in Hamiltonian dynamics [GG19]. It is a version of the classical Poincaré recurrence theorem, but in the setting of Lagrangian submanifolds instead of sets of positive measure. Note that except for the case of surfaces, this is a purely symplectic question, since Lagrangian submanifolds do not bound and they have zero measure.

Let $\Lambda_r \subset M_a$ be a Lagrangian torus as in § 1.3. For an arbitrary Hamiltonian diffeomorphism $\phi \in \text{Ham}(M_a)$, consider the *recurrence set*

$$\mathcal{R}_\phi := \{n \in \mathbb{N} : \phi^n \Lambda_r \cap \Lambda_r \neq \emptyset\}.$$

THEOREM D (Lagrangian recurrence). *The lower density of \mathcal{R}_ϕ is at least $1/k$.*

We remark that while in [GG18] a similar statement was proven for specific rigid Hamiltonian diffeomorphisms of complex projective spaces (pseudo-rotations) and arbitrary Lagrangians, we provide the first non-trivial higher-dimensional examples of Lagrangian submanifolds satisfying the recurrence property for *all* Hamiltonian diffeomorphisms.

1.5 Area-preserving homeomorphisms of S^2

It has been established by Cristofaro-Gardiner, Humilière and Seyfaddini [CHS21] that the group G_{S^2} of symplectic homeomorphisms of the sphere possesses a non-trivial normal subgroup $G_{S^2}^F$ of homeomorphisms of finite energy. These are the homeomorphisms ϕ for which there exists a sequence of Hamiltonian diffeomorphisms ψ_j and a constant $C > 0$ such that $d_{C^0}(\psi_j, \phi) \rightarrow 0$ and $d_{\text{Hofer}}(\psi_j, \text{id}) \leq C$. This is achieved by showing that radially symmetric Hamiltonian homeomorphisms of bounded energy satisfying a certain monotone twist condition must, in a precise sense, have finite Calabi invariant. We extend this result as follows.

Let $z : S^2 \rightarrow [-1/2, 1/2]$ be the moment map for the standard S^1 -action on S^2 . It is the height function for the standard embedding of S^2 in \mathbb{R}^3 scaled by a factor of $1/2$. Let $h : [-1/2, 1/2) \rightarrow \mathbb{R}$

be a smooth function that vanishes on $[-1/2, 0]$. Consider $\phi \in G_{S^2}$ generated by $H = h \circ z$: it is the C^0 -limit of $\phi_i = \phi_{H_i}^1 \in \text{Ham}_c(\mathbb{D}^2) \subset \text{Ham}(S^2)$ for $H_i = h_i \circ z$ for h_i approximations to h constant near $1/2$ which agree with h on $[-1/2, 1/2 - 1/i]$.

THEOREM E. *If $\phi \in G_{S^2}^F$, then the primitive of h is a bounded function.*

The proof is based on a combination of [CHS21, Lemma 3.11], an interesting soft result relating C^0 -smallness, supports, and the Hofer metric, with our Lagrangian spectral estimators. It turns out that suitable linear combination of these invariants are continuous with respect to the C^0 topology on $\text{Ham}(S^2)$ and extend to the group G_{S^2} . In fact, our invariants give rise to an infinite series of pairwise distinct normal subgroups of G_{S^2} containing $G_{S^2}^F$, see §5 for a precise formulation.

Remark 2. It is easy to see that if h is bounded, then $\phi \in G_{S^2}^F$. It would be very interesting to bridge the discrepancy between this observation and Theorem E further. Note that there do exist unbounded functions h whose primitives are bounded. We expect that arguments along the lines of Sikorav's trick as in §3.4 and lower bounds in terms of further linear combinations of the $c_{k,B}$ invariants could be useful to this end.

Remark 3. We note that a suitable generalized limit construction (e.g. the Banach limit or the limit with respect to a non-principal ultrafilter) yields the existence of many homomorphisms $\phi \mapsto \mathcal{C}(\phi) \in \mathbb{R}$ on the group of ϕ as in Theorem E that coincide with the Calabi invariant of $\phi \in \text{Ham}_c(\mathbb{D}^2)$ if $H = h \circ z$ extends smoothly to S^2 .

1.6 Organization of the paper

In §2 we introduce and list the properties of Lagrangian estimators, our main technical tool. The detailed construction of the estimators via Lagrangian Floer theory in symmetric products, as well as the proof of their properties is presented in §6.

Section 3 deals with flats in Hofer geometry. We prove Theorems A and B, and present a result on infinite-dimensional flats in subgroups of Hamiltonian diffeomorphisms of the disk with vanishing Calabi invariant. In addition, we derive an estimate on the asymptotic Hofer norm which is used in the proof of Theorem C on Lagrangian packing. This proof can be found in §3.6.

In §4 we deduce Theorem D on Lagrangian recurrence from the packing constraint by a combinatorial argument.

In §5 we prove the C^0 -continuity of certain linear combinations of our invariants and present applications to normal subgroups of the group of area-preserving homeomorphisms of S^2 .

We conclude the paper with a discussion of further directions in §7.

2. Lagrangian estimators

In this section we introduce three slightly different flavors of *Lagrangian estimators*, the main technical tool of the present paper. These are functionals with a number of remarkable properties defined on the space of time-dependent Hamiltonians (*spectral estimators*), on the group of Hamiltonian diffeomorphisms (*group estimators*), and on the Lie algebra of functions on a symplectic manifold (*algebra estimators*). They somewhat resemble, but are different from, partial symplectic quasi-morphisms and quasi-states introduced in [EP06], respectively (see Remark 7).

We rely on the theory of Lagrangian spectral invariants [Lec08, LZ18, FOOO19] in the setting of bulk-deformed Lagrangian Floer homology [FOOO09] for symplectic orbifolds [CP14]

and crucially its recent investigation [MS21] in the context of Lagrangian links in symplectic four-manifolds. The output of our construction is a new invariant of a Hamiltonian diffeomorphism of S^2 that instead of being supported on a single non-displaceable Lagrangian circle is supported on a non-displaceable configuration of pairwise disjoint and (in general) individually displaceable Lagrangian circles.

We start with a couple of preliminary notions and notation. Let $z : S^2 \rightarrow [-1/2, 1/2]$ be the moment map for the standard $S^1 = \mathbb{R}/\mathbb{Z}$ -action on S^2 . It is instructive to think that $S^2 = S^2(1)$ is the round sphere of radius $1/2$ in \mathbb{R}^3 equipped with the standard area form divided by π , and z is simply the vertical Euclidean coordinate.

Let $k \geq 1$ be a positive integer, and let $0 < C < B$ be two positive rational numbers such that $2B + (k - 1)C = 1$.

Denote by $\mathbb{L}_{k,B}^0 \subset S^2$ be the configuration of k disjoint circles given by $\mathbb{L}_{k,B}^0 = \bigcup_{0 \leq j < k} L_{k,B}^{0,j}$, where

$$L_{k,B}^{0,j} = (z)^{-1}(-1/2 + B + jC). \tag{1}$$

Let $0 < a < B - C$ be a rational number. Consider the symplectic manifold $M_a = S^2 \times S^2(2a)$. Denote by S the equator of $S^2(2a)$ and put $\mathbb{L}_{k,B} = \mathbb{L}_{k,B}^0 \times S$, $L_{k,B}^j = L_{k,B}^{0,j} \times S$ for $0 \leq j < k$.

For an open subset U of a $2n$ -dimensional symplectic manifold and a Hamiltonian H supported in $[0, 1] \times U$ we define the Calabi invariant as

$$\text{Cal}(\{\phi_H^t\}) = \int_0^1 \int_U H_t \omega^n.$$

Moreover, Cal defines a homomorphism $\text{Cal} : \widetilde{\text{Ham}}_c(U) \rightarrow \mathbb{R}$.

Finally, we call an open set U *displaceable from a subset V* if there exists a Hamiltonian diffeomorphism θ such that $\theta(U) \cap V = \emptyset$.

Now we are ready to formulate the main result of the present section.

THEOREM F (Lagrangian spectral estimators). *For each k, B, a as above, with B, a rational, there exists a map*

$$c_{k,B} : C^\infty([0, 1] \times M_a, \mathbb{R}) \rightarrow \mathbb{R}$$

satisfying the following properties.

1. (Hofer–Lipschitz) *For each $G, H \in C^\infty([0, 1] \times M_a, \mathbb{R})$,*

$$|c_{k,B}(G) - c_{k,B}(H)| \leq \int_0^1 \max |G_t - H_t| dt.$$

2. (Monotonicity) *If $G, H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ satisfy $G \leq H$ as functions, then*

$$c_{k,B}(G) \leq c_{k,B}(H).$$

3. (Normalization) *For each $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ and $b \in C^\infty([0, 1], \mathbb{R})$,*

$$c_{k,B}(H + b) = c_{k,B}(H) + \int_0^1 b(t) dt.$$

4. (Lagrangian control) *If $(H_t)|_{L_{k,B}^j} \equiv c_j(t) \in \mathbb{R}$ for all $0 \leq j < k$ and $t \in [0, 1]$, then*

$$c_{k,B}(H) = \frac{1}{k} \sum_{0 \leq j < k} \int_0^1 c_j(t) dt.$$

5. (Independence of Hamiltonian) For $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ with zero mean, the value

$$c_{k,B}(H) = c_{k,B}(\phi_H)$$

depends only on the class

$$\phi_H = [\{\phi_H^t\}] \in \widetilde{\text{Ham}}(M_a)$$

in the universal cover of $\text{Ham}(M_a)$ generated by H .

6. (Subadditivity) For all $\phi, \psi \in \widetilde{\text{Ham}}(M_a)$,

$$c_{k,B}(\phi\psi) \leq c_{k,B}(\phi) + c_{k,B}(\psi).$$

7. (Calabi property) If a Hamiltonian $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ is supported in an open set of the form $[0, 1] \times U$, where $U \subset M_a$ is disjoint from $\mathbb{L}_{k,B}$, then

$$c_{k,B}(\phi_H) = -\frac{1}{\text{vol}(M_a)} \text{Cal}(\{\phi_H^t\}).$$

8. (Controlled additivity) If $\psi = \phi_H \in \widetilde{\text{Ham}}(M_a)$ for a Hamiltonian $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ supported in $[0, 1] \times U$ for an open set $U \subset M_a$ disjoint from $\mathbb{L}_{k,B}$, then for all $\phi \in \widetilde{\text{Ham}}(M_a)$

$$c_{k,B}(\phi\psi) = c_{k,B}(\phi) + c_{k,B}(\psi).$$

Remark 4. In fact, the controlled additivity property of $c_{k,B}$ holds under the more general assumption that

$$(H_t)|_{L_{k,B}^j} \equiv c_j(t) \in \mathbb{R}$$

for all $0 \leq j < k$ and $t \in [0, 1]$. However, since we do not use this stronger version, we chose to omit it for simplicity of exposition.

In turn, this implies by homogenization that the following holds.

THEOREM G (Lagrangian group estimators). For each k, B, a as above, with B, a rational, there exists a map

$$\mu_{k,B} : C^\infty([0, 1] \times M_a, \mathbb{R}) \rightarrow \mathbb{R}$$

satisfying the following properties.

1. (Hofer–Lipschitz) For each $G, H \in C^\infty([0, 1] \times M_a, \mathbb{R})$,

$$|\mu_{k,B}(G) - \mu_{k,B}(H)| \leq \int_0^1 \max |G_t - H_t| dt.$$

2. (Monotonicity) If $G, H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ satisfy $G \leq H$ as functions, then

$$\mu_{k,B}(G) \leq \mu_{k,B}(H).$$

3. (Normalization) For each $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ and $b \in C^\infty([0, 1], \mathbb{R})$,

$$\mu_{k,B}(H + b) = \mu_{k,B}(H) + \int_0^1 b(t) dt.$$

4. (Lagrangian control) If $(H_t)|_{L_{k,B}^j} \equiv c_j(t) \in \mathbb{R}$ for all $0 \leq j < k$ and $t \in [0, 1]$, then

$$\mu_{k,B}(H) = \frac{1}{k} \sum_{0 \leq j < k} \int_0^1 c_j(t) dt.$$

5. (Independence of Hamiltonian) For $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ with zero mean, the value

$$\mu_{k,B}(H) = \mu_{k,B}(\phi_H)$$

depends only on the class

$$\phi_H = [\{\phi_H^t\}] \in \widetilde{\text{Ham}}(M_a)$$

in the universal cover of $\text{Ham}(M_a)$ generated by H .

6. (Conjugation invariance) For all $\phi, \psi \in \widetilde{\text{Ham}}(M_a)$ we have

$$\mu_{k,B}(\psi\phi\psi^{-1}) = \mu_{k,B}(\phi).$$

7. (Positive homogeneity) For all $\phi \in \widetilde{\text{Ham}}(M_a)$ and $m \in \mathbb{Z}_{\geq 0}$

$$\mu_{k,B}(\phi^m) = m \cdot \mu_{k,B}(\phi).$$

8. (Commutative subadditivity) If $\phi, \psi \in \widetilde{\text{Ham}}(M_a)$ commute, $\phi\psi = \psi\phi$, then

$$\mu_{k,B}(\phi\psi) \leq \mu_{k,B}(\phi) + \mu_{k,B}(\psi).$$

9. (Calabi property) If a Hamiltonian $H \in C^\infty([0, 1] \times M_a, \mathbb{R})$ is supported in an open set of the form $[0, 1] \times U$, where $U \subset M_a$ is displaceable from $\mathbb{L}_{k,B}$, then

$$\mu_{k,B}(\phi_H) = -\frac{1}{\text{vol}(M_a)} \text{Cal}(\{\phi_H^t\}).$$

The proof is given in § 6.3.4.

Remark 5. The rationality of B, a is necessary for the Lagrangian control property and the Calabi property. In our applications to Hofer’s geometry this will not lead to a loss of generality because \mathbb{Q} is dense in \mathbb{R} . Note also that $\psi\phi\psi^{-1}$ in the conjugation invariance property (Property 6 of Theorem G) depends only on ϕ and the image of ψ under the natural map $\widetilde{\text{Ham}}(M_a) \rightarrow \text{Ham}(M_a)$. We note that if $\psi^2 = \text{id}$ and $\phi\psi = \psi\phi$ in $\widetilde{\text{Ham}}(M_a)$, then

$$\mu_{k,B}(\phi\psi) = \mu_{k,B}(\phi). \tag{2}$$

Indeed commutative subadditivity implies that $-\mu(\psi) \leq \mu(\phi\psi) - \mu(\phi) \leq \mu(\psi)$, while positive homogeneity yields $\mu(\psi) = 0$. Moreover, note that by the Hofer–Lipschitz property one can naturally extend $\mu_{k,B}$ to a map $C^0([0, 1] \times M, \mathbb{R}) \rightarrow \mathbb{R}$ satisfying a directly analogous list of properties.

Remark 6. While it is not directly pertinent to our applications in this paper, it would be interesting to explicitly calculate the restriction of $c_{k,B}$ and $\mu_{k,B}$ to $\pi_1(\text{Ham}(M_a))$. It will be determined by the valuation of a suitable Seidel representation evaluated on Gromov’s loop of infinite order in $\pi_1(\text{Ham}(M_a))$ and respectively its homogenization.

It turns out to be useful to consider the restriction of $\mu_{k,B}$ to the space $C^\infty(M_a, \mathbb{R})$ of Hamiltonians which do not depend on time. We recall that the Poisson bracket of two functions $F, G \in C^\infty(M_a, \mathbb{R})$ is defined as $\{F, G\} = dF(X_G)$. The following list of properties is a direct consequence of those in Theorem G: note that quasi-additivity and vanishing follow from commutative subadditivity and the Calabi property of $\mu_{k,B}$.

THEOREM H (Lagrangian algebra estimators). *The map $C^\infty(M_a, \mathbb{R}) \rightarrow C^\infty([0, 1] \times M_a, \mathbb{R})$, $F \mapsto H(t, x) = F(x)$, induces a map*

$$\begin{aligned}\zeta_{k,B} &: C^\infty(M_a, \mathbb{R}) \rightarrow \mathbb{R} \\ \zeta_{k,B}(F) &= \mu_{k,B}(H)\end{aligned}$$

which satisfies the following properties.

1. (*C^0 -Lipschitz*) For each $G, H \in C^\infty(M_a, \mathbb{R})$,

$$|\zeta_{k,B}(G) - \zeta_{k,B}(H)| \leq |G - H|_{C^0}.$$

2. (*Monotonicity*) If $G, H \in C^\infty(M_a, \mathbb{R})$ satisfy $G \leq H$ as functions, then

$$\zeta_{k,B}(G) \leq \zeta_{k,B}(H).$$

3. (*Normalization*)

$$\zeta_{k,B}(1) = 1.$$

4. (*Lagrangian control*) If $H|_{L_{k,B}^j} \equiv c_j \in \mathbb{R}$ for all $0 \leq j < k$, then

$$\zeta_{k,B}(H) = \frac{1}{k} \sum_{0 \leq j < k} c_j.$$

5. (*Invariance*) For all $\psi \in \text{Ham}(M_a)$ and $H \in C^\infty(M_a, \mathbb{R})$ we have

$$\zeta_{k,B}(H \circ \psi^{-1}) = \zeta_{k,B}(H).$$

6. (*Positive homogeneity*) For all $H \in C^\infty(M_a, \mathbb{R})$ and $t \in \mathbb{R}_{\geq 0}$,

$$\zeta_{k,B}(t \cdot H) = t \cdot \zeta_{k,B}(H).$$

7. (*Quasi-additivity and vanishing*) If $F, G \in C^\infty(M_a, \mathbb{R})$ Poisson-commute, $\{F, G\} = 0$, then

$$\zeta_{k,B}(F + G) \leq \zeta_{k,B}(F) + \zeta_{k,B}(G)$$

and if in addition G is supported in U displaceable from $\mathbb{L}_{k,B}$, then

$$\zeta_{k,B}(F + G) = \zeta_{k,B}(F) + \zeta_{k,B}(G) = \zeta_{k,B}(F).$$

Finally, we consider the restriction of the invariants $\mu_{k,B}, \zeta_{k,B}$ to Hamiltonians on S^2 by means of the stabilization by the zero Hamiltonian.

THEOREM I. *The map $C^\infty([0, 1] \times S^2) \rightarrow C^\infty([0, 1] \times M_a)$, $F \mapsto H = F \oplus 0$, that is, $H(t, x, y) = F(t, x)$ induces maps*

$$\begin{aligned}c_{k,B}^0 &: C^\infty([0, 1] \times S^2, \mathbb{R}) \rightarrow \mathbb{R}, \\ \mu_{k,B}^0 &: C^\infty([0, 1] \times S^2, \mathbb{R}) \rightarrow \mathbb{R}, \\ \zeta_{k,B}^0 &: C^\infty(S^2, \mathbb{R}) \rightarrow \mathbb{R},\end{aligned}$$

by means of

$$c_{k,B}^0(F) = c_{k,B}(H), \quad \mu_{k,B}^0(F) = \mu_{k,B}(H), \quad \zeta_{k,B}^0(F) = \zeta_{k,B}(H).$$

These maps satisfy the corresponding lists of properties as in Theorems F–H with M_a replaced by S^2 , $L_{k,B}^j$ by $L_{k,B}^{0,j}$, and $\mathbb{L}_{k,B}$ by $\mathbb{L}_{k,B}^0$ everywhere.

In addition, $c_{k,B}^0$ and $\mu_{k,B}^0$ satisfy the following stronger independence of the Hamiltonian property: $c_{k,B}^0(H)$ and $\mu_{k,B}^0(H)$ for a mean-normalized Hamiltonian $H \in C^\infty([0, 1] \times S^2, \mathbb{R})$ depend only on $\phi = \phi_H^1 \in \text{Ham}(S^2)$.

As a function

$$c_{k,B}^0 : \text{Ham}(S^2) \rightarrow \mathbb{R}$$

it satisfies the subadditivity, Calabi, and controlled additivity properties. As a function

$$\mu_{k,B}^0 : \text{Ham}(S^2) \rightarrow \mathbb{R}$$

it satisfies the conjugation invariance, positive homogeneity, commutative subadditivity, and the Calabi properties. In particular,

$$|\zeta_{k,B}^0(H)| \leq d_{\text{Hofer}}(\phi_H^1, \text{id})$$

for all $H \in C^\infty(S^2, \mathbb{R})$.

Proof. The proof of all statements is immediate, except for stronger independence of the Hamiltonian. To this end, we observe that by a classical result of Smale $\pi_1(\text{Ham}(S^2)) \cong \mathbb{Z}/2\mathbb{Z}$. Let ψ be its generator. Since $\pi_1(\text{Ham}(S^2))$ lies in the center of $\widetilde{\text{Ham}}(S^2)$, the result for $\mu_{k,B}^0$ follows from (2). For $c_{k,B}^0$ we have $c_{k,B}^0(\psi) = 0$ by Lagrangian control, since ψ is generated by the mean-zero Hamiltonian $F = z : S^2 \rightarrow [-1/2, 1/2]$. Hence, by subadditivity, for all $\phi \in \widetilde{\text{Ham}}(S^2)$ we have $c_{k,B}^0(\phi\psi) \leq c_{k,B}^0(\phi)$. Replacing ϕ by $\phi\psi$ and using $\psi^2 = \text{id}$, we obtain the inequality $c_{k,B}^0(\phi) \leq c_{k,B}^0(\phi\psi)$ in the reverse direction. \square

Remark 7. We observe that the maps $\zeta_{k,B}^0$, and hence also $\zeta_{k,B}$, are not partial symplectic quasi-states for $k > 1$. Indeed, $\zeta_{k,B}^0$ equals $1/k$ for the cut-off of the indicator function of a small neighborhood of $L_{k,B}^{0,0}$, a circle of our configuration having the smallest area. However, this circle is displaceable, a contradiction with the vanishing axiom for quasi-states. However, it is not hard to see that the choices involved in the Floer data defining $\zeta_{1,1/2}^0$ can be chosen in such a way that it coincides with the symplectic quasi-state ζ_0 on S^2 (which is, in fact, unique: see [PR14, Exercise 5.4.29]).

3. Hofer’s geometry: proofs and further results

Here we apply the techniques of Lagrangian estimators to the proof of our main applications to Hofer’s geometry. Note that for proofs of Theorems A and B we need the simplest Lagrangian configurations consisting of two circles.

3.1 Proof of Theorem A

Step 1: construction. We start with a more explicit formulation of the theorem. Consider a symmetric interval $I = (-b, b)$ for $b < 1/6$. Let

$$\mathcal{F}_I \subset C_c^\infty(I)$$

be the space of even compactly supported smooth functions on I . In other words, \mathcal{F}_I consists of functions $h \in C_c^\infty(I)$ satisfying $h(x) = h(-x)$ for all $x \in I$. Endow \mathcal{F}_I with the C^0 norm

$$\|h\|_{C^0} = \max_I |h|,$$

which induces the distance function $d_{C^0}(h_1, h_2) = \|h_1 - h_2\|_{C^0}$. Observe that the group $\mathcal{G} = C_c^\infty((0, b))$ with the C^0 distance naturally embeds into \mathcal{F}_I by even extension.

Consider the standard symplectic sphere (S^2, ω) of total area 1. It admits a Hamiltonian S^1 -action whose zero-mean normalized moment map $z : S^2 \rightarrow \mathbb{R}$ has image $[-1/2, 1/2]$. We define the following embedding

$$\Phi : \mathcal{F}_I \rightarrow \text{Ham}(S^2),$$

keeping in mind that the Hofer metric works with zero-mean Hamiltonians. We first build an embedding of \mathcal{F}_I to the space \mathcal{I} of even functions of integral zero in $C^\infty([-1/2, 1/2])$. To a function $h \in \mathcal{F}_I$ we associate $h^\# \in \mathcal{I}$ defined by $h^\#|_{(-b, b)} = h$, $h^\#|_{(1/2-b, 1/2]}(x) = -h(1/2 - x)$, $h^\#|_{[-1/2, -1/2+b)}(x) = -h(-1/2 - x)$, extended by zero to $[-1/2, 1/2]$. Now, for $h \in \mathcal{F}_I$ we consider the mean-zero Hamiltonian $\Gamma(h) \in C^\infty(S^2, \mathbb{R})$ given by

$$\Gamma(h) = h^\# \circ z,$$

and let

$$\Phi(h) = \phi_{\Gamma(h)}^1$$

be the time-one map of $\Gamma(h)$. It is immediate by construction that this map $\Phi : \mathcal{F}_I \rightarrow \text{Ham}(S^2)$ is a homomorphism and for all $h_1, h_2 \in \mathcal{F}$

$$d_{\text{Hofer}}(\Phi(h_1), \Phi(h_2)) = d_{\text{Hofer}}(\Phi(h_1 - h_2), \text{id}) \leq |h_1 - h_2|_{C^0}. \quad (3)$$

We *claim* that the monomorphism of groups

$$\Phi : (\mathcal{F}_I, d_{C^0}) \hookrightarrow (\text{Ham}(S^2), d_{\text{Hofer}})$$

is an isometric embedding.

Step 2: proof of the claim. Note that by (3), the main step in the proof of the claim is the inequality

$$d_{\text{Hofer}}(\Phi(h), \text{id}) \geq |h|_{C^0} \quad \forall h \in \mathcal{F}_I. \quad (4)$$

As $|(-h)|_{C^0} = |h|_{C^0}$ and

$$d_{\text{Hofer}}(\Phi(-h), \text{id}) = d_{\text{Hofer}}(\Phi(h)^{-1}, \text{id}) = d_{\text{Hofer}}(\Phi(h), \text{id})$$

for all $h \in \mathcal{F}_I$, it is sufficient to prove (4) under the assumption that $|h|_{C^0} = h(x_0) > 0$. Note that in this case $h(x_0) = h(-x_0)$, and hence either $x_0 = 0$, or we can assume that $x_0 \in (0, b)$. Consider $B = 1/2 - x_0$. For¹ $x_0 \in [0, b)$ we consider $\zeta_i = \zeta_{2, B_i}^0$, where $B_i = 1/2 - x_i$, and $x_i \in (0, b)$ is a sequence of rational numbers converging to x_0 . Here we fix a rational parameter $0 < a < 1/2 - 3b$ for defining ζ_{2, B_i}^0 as in Theorem I. By continuity of h we now have

$$h(x_i) \xrightarrow{i \rightarrow \infty} h(x_0) = |h|_{C^0}.$$

Now by the Lagrangian control property of ζ_i we have

$$\zeta_i(\Gamma(h)) = h(x_i),$$

and by the Hofer–Lipschitz and independence of Hamiltonian properties

$$d_{\text{Hofer}}(\Phi(h)) \geq \zeta_i(\Gamma(h))$$

for all i . Therefore, taking limits as $i \rightarrow \infty$ we obtain

$$d_{\text{Hofer}}(\Phi(h)) \geq |h|_{C^0}$$

as required. This finishes the proof. \square

¹ When $x_0 = 0$, (4) follows from a result of the first author [Pol98]. See [EP03, EP09] and [LZ18] for alternative proofs.

Remark 8. We note that the interval $I = (-1/6, 1/6)$ in the claim of Step 1 is the best possible using the method we describe in this paper. Indeed, the disconnected Lagrangian given by $z^{-1}(\{\pm(1/6 + \delta)\})$ yields a Lagrangian in the symmetric square that is displaceable from itself. (See § 6 for a description of the framework.) Moreover, Sikorav’s trick in § 3.4 shows that this embedding is not isometric for $b > 1/6$.

Remark 9. Note also that for any odd function $h(z) \in C^\infty(S^2, \mathbb{R})$, that is $h(-z) = -h(z)$, ϕ_h^1 is conjugate to its inverse. Indeed, for the involution $R \in \text{Ham}(S^2)$ given by $R(x, y, z) = (x, -y, -z)$ we have $h \circ R^{-1} = -h$ so $(\phi_h^1)^{-1} = \phi_{-h}^1 = \phi_{h \circ R^{-1}}^1 = R \circ \phi_h^1 \circ R^{-1}$. Therefore, $d_{\text{Hofer}}(\phi_{2h}^1, \text{id}) \leq C = 2d_{\text{Hofer}}(R, \text{id})$. Hence, all such odd autonomous Hamiltonians generate one-parametric subgroups in the ball of Hofer of radius C around the identity. From this perspective, it is natural that our construction is based on even functions.

3.2 Proof of Theorem B

Let $I = (-b, b)$ for $b < 1/6$ and $0 < a < 1/2 - 3b$. Arguing as in the proof of Theorem A we get that the monomorphism $\widetilde{\Psi} : \mathcal{F}_I \rightarrow \widetilde{\text{Ham}}(M_a)$ covering Ψ is an isometric embedding.

In order to pass from the universal cover to the group itself, we shall restrict ourselves to the subspace $\mathcal{F}_I^0 \subset \mathcal{F}_I$ consisting of functions h with $h(0) = 0$. Note that the image of $\mathcal{G} = C_c^\infty((0, b))$ in \mathcal{F}_I lies in \mathcal{F}_I^0 .

Let us show that the monomorphism of groups

$$\Psi = \iota \circ \Phi : (\mathcal{F}_I^0, d_{C^0}) \hookrightarrow (\text{Ham}(M_a), d_{\text{Hofer}})$$

is a bi-Lipschitz embedding. It suffices to show that there exists $K \geq 1$ such that for all $h \in \mathcal{F}_I$

$$d_{\text{Hofer}}(\text{id}, \Psi(h)) \geq \frac{1}{K} |h|_{C^0}. \tag{5}$$

Suppose without loss of generality that $\|h\|_{C^0} = h(r) > 0$ with $r \in (0, b)$.

Consider the invariant $\mu_{2,B} : \widetilde{\text{Ham}}(M_a) \rightarrow \mathbb{R}$ provided by Theorem G for

$$1/2 > B = 1/2 - r \geq 1/2 - b.$$

In order to proceed further, we have to understand the effect of the fundamental group $\pi_1(\text{Ham}(M))$. It is a finitely generated abelian group. In fact, by [AM00, Theorem 1.1] we have $\pi_1(\text{Ham}(M)) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The $\mathbb{Z}/2\mathbb{Z}$ terms appear from the natural maps $\pi_1(\text{Ham}(S^2)) \rightarrow \pi_1(\text{Ham}(M))$ corresponding to the two components of $M = S^2 \times S^2$. The \mathbb{Z} term is the well-known Gromov loop [Gro85], investigated in detail by Abreu and McDuff [McD87, AM00]. Set $\mathcal{T} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ for the torsion part of $\mathcal{G} = \pi_1(\text{Ham}(M))$, and let $\mathcal{A} = \mathcal{G}/\mathcal{T} \cong \mathbb{Z}$ be its free part. Note that $\pi_1(\text{Ham}(M)) \subset Z(\widetilde{\text{Ham}}(M))$ is a central subgroup. As in the proof of Theorem A it is easy to see that $\mu_{2,B}$ vanishes on \mathcal{T} and by commutative subadditivity descends to $\widetilde{\text{Ham}}(M)/\mathcal{T}$. However, the same is *not* clear for \mathcal{A} . We proceed differently.

Observe that by [Ost06, Theorem 1.2] there exists a homogeneous Calabi quasi-morphism $\rho : \widetilde{\text{Ham}}(M) \rightarrow \mathbb{R}$ that is 1-Lipschitz in the Hofer metric and restricts to a non-trivial homomorphism $\mathcal{G} \rightarrow \mathbb{R}$. It vanishes on the torsion \mathcal{T} so we can consider it to be a homomorphism $\rho : \mathcal{A} \rightarrow \mathbb{R}$. Choosing a generator g of \mathcal{A} , we have $\rho(g^k) = k \cdot \rho(g)$ for a positive constant $\rho(g)$. It is also known that $\rho(\Psi(h)) = h(0) = 0$ for $h \in \mathcal{F}_I^0$. This follows from ρ yielding a symplectic quasi-state on $C^\infty(M_a, \mathbb{R})$ and $S \times S \subset M_a$, where $S \subset S^2$ is the equator, being a stem (see [EP06, EP09, PR14]).

Define the map

$$\begin{aligned}\nu_r : \widetilde{\text{Ham}}(M) &\rightarrow \mathbb{R}^2, \\ \nu_r(\tilde{\phi}) &= (\mu_{2,B}(\tilde{\phi}), \rho(\tilde{\phi})).\end{aligned}$$

Let $\tilde{\Psi} : \mathcal{F}_I^0 \rightarrow \widetilde{\text{Ham}}(M)$ be the homomorphism covering Ψ .

Observe that ν_r is 1-Lipschitz in Hofer's metric, where \mathbb{R}^2 is endowed with the l_∞ norm.

By the above-mentioned properties of ρ , and by Lagrangian control of $\mu_{2,B}$, we can calculate for an arbitrary element $\tilde{\Psi}(h)g^k f$ covering $\Psi(h)$, where $f \in \mathcal{T}$, that

$$\nu_r(\tilde{\Psi}(h)g^k f) = \nu_r(\tilde{\Psi}(h)g^k) = (\mu_{2,B}(\tilde{\Psi}(h)g^k), \rho(\tilde{\Psi}(h)g^k)) = (\mu_{2,B}(\tilde{\Psi}(h)g^k), k\rho(g)).$$

This implies that

$$|\nu_r(\tilde{\Psi}(h)g^k f)| \geq \max\{|h|_{C^0} - |k|d_{\text{Hofer}}(g, \text{id}), |k|\rho(g)\} \geq C_2|h|_{C^0},$$

for $C_2 = \rho(g)/(\rho(g) + d_{\text{Hofer}}(g, \text{id}))$, the last step being an easy optimization in $|k|$.

Therefore, we have

$$C_2|h|_{C^0} \leq |\nu_r(\tilde{\Psi}(h)g^k f)| \leq d_{\text{Hofer}}(\tilde{\Psi}(h)g^k f, \text{id}),$$

and, hence,

$$d_{\text{Hofer}}(\Psi(h), \text{id}) \geq \frac{1}{K}|h|_{C^0}$$

for $K = 1/C_2 \geq 1$. This finishes the proof. \square

3.3 Almost flats in the kernel of Calabi

We call a map

$$f : (X, d_X) \rightarrow (Y, d_Y)$$

of metric spaces *almost isometric* if there exists a constant $D \geq 0$ such that

$$d_X(p, q) - D \leq d_Y(f(p), f(q)) \leq d_X(p, q) + D$$

for all $p, q \in X$.

Fix the interval

$$I_k = (-1/2 + 1/(k+1), -1/2 + 1/k).$$

Consider $\mu_{k,B}$ with $1/k > B > 1/(k+1)$. Note that in this case $B > C$, so $\mu_{k,B}$ is well-defined. We prove that the space of all functions $\mathcal{G}_k \subset C_c^\infty(I_k)$ with zero mean admits an almost isometric embedding into $(\text{Ham}(S^2), d_{\text{Hofer}})$. Recall that for a proper open subset $U \subset S^2$ its symplectic form is exact, and the Calabi homomorphism

$$\text{Cal}_U : \text{Ham}_c(U) \rightarrow \mathbb{R}$$

is defined as $\text{Cal}_U(\phi) = \text{Cal}_U(\{\phi_H^t\})$ for any $H \in C_c^\infty([0, 1] \times U, \mathbb{R})$ with $\phi = \phi_H^1$. Observe that by the natural constant extension we have the inclusion $\text{Ham}_c(U) \rightarrow \text{Ham}(S^2)$. In particular, $\ker(\text{Cal}_U)$ can and shall be considered to be a subgroup of $\text{Ham}(S^2)$. Starting from the following result we require configurations $\mathbb{L}_{k,B}$ for all values of k .

THEOREM J. *Let $\Phi_k : (\mathcal{G}_k, d_{C^0}) \rightarrow (\text{Ham}(S^2), d_{\text{Hofer}})$ be given by*

$$\Phi_k(h) = \phi_H^1$$

for $H = k \cdot h \circ z$. Then Φ_k is a bi-Lipschitz almost isometric group embedding, whose image lies in $\ker(\text{Cal}_D)$, where $D = D_{1/k}$ is the open cap of area $1/k$ around the south pole.

Furthermore, for each proper open set $U \subset S^2$ there is a bi-Lipschitz almost isometric embedding of $C_c^\infty(I)$ of an open interval I into $\ker(\text{Cal}_U)$.

A similar result is proved in [CHS21].

The proof of this statement is similar to that of Theorem A, with the additional observation that by the conjugation invariance of $\mu_{k,B}$ we may suppose that the open set U contains $D_{1/k}$ for k sufficiently large. Moreover, passing from smooth functions on an interval, say $C_c^\infty((-\frac{1}{2} + 1/(k + 1), -\frac{1}{2} + 1/2(k + 1) + 1/2k))$ to functions in \mathcal{G}_k can be easily carried out by odd extension about the midpoint $v_k = -\frac{1}{2} + 1/2(k + 1) + 1/2k$ of I_k . On the one hand, by use of Lagrangian estimators for the lower bound, and the evident upper bound on the Hofer norm via the Hamiltonian, we obtain

$$|h_1 - h_2|_{C^0} \leq d_{\text{Hofer}}(\Phi_k(h_1), \Phi_k(h_2)) \leq k|h_1 - h_2|_{C^0}$$

for all $h_1, h_2 \in \mathcal{G}_k$. Hence, Φ_k is bi-Lipschitz. On the other hand, estimating $d_{\text{Hofer}}(\Phi_k(h_1), \Phi_k(h_2)) = d_{\text{Hofer}}(\Phi_k(h_1 - h_2), \text{id})$ via Sikorav’s trick (see §3.4) we obtain

$$d_{\text{Hofer}}(\Phi_k(h_1), \Phi_k(h_2)) \leq |h_1 - h_2|_{C^0} + D_k$$

for a constant D_k depending only on k . This proves that Φ_k is almost isometric. (We remark that one can show that $D_k < 2$ for all k .)

Finally, analogously to the proof of Theorem B, we can show that these almost isometric embeddings extend to stabilizations on the level of universal covers and remain bi-Lipschitz embeddings on the level of groups.

THEOREM K. *Let $k \geq 2$, $1/(k + 1) < B < 1/k$ and $0 < a < ((k + 1)/(k - 1))B - 1/(k - 1)$. The monomorphism of groups*

$$\Psi_k = \iota \circ \Phi_k : (\mathcal{G}_k, d_{C^0}) \hookrightarrow (\text{Ham}(M_a), d_{\text{Hofer}})$$

is a bi-Lipschitz embedding. In fact, there exists $K \geq 1$ such that for all $h_1, h_2 \in \mathcal{G}_k$

$$\frac{1}{K}|h_1 - h_2|_{C^0} \leq d_{\text{Hofer}}(\Psi_k(h_1), \Psi_k(h_2)) \leq k|h_1 - h_2|_{C^0}.$$

At the same time, the monomorphism $\widetilde{\Psi}_k : \mathcal{G}_k \rightarrow \widetilde{\text{Ham}}(M_a)$ covering Ψ_k is an almost isometric embedding.

Question 10. In [FOOO12] a family of non-displaceable Lagrangian tori was constructed on $S^2 \times S^2$. It is possible to prove that these tori yield the existence of large flats in $\text{Ham}(S^2 \times S^2)$. These flats do not come by stabilization from S^2 . Is it possible to prove that they cannot be at a finite Hofer distance from a flat supported in an arbitrarily small neighborhood of a symplectic divisor of the form $\{pt\} \times S^2$?

3.4 The asymptotic Hofer norm

Let (M, ω) be a closed symplectic manifold. The asymptotic Hofer (pseudo-)norm $\nu(\widetilde{\phi})$ of an element $\widetilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ is defined as follows:

$$\nu(\widetilde{\phi}) = \lim_{m \rightarrow \infty} \frac{1}{m} \widetilde{d}_{\text{Hofer}}(\widetilde{\phi}^m, \text{id}),$$

where $\widetilde{d}_{\text{Hofer}}$ is the Hofer pseudo-norm on $\widetilde{\text{Ham}}(M, \omega)$. For $\phi \in \text{Ham}(M, \omega)$ we define its asymptotic Hofer norm $\nu(\phi)$ similarly, or alternatively $\nu(\phi) = \inf \nu(\widetilde{\phi})$ where the infimum runs over all $\widetilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ covering ϕ .

The Hofer norm is known to yield quantitative invariants of subsets in symplectic manifolds, for instance the displacement energy [Hof90] (see also [Pol01]). It turns out that the asymptotic Hofer norm produces an invariant of subsets of symplectic manifolds controlling their packing number.

Let $A \subset M$ be a compact subset. Its packing number $k(A) \in \mathbb{N} \cup \{\infty\}$ is defined as the maximal k such that there exist $\theta_1, \dots, \theta_k \in \text{Ham}(M, \omega)$ with $\theta_1(A), \dots, \theta_k(A)$ pairwise disjoint ($\theta_i(A) \cap \theta_j(A) = \emptyset$ for all $i \neq j$).

Let $C_0^\infty(M; \mathbb{R})$ denote the space of mean-zero smooth functions on M . For $f \in C_0^\infty(M; \mathbb{R})$, set $\tilde{\phi}_f \in \widetilde{\text{Ham}}(M, \omega)$ for the class of the time-one Hamiltonian isotopy which it generates.

DEFINITION 11. Let $A \subset M$ be a proper compact set. Its α -invariant is

$$\alpha(A) = \inf_{U \supset A} \sup \{ \nu(\tilde{\phi}_f) \mid f \in C_0^\infty(M; \mathbb{R}), |f|_{C^0} = 1, \text{supp}(f) \subset U, f|_A \equiv 1 \},$$

where the infimum runs over all open neighborhoods U of A .

Remark 12. A more elaborate alternative invariant $\alpha'(A)$ would be the infimum over open neighborhoods U of A of constants $a > 0$ such that the map $\iota_U : \ker(\text{Cal}_U) \rightarrow \widetilde{\text{Ham}}(M, \omega)$ is coarse Lipschitz with constant $a > 0$, that is

$$d_{\text{Hofer}}(\iota_U(\tilde{\phi}), \iota_U(\tilde{\psi})) \leq a \cdot d_{\text{Hofer}}(\tilde{\phi}, \tilde{\psi}) + b$$

for all $\tilde{\phi}, \tilde{\psi} \in \ker(\text{Cal}_U)$ and $b \geq 0$ independent of $\tilde{\phi}, \tilde{\psi}$. It is easy to see that $\alpha(A) \leq \alpha'(A)$ and Proposition 13 below applies to $\alpha'(A)$ as well.

In view of *Sikorav's trick* [Sik90, HZ94], which we recall below in detail for the reader's convenience, we obtain the following.

PROPOSITION 13. *The packing number of A satisfies*

$$k(A) \leq \left\lfloor \frac{1}{\alpha(A)} \right\rfloor,$$

where the right-hand side is by convention ∞ if $\alpha(A) = 0$.

Proof of Proposition 13. It is sufficient to prove that $k(A) \leq 1/\alpha(A)$. If $\alpha(A) = 0$, there is nothing to prove. Hence, we suppose that $\alpha(A) > 0$. Then we can prove the equivalent statement that $\alpha(A) \leq 1/k(A)$.

Suppose that M can be packed by l copies of A . Namely, let $\theta_1 = \text{id}, \dots, \theta_l \in \text{Ham}(M, \omega)$ yield an l -packing of A in M : $\{\theta_j(A)\}_{1 \leq j \leq l}$ are pairwise disjoint. Let $U \supset A$ be a sufficiently small open neighborhood of A so that $\{\theta_j(U)\}_{1 \leq j \leq l}$ are pairwise disjoint. Set $U_j = \theta_j(U)$ for $1 \leq j \leq l$. Let $f \in C_0^\infty(M; \mathbb{R})$, $|f|_{C^0} = 1$, $\text{supp}(f) \subset U$, $f|_A \equiv 1$ be as in the definition of $\alpha(A)$. For $1 \leq j \leq l$, $f_j = f \circ \theta_j^{-1}$ satisfies $\text{supp}(f_j) \subset U_j$.

Let us show that $\nu(\tilde{\phi}_f, \text{id}) \leq 1/l$, as calculated in $\widetilde{\text{Ham}}(M, \omega)$, which implies our claim. Indeed, by the conjugation invariance of Hofer's pseudo-metric on $\widetilde{\text{Ham}}(M, \omega)$,

$$d_{\text{Hofer}}(\phi_f^{lt}, \phi_{f_1}^t \cdots \phi_{f_l}^t) \leq C,$$

where $C \leq 2(d_{\text{Hofer}}(\theta_1, \text{id}) + \cdots + d_{\text{Hofer}}(\theta_l, \text{id}))$ is independent of t . However,

$$\phi_{f_1}^t \cdots \phi_{f_l}^t = \phi_F^t,$$

$$F = f_1 + \cdots + f_l.$$

As $|F|_{C^0} = 1$ we obtain that $\nu(\phi_f) = (1/l)\nu(\phi_F) \leq 1/l$, which proves the proposition. \square

Next, we study the asymptotic Hofer norm in the following situation. Let $M = S^2$ with a round area form of total area 1. We write z for the vertical coordinate scaled by a factor of $1/2$. Let us fix $B > C > 0$ with $2B + (k - 1)C = 1$. Consider a Lagrangian configuration $\sqcup_{1 \leq j \leq k} \{z = z_j\}$, where

$$z_j = -1/2 + B + (j - 1)C, \quad j = 1, \dots, k. \tag{6}$$

Denote by $\sigma_{B,C}$ the measure $(1/k) \sum_{j=1}^k \delta_{z_j}$. By using Lagrangian estimators $\mu_{B_i,k}$ as in Theorem G, for rational $B_i \xrightarrow{i \rightarrow \infty} B$, we get that for every smooth function $h = h(z)$ on S^2 with zero mean the asymptotic Hofer norm satisfies

$$\nu(\phi_h) = \lim_{t \rightarrow +\infty} \frac{d_{\text{Hofer}}(\phi_h^t, \text{id})}{t} \geq \int h d\sigma_{B,C}. \tag{7}$$

Let us emphasize that in this definition the flow ϕ_h^t naturally lifts to the universal cover $\widetilde{\text{Ham}}$ and we consider Hofer’s metric there.

Question 14. Is this estimate sharp?

As a test, we fix small $\delta > 0$ and

$$r \in \left(\frac{1}{k+1}, \frac{1}{k} \right),$$

and consider $h_{r,\delta}$ to be a smoothing of the indicator function of

$$[-1/2 + r - \delta, -1/2 + r + \delta],$$

which we arrange to have zero mean by extending it to $[-1/2 + r - \delta, -1/2 + r + 5\delta]$ as an odd function about the midpoint $-1/2 + r + 2\delta$ of the interval. Choose $B = r$, put

$$C = \frac{(1 - 2B)}{(k - 1)}.$$

Note that $h_{r,\delta}$ has $|h_{r,\delta}|_{C^0} = 1$ and equals 1 on the circle $K_r := L_{k,B}^{0,0}$ of area r and is supported in its δ -neighborhood. Since $B > C$, we can apply inequality (7) with the measure $\sigma_{B,C}$ and get

$$u(r) := \liminf_{\delta \rightarrow 0} \nu(\phi_{h_{r,\delta}}) \geq \frac{1}{k}.$$

At the same time, note that there exists a packing of S^2 by k copies of the support of h , so by Proposition 13 $u(r) \leq 1/k$. Thus, $u(r) = 1/k$, so the estimate is sharp.

Presumably, progress in the direction outlined in § 7.1 will yield efficient lower bounds on the asymptotic Hofer norm for more general autonomous Hamiltonians.

3.5 Stabilization

Consider now the *stabilization* of the Hamiltonian diffeomorphism $\phi_{h_{r,\delta}}$ constructed in § 3.4. Fix a closed symplectic manifold (P, Ω) and put

$$\Phi_{r,\delta} := \phi_{h_{r,\delta}} \times \text{id} \in \text{Ham}(S^2 \times P).$$

Put

$$u_P(r) = \lim_{\delta \rightarrow 0} \nu(\Phi_{r,\delta}).$$

Question 15. Is $u_P(r) > 0$?

If the answer is affirmative, it would be interesting to calculate or estimate this quantity. Our method, based on spectral invariants in Lagrangian Floer theory on orbifolds, yields ‘yes’ when (P, Ω) is a 2-sphere of area $2a$ with $B + a > C$. In fact, we have in this case $u_P(r) = 1/k$ as above.

Now we discuss another version of the stabilization. Let $S \subset P$ be a closed Lagrangian submanifold, and let f_W be a smoothing of the characteristic function of a small Weinstein neighborhood W of S . Consider the Hamiltonian $h_{r,\delta}f_W$. We denote by $u_S(r)$ the lower limit of the corresponding asymptotic Hofer norm when $r \rightarrow 0$ and W shrinks to S .

Question 16. Under which assumptions on S does one have $u_S(r) > 0$?

For instance, when P is S^2 of area $2a$ with $B > C + a$ and S is the equator, Hamiltonians $h_{r,\delta}f_W$, are concentrated near $\Lambda_r := K_r \times S$, and the same argument based on Theorem G and Proposition 13 yields

$$u_S(r) = 1/k. \quad (8)$$

3.6 Lagrangian packing via Hofer's geometry

Here we use identity (8) for the asymptotic Hofer norm in order to deduce Theorem C.

Proof of Theorem C. Equation (8) implies that $\alpha(\Lambda_r) \geq 1/k$, whence by Proposition 13, $k(\Lambda_r) \leq k$. \square

Remark 17. Theorem C can be interpreted as follows.² Consider $m = k + 1$ Hamiltonian copies $L_1 = \phi_1(\Lambda_r), \dots, L_m = \phi_m(\Lambda_r)$, where $\phi_j \in \text{Ham}(M_a)$ for all $1 \leq j \leq m$, of Λ_r in M_a . Then the collection $\{L_j\}_{1 \leq j \leq m}$ has the following ‘Borromean’ property: for every proper subset I of $\{1, \dots, m\}$ there exist $\{\theta_j \in \text{Ham}(M_a)\}_{j \in I}$ such that $\{\theta_j(L_j)\}_{j \in I}$ are pairwise disjoint, but for $I = \{1, \dots, m\}$ such a disjoinment does not exist. Of course, this is part of a general phenomenon, which arises for any compact set $A \subset M$ with finite packing number $k(A)$, if we take $m = k(A) + 1$.

Question 18. Is it possible to prove that for a larger than $B - C$ one can pack $k + 1$ Hamiltonian images of Λ or more? The recent methods of Hind and Kerman [HK21] might help produce packings of this kind.

4. Lagrangian Poincaré recurrence: proof

Proof of Theorem D. Denote by K the complete graph with vertices $\mathbb{Z}_{\geq 0}$. Edge-color K as follows: the edge ij is blue if $\phi^i \Lambda \cap \phi^j \Lambda \neq \emptyset$, and it is red otherwise. By Theorem C, this coloring does not possess any red complete subgraph with $k + 1$ vertices. Fix a maximal red complete subgraph, say, Q .

Since Q is maximal, each vertex outside Q is connected to some vertex in Q by a blue edge. Put $N = mk$, and consider the graph B_N with vertices $\{0, \dots, N\}$ connected only by the blue edges. The positive integer m will play the role of the large parameter in the proof. In particular, we assume that the maximal element q of Q is $\leq mk$.

The number of vertices in $B_N \setminus Q$ is at least $(m - 1)k + 1$. Denote by d the maximal degree of a vertex from Q in B_N . Then, by counting outgoing blue edges from Q we get $(m - 1)k + 1 \leq dk$ which yields $d \geq m$. It follows that some vertex $p \in Q$ has at least m blue outgoing edges in B_N .

Note now that the coloring is invariant under positive translations. It follows that 0 has at least $m - q$ outgoing blue edges (we can lose at most q edges as roughly speaking the corresponding vertices will become negative after the shift by $-p$), yielding

$$|\mathcal{R}_\phi \cap [0, mk]| \geq m - q.$$

We conclude the proof by noticing that $(m - q)/mk \rightarrow 1/k$ as $m \rightarrow +\infty$. \square

² We thank Ivan Smith for bringing this interpretation to our attention.

Furthermore, we observe that the same estimates work in general whenever a subset $\Lambda \subset X$ of a set X cannot be $(k + 1)$ -packed into X by powers of a given invertible map $\phi : X \rightarrow X$. For instance, this is the case when (X, ν) is a measure space of total measure 1, Λ is a subset of positive measure, $k = \lfloor 1/\nu(\Lambda) \rfloor$, and ϕ is any invertible measure-preserving transformation. In this setting a stronger statement follows from the Ergodic Theorem [Ber85, Theorem 1.2]: there exists a sequence i_m of density $\geq 1/\nu(\Lambda)$ such that $\nu(\Lambda \cap \phi^{-i_1}\Lambda \cap \dots \cap \phi^{-i_m}\Lambda) > 0$ for all m . However, in our Lagrangian situation the same method is not applicable: indeed our Lagrangian Λ is a measure-zero subset which does not bound a positive-measure subset. For a different example in symplectic topology, we could consider X to be a symplectic manifold, $\phi \in \text{Symp}(X)$ a symplectomorphism, Λ an open ball of a given capacity, and k its packing number. That this sometimes gives sharper bounds than simply the volume constraint is one of the paradigms of modern symplectic topology, initiated in [Gro85, MP94, Bir01].

5. C^0 -continuity and non-simplicity

We observe that the collection of Lagrangian spectral estimators $c_{k,B}^0$ contains sufficient data to provide new C^0 -continuous invariants on $\text{Ham}(S^2)$ that extend to the group of area-preserving homeomorphisms G_{S^2} of the two-sphere. Following the strategy of [CHS20, CHS21] this is shown to yield alternative proofs of the non-simplicity of the group of compactly supported area-preserving homeomorphisms of the two-disk, known as the ‘simplicity conjecture’ [MS17, Problem 42] by way of proving the ‘infinite twist conjecture’ [MS17, Problem 43], as well as that of G_{S^2} . We refer to the above references for the original proofs of these conjectures by using periodic Floer homology, as well as for ample further information about the non-simplicity questions and their historical context. We shall also rely on the following result [CHS21, Lemma 3.11] proven by soft fragmentation methods.

LEMMA 19. *Let $U \subset S^2$ be a disk. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\phi \in \text{Ham}(S^2)$ satisfies $d_{C^0}(\phi, \text{id}) < \delta$ then there exists ψ supported in U such that $d_{\text{Hofer}}(\phi, \psi) < \epsilon$.*

This allows us to prove the following C^0 -continuity result. Consider the C^0 -closure groups $G_{S^2} = \text{Ham}(S^2)^{C^0}$ and $G_{\mathbb{D}^2} = \text{Ham}_c(\mathbb{D}^2)^{C^0}$ inside the corresponding homeomorphism groups. It is a well-known fact that these groups coincide with the groups of orientation and area-preserving homeomorphisms of S^2 and of area-preserving homeomorphisms of \mathbb{D}^2 with compact support (see, e.g., [Mül14] and references therein).

THEOREM L. *The map*

$$\begin{aligned} \tau_{k,k',B,B'} : \text{Ham}(S^2) &\rightarrow \mathbb{R}, \\ \tau_{k,k',B,B'} &= c_{k,B}^0 - c_{k',B'}^0 \end{aligned}$$

is 2-Lipschitz in Hofer’s metric, is C^0 -continuous, and extends to G_{S^2} .

Proof. Let U be the open disk of area $\min\{B, B'\}$ disjoint from

$$\mathbb{L}_{k,k',B,B'}^0 = \mathbb{L}_{k,B}^0 \cup \mathbb{L}_{k',B'}^0.$$

Then by the Calabi property we obtain that $\tau = \tau_{k,k',B,B'}$ satisfies $\tau(\psi) = 0$ for all ψ supported in U . (Indeed, in this case it is easy to find a Hamiltonian H supported in U that generates ψ .) For $\epsilon > 0$ consider $\delta > 0$ provided by Lemma 19. For $\theta \in \text{Ham}(S^2)$ define the C^0 -neighborhood $\mathcal{U}_{\theta,\delta} = \{\theta\phi \mid d_{C^0}(\phi, \text{id}) < \delta\}$. Then for each $\theta\phi \in \mathcal{U}_{\theta,\delta}$

$$|\tau(\theta\phi) - \tau(\theta\psi)| \leq 2\epsilon$$

for ψ supported in U provided by Lemma 19. However, in view of the controlled additivity property and the above vanishing property of τ , we have $\tau(\theta\psi) = \tau(\theta)$. Hence,

$$|\tau(\theta\phi) - \tau(\theta)| \leq 2\epsilon,$$

which proves the C^0 -continuity of τ . Note that it proves more: in fact, τ is uniformly continuous with respect to the uniform structure on $\text{Ham}(S^2)$ given by the neighborhoods $\mathcal{V}_\delta = \{(\theta, \psi) \mid d_{C^0}(\theta^{-1}\psi, \text{id}) < \delta\}$, for $\delta > 0$, of the diagonal in $\text{Ham}(S^2) \times \text{Ham}(S^2)$. Therefore, τ extends to G_{S^2} . \square

It is convenient to consider $\mathbb{L}_{1,1/2}^0 = S \subset S^2$ to be the standard equator. Let $\tau_{k,B} = \tau_{k,1,B,1/2}$, where $c_{1,1/2}^0$ is the Lagrangian spectral invariant of $S \subset S^2$. By Theorem L, $\tau_{k,B}$ extends to $G_{S^2}^F$. We remark that with minor modifications the argument below also works for $\tau_{k,k',B,B'}$ with k', B' fixed.

Proof of Theorem E. Recall that we are given a smooth function $h : [-1/2, 1/2] \rightarrow \mathbb{R}$ vanishing on $[-1/2, 0]$. Consider $\phi \in G_{\mathbb{D}^2} \subset G_{S^2}$ generated by $H = h \circ z$. Assume that a homeomorphism ϕ , generated by the Hamiltonian H , lies in $G_{S^2}^F$, i.e. ϕ is the C^0 -limit of Hamiltonian diffeomorphisms of Hofer's norm $\leq C$. Then $|\tau_{k,B}(\phi)| \leq 2C$ for all k . Observe now that by Lagrangian control and normalization properties, for every rational $B \in (0, 1/2)$,

$$\left| \int_0^{1/2-B} h(s) ds \right| = (1 - 2B) \lim_{k \rightarrow +\infty} |\tau_{k,B}(\phi)| \leq 2C. \quad (9)$$

Hence, h has bounded primitive. \square

Question 20. Formula (9) shows that one can reconstruct the Calabi invariant of rotationally symmetric Hamiltonian diffeomorphisms by using invariants $\tau_{k,B}$. Does such a reconstruction exist for more general symplectomorphisms?

Now we prove the following consequence of Theorem E (compare [LeR10]). Recall first that, as explained in [CHS20, Proposition 2.2] by a well-known argument of Epstein and Higman [Hig54, Eps70], the group $G_{S^2}^F$ contains the commutator subgroup $[G_{S^2}, G_{S^2}]$ of G_{S^2} . Therefore, the quotient group $Q = G_{S^2}/G_{S^2}^F$ is abelian. Hence, every subgroup $H_0 \subset Q$ of Q is normal and its preimage $H = \pi^{-1}(H_0)$ under the natural projection $\pi : G_{S^2} \rightarrow Q$ is a normal subgroup of G_{S^2} containing $G_{S^2}^F$. The results [CHS20, CHS21] on the non-simplicity of G_{S^2} are equivalent to the construction of various explicit embeddings $\mathbb{R} \hookrightarrow Q$. Using our spectral invariants, we provide an explicit embedding of a large function space into Q whose image contains all these copies of \mathbb{R} .

Let $\mathcal{G} \subset C^1((0, 1/2])$ denote C^1 -functions constant near $1/2$, and let \mathcal{G}_b denote bounded such functions.

THEOREM M. *The map $j : \mathcal{G} \rightarrow G_{S^2}$, $\rho \mapsto \phi$, where ϕ is given by $H = h \circ z$ as above where $h(s) = -\rho'(1/2 - s)$ for $s \in (0, 1/2)$ and $h(s) = 0$ for $s \leq 0$, induces a monomorphism*

$$\mathcal{G}/\mathcal{K} \hookrightarrow G_{S^2}/G_{S^2}^F$$

for a subgroup $\mathcal{K} \subset \mathcal{G}_b$. In particular, every subgroup of $\mathcal{G}/\mathcal{G}_b$ yields a normal subgroup of G_{S^2} containing $G_{S^2}^F$.

Proof. From the homomorphism $j : \mathcal{G} \rightarrow G_{S^2}$ we immediately obtain a homomorphism $j_1 = \pi \circ j : \mathcal{G} \rightarrow Q = G_{S^2}/G_{S^2}^F$. It suffices to prove that $\mathcal{K} = \text{Ker}(j_1) \subset \mathcal{G}_b$. Now if $\rho \in \text{Ker}(j_1)$, then $\phi = j(\rho) \in G_{S^2}^F$. By Theorem E this implies that ρ is bounded, that is, $\rho \in \mathcal{G}_b$. \square

Remark 21. It is easy to see by construction that direct analogues of Theorems E and M hold for $G_{\mathbb{D}^2}$ and $G_{\mathbb{D}^2}^F$.

Remark 22. It would be interesting to compare \mathcal{K} and \mathcal{G}_b . See Remark 2.

Remark 23. Finally, it is easy to see that, for instance, Theorem J about the existence of flats in $\text{Ham}(S^2)$ extends naturally to the groups $\text{Hameo}(S^2), \text{Hameo}_c(\mathbb{D}^2)$ from [OM07] with their respective Hofer’s metrics, and possibly further to $G_{S^2}^F, G_{\mathbb{D}^2}^F$.

6. Lagrangian spectral invariants and estimators

Here we construct Lagrangian estimators described in § 2 by using Lagrangian spectral invariants in symmetric products, and prove Theorems F and G. Interestingly enough, a remarkable Toeplitz tridiagonal matrix, the A_k Cartan matrix, naturally appears in the course of our calculation of the critical points of the Landau–Ginzburg superpotential combined from smooth and orbifold terms.

6.1 Lagrangian Floer homology with bounding cochains and bulk deformation

We briefly discuss the general algebraic properties of the Lagrangian Floer homology theory with weak bounding cochains and bulk. We refer to [MS21] for a slightly more detailed discussion, and to the original work [FOOO09, FOOO10, FOOO11, FOOO12, FOOO19] for all detailed definitions. We also remark that the Fukaya algebra of a Bohr–Sommerfeld Lagrangian submanifold in a rational symplectic manifold, that is when $[\omega]$ is contained in the image of $H^2(M; \mathbb{Q})$ inside $H^2(M; \mathbb{R})$, was constructed in [CW22] by classical transversality techniques. Hence, one could feasibly carry out all constructions in this paper by the above techniques, when restricted to the rational setting, which is sufficient for our purposes. We also note that we establish our result as a rather formal consequence of the methods of [FOOO09, FOOO19, MS21], hence it applies with whichever perturbation schemes these papers do.

For a subgroup $\Gamma \subset \mathbb{R}$, define the Novikov field with coefficients in the field $\mathbb{K} = \mathbb{C}$ as

$$\Lambda_\Gamma = \left\{ \sum_j a_j T^{\kappa_j} \mid a_j \in \mathbb{K}, \kappa_j \in \Gamma, \kappa_j \rightarrow +\infty \right\}.$$

This field possesses a non-Archimedean valuation $\nu : \Lambda_\Gamma \rightarrow \mathbb{R} \cup \{+\infty\}$ given by $\nu(0) = +\infty$, and

$$\nu\left(\sum a_j T^{\kappa_j}\right) = \min\{\kappa_j \mid a_j \neq 0\}.$$

For now we may assume that $\Gamma = \mathbb{R}$, but later it will be important to choose a smaller subgroup. We often omit the subscript Γ , and write Λ for Λ_Γ . Set $\Lambda_0 = \nu^{-1}([0, +\infty)) \subset \Lambda$ to be the subring of elements of non-negative valuation, and $\Lambda_+ = \nu^{-1}((0, +\infty)) \subset \Lambda_0$ the ideal of elements of positive valuation.

Given a closed connected oriented spin Lagrangian submanifold $L \subset M$, and an ω -tame almost complex structure on M , considering the moduli spaces of J -holomorphic disks with boundary on L and with boundary and interior punctures, and suitable virtual perturbations required to regularize the problem, as well as suitable homological perturbation techniques, yields the following maps. First, considering only $k + 1$ boundary punctures, for $k \geq 0$, we have the maps

$$m_k : H^*(L; \Lambda)^{\otimes k} \rightarrow H^*(L; \Lambda).$$

These maps satisfy the relations of a curved filtered A_∞ algebra. Furthermore, these maps decompose as $m_k = \sum m_{k,\beta} T^{\omega(\beta)}$, where the sum runs over the relative homology classes

$\beta \in H_2(M, L; \mathbb{Z})$ of disks in M with boundary on L . Moreover, considering also l interior punctures yields maps

$$q_{l,k} : H^*(M; \Lambda)^{\otimes l} \otimes H^*(L; \Lambda)^{\otimes k} \rightarrow H^*(L; \Lambda).$$

Given a ‘bulk’ class $\mathbf{b} \in H_*(M; \Lambda_+)$ we can deform the m_k operations to

$$m_k^{\mathbf{b}}(x_1 \otimes \cdots \otimes x_k) = \sum_{r \geq 0} q_{r,k}(\mathbf{b}^{\otimes r} \otimes x_1 \otimes \cdots \otimes x_k).$$

The operations $m_k^{\mathbf{b}}$ for $k \geq 0$ also satisfy the relations of a curved filtered A_∞ algebra, and decompose into a sum $m_k^{\mathbf{b}} = \sum m_{k,\beta}^{\mathbf{b}} T^{\omega(\beta)}$ as above, over the relative homology classes $\beta \in H_2(M, L; \mathbb{Z})$ of disks in M with boundary on L .

Furthermore, given a ‘cochain’ class $b = b_0 + b_+$ for $b_0 \in H^1(L; \mathbb{C})$ and $b_+ \in H^1(L; \Lambda_+)$, we define the b -twisted A_∞ operations with bulk $\mathbf{b} \in H_*(M; \Lambda_+)$ as follows: first set

$$m_{k,\beta}^{\mathbf{b},b_0} = e^{\langle b_0, \partial\beta \rangle} m_{k,\beta}^{\mathbf{b}}.$$

These operations again satisfy the curved filtered A_∞ equations. Now, to further deform by $b_+ \in H^1(L; \Lambda_+)$, we set

$$m_{k,\beta}^{\mathbf{b},b}(x_1 \otimes \cdots \otimes x_k) = \sum_{l \geq 0} \sum m_{k+l,\beta}^{\mathbf{b},b_0}(b_+ \otimes \cdots \otimes b_+ \otimes x_1 \otimes b_+ \otimes \cdots \otimes b_+ \otimes x_k \otimes b_+ \otimes \cdots \otimes b_+),$$

where the interior sum runs over all possible positions of the k symbols x_1, \dots, x_k appearing in this order, and l symbols b_+ , and finally we set

$$m_k^{\mathbf{b},b} = \sum_{\beta} T^{\omega(\beta)} m_{k,\beta}^{\mathbf{b},b}.$$

The operations $m_k^{\mathbf{b},b}$ also satisfy the curved filtered A_∞ equations. Moreover, if $b_0 - b'_0 \in H^1(L; 2\pi\sqrt{-1}\mathbb{Z})$, then $m_k^{\mathbf{b},b} = m_k^{\mathbf{b},b'}$, $b = b_0 + b_+$, $b' = b'_0 + b_+$.

Given $\mathbf{b} \in H_*(M; \Lambda_+)$, we say that $b = b_0 + b_+ \in H^1(L; \Lambda_0)$ is a *weak bounding cochain* for $\{m_k^{\mathbf{b}}\}$ if there exists a constant $c \in \Lambda_+$ such that

$$\sum_k m_k^{\mathbf{b},b_0}(b_+ \otimes \cdots \otimes b_+) = c \cdot 1_L,$$

where $1_L \in H^0(L; \Lambda_0)$ is the cohomological unit. We call $c = W^{\mathbf{b}}(b)$ the potential function of the weak bounding cochain b . Note that it depends only on the class of b in $H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z})$, which is well-defined as there is no torsion in $H^1(L; \mathbb{Z})$. We shall henceforth consider bounding cochains as elements of $H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z})$.

Now we have the following result regarding Lagrangian tori, which was first proven in the context of Lagrangian torus fibers of toric manifolds in [FOOO10, Theorem 4.10] and [FOOO11, Theorem 3.16].

THEOREM N [FOOO12, Theorem 2.3]. *If $L \cong T^n$ is a Lagrangian torus and all elements of $H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z})$ are weak bounding cochains for $\{m_k^{\mathbf{b}}\}$, then if b is a critical point of the potential function*

$$W^{\mathbf{b}} : H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) \rightarrow \Lambda_+$$

with $H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) \cong (\Lambda_0/2\pi\sqrt{-1}\mathbb{Z})^n$ identified with $\Lambda_0 \setminus \Lambda_+$ by the exponential function, then $m_1^{\mathbf{b},b} = 0$ and hence the (\mathbf{b}, b) -deformed Floer cohomology of L is isomorphic to $H^(L; \Lambda)$. Moreover, this implies that L is non-displaceable by Hamiltonian isotopies in M .*

Finally, we note that $H^*(L; \Lambda) \cong H^*(L; \mathbb{C}) \otimes_{\mathbb{C}} \Lambda$ possesses a non-Archimedean filtration function \mathcal{A} determined by requiring that the basis $E = E_0 \otimes 1$, for $E_0 = (e_1, \dots, e_B)$ a basis of $H^*(L; \mathbb{C})$ be orthonormal in the sense that $\mathcal{A}(e_j \otimes 1) = 0$ for all $1 \leq j \leq B$, $\mathcal{A}(0) = -\infty$, and for all $(\lambda_1, \dots, \lambda_B) \in \Lambda^B$,

$$\mathcal{A}\left(\sum \lambda_j e_j \otimes 1\right) = \max\{\mathcal{A}(e_j \otimes 1) - \nu(\lambda_j)\}.$$

It is important to note that the A_∞ algebra $\{m_k^{\mathbf{b}, b}\}$ from Theorem N has unit $1_L \in H^*(L; \Lambda)$ of non-Archimedean filtration level $\mathcal{A}(1_L) = 0$.

6.2 Lagrangian spectral invariants

In this section we discuss Lagrangian spectral invariants. While essentially going back to Viterbo [Vit92], they were defined in Lagrangian Floer homology by a number of authors in varying degrees of generality, starting with Oh [Oh97], Leclercq [Lec08], and Monzner, Vichery, and Zapolsky [MVZ12]. However, the two main contributions that are relevant to our goals are the paper of Leclercq and Zapolsky [LZ18] in the context of monotone Lagrangians, and of Fukaya, Oh, Ohta, and Ono [FOOO19, Definition 17.15] in the context of Floer homology with bounding cochains and bulk deformation.

6.2.1 *Discrete submonoids and their associated subgroups.* First we formulate the spaces of possible values of our spectral invariants. Following [FOOO19], consider elements $\mathbf{b} \in H_*(M; \Lambda_0)$, $b \in H^1(L; \Lambda_0)$. We say that they are *gapped* if they can be written as

$$\begin{aligned} \mathbf{b} &= \sum_{g \in G(\mathbf{b})} \mathbf{b}_g T^g, & \mathbf{b}_g &\in H_*(M; \mathbb{C}), \\ b &= \sum_{g \in G(b)} b_g T^g, & b_g &\in H^1(L; \mathbb{C}), \end{aligned}$$

where $G(\mathbf{b}), G(b)$ are discrete submonoids of $\mathbb{R}_{\geq 0}$. In practice, all relevant elements \mathbf{b}, b will be gapped, so we assume that they are for the rest of this section.

Given an ω -tame almost complex structure J on M define the submonoid $G(L, \omega, J)$ to be generated by the areas $\omega([u])$ of all J -holomorphic disks u with boundary on L . It is discrete by Gromov compactness. Note that in the orbifold setting below, one includes both areas of smooth holomorphic disks and those of orbifold holomorphic disks.

DEFINITION 24. Let $G(L, \mathbf{b}, b) \subset \mathbb{R}_{\geq 0}$ be the discrete submonoid generated by the union $G(\mathbf{b}) \cup G(b) \cup G(L, \omega, J)$. Furthermore, let $\Gamma(\mathbf{b}), \Gamma(b), \Gamma(L, \omega, J), \Gamma(L, \mathbf{b}, b)$ be the *subgroups* of \mathbb{R} generated by the monoids $G(\mathbf{b}), G(b), G(L, \omega, J), G(L, \mathbf{b}, b)$, respectively. We call (L, \mathbf{b}, b) *rational* if $\Gamma(L, \mathbf{b}, b)$ is a discrete subgroup of \mathbb{R} .

Given a Hamiltonian $H \in C^\infty([0, 1] \times M, \mathbb{R})$ we consider the chords $x : [0, 1] \rightarrow M$ with $x(0), x(1) \in L$, satisfying $\dot{x}(t) = X_H^t(x(t))$ for all $t \in [0, 1]$. Furthermore, we restrict attention to only those chords that are contractible relative to L . Set $\text{Spec}(H, L)$ to be the set of all actions of pairs (x, \bar{x}) each consisting of a chord x and its contraction \bar{x} to L , called a capping, $\bar{x} : \mathbb{D} \rightarrow M, \bar{x}|_{\partial\mathbb{D} \cap \{\Im(z) \geq 0\}} = x, \bar{x}(\partial\mathbb{D} \cap \{\Im(z) \leq 0\}) \subset L$. The action of (x, \bar{x}) is given by

$$\mathcal{A}_{H,L}(x, \bar{x}) = \int_0^1 H(t, x(t)) dt - \int_{\bar{x}} \omega.$$

DEFINITION 25. Define the (\mathbf{b}, b) -deformed spectrum $\text{Spec}(H, L, \mathbf{b}, b)$ of H, L by

$$\text{Spec}(H, L, \mathbf{b}, b) = \text{Spec}(H, L) + \Gamma(L, \mathbf{b}, b).$$

We recall that for two subsets A, B of \mathbb{R} , $A + B$ is the subset of \mathbb{R} given by

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

6.2.2 *Filtered Floer complex and spectral invariants.* Consider a Hamiltonian H and $L \subset M$ a Lagrangian as above. Let (\mathbf{b}, b) be a weak bounding cochain with bulk deformation. Assuming that (H, L) is non-degenerate, that is $\phi_H^1(L)$ intersects L transversely, and $\{J_t\}_{t \in [0,1]}$ a time-dependent ω -compatible almost complex structure on M , following [FOOO09, FOOO19] one constructs a filtered finite rank Λ_0 -complex $CF(H, L, \mathbf{b}, b; \Lambda_0)$, where $\Lambda = \Lambda_\Gamma$ for $\Gamma = \Gamma(L, \mathbf{b}, b)$.

We set $CF(H, L, \mathbf{b}, b) = CF(H, L, \mathbf{b}, b; \Lambda_0) \otimes_{\Lambda_0} \Lambda$. The module $CF(H, L, \mathbf{b}, b)$ comes with an non-Archimedean filtration function $\mathcal{A}_{H,L}$ whose values are contained in

$$\text{Spec}(H, L, \mathbf{b}, b) \cup \{-\infty\}.$$

Indeed, as a Λ -module $CF(H, L, \mathbf{b}, b)$ is given by the completion with respect to the action functional of the vector space generated by pairs (x, \bar{x}) of Hamiltonian H -chords from L to L contractible relative to L , where we identify between $(x, \bar{x}), (x, \bar{x}')$ if $v = \bar{x}' \# \bar{x}^- : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L)$, defined by gluing suitably reparametrized \bar{x}' and $\bar{x}^-(z) = \bar{x}(\bar{z})$ along their common boundary chord, satisfies $\langle [\omega], [v] \rangle = 0$. In this case the filtration \mathcal{A} is defined by declaring any Λ -basis $[(x_i, \bar{x}_i)]$, for $\{x_i\}$ the finite set of contractible H -chords from L to L an orthogonal Λ -basis of $CF(H, L, \mathbf{b}, b)$. Finally, the homology $HF(H, L, \mathbf{b}, b)$ of $CF(H, L, \mathbf{b}, b)$ is naturally isomorphic to self-Floer homology $HF(L, \mathbf{b}, b) = HF((L, \mathbf{b}, b), (L, \mathbf{b}, b))$ of L deformed by (\mathbf{b}, b) . We write $\Phi_H : HF(L, \mathbf{b}, b) \rightarrow HF(H, L, \mathbf{b}, b)$ for this isomorphism. In the setting of Theorem N there is a natural isomorphism between $HF(H, L, \mathbf{b}, b)$ and $H^*(L, \Lambda)$, which, in particular, explains why L is not Hamiltonianly displaceable: if $\phi_H^1(L) \cap L = \emptyset$ then $CF(H, L, \mathbf{b}, b) = 0$ and, hence, one must have $HF(H, L, \mathbf{b}, b) = 0$.

For $a \in \mathbb{R}$ the subspace $CF(H, L, \mathbf{b}, b)^a$ generated over Λ_0 by all $[(x, \bar{x})]$ satisfying $\mathcal{A}_{H,L}(x, \bar{x}) < a$ forms a subcomplex of $CF(H, L, \mathbf{b}, b)$. We denote its homology by $HF(H, L, \mathbf{b}, b)^a$. It comes with a natural map $HF(H, L, \mathbf{b}, b)^a \rightarrow HF(H, L, \mathbf{b}, b)$.

Given a class $z \in HF(L, \mathbf{b}, b)$, we define its spectral invariant with respect to H as in [FOOO19, Definition 17.15] by

$$c(L, \mathbf{b}, b; z, H) = \inf\{a \in \mathbb{R} \mid \Phi_H(z) \in \text{im}(HF(H, L, \mathbf{b}, b)^a \rightarrow HF(H, L, \mathbf{b}, b))\}.$$

In the following, we will primarily work with $z = 1_L$, the unit of the algebra on $HF(L, \mathbf{b}, b)$ induced by the $m_2^{\mathbf{b}, b}$ operation. Following the arguments of [FOOO19, Theorem 7.2], [LZ18, Theorem 35] together with the argument in [PR14, Remark 4.3.2] for rational spectrality, it is straightforward to show the following properties of the spectral invariant $c(L, \mathbf{b}, b; z, H)$.

- (1) *Non-degenerate spectrality:* for each $z \in HF(L, \mathbf{b}, b) \setminus \{0\}$, and Hamiltonian H such that H, L is non-degenerate,

$$c(L, \mathbf{b}, b; z, H) \in \text{Spec}(H, L, \mathbf{b}, b).$$

- (2) *Rational spectrality:* if $\Gamma(H, \mathbf{b}, b)$ is rational, then for $z \in HF(L, \mathbf{b}, b) \setminus \{0\}$, the condition

$$c(L, \mathbf{b}, b; z, H) \in \text{Spec}(H, L, \mathbf{b}, b)$$

holds for each Hamiltonian H .

(3) *Non-Archimedean property:* for each Hamiltonian H , $c(L, \mathbf{b}, b; -, H)$ is a non-Archimedean filtration function on $HF(L, \mathbf{b}, b)$ as a module over the Novikov field Λ with its natural valuation.

(4) *Hofer–Lipschitz:* for each $z \in HF(L, \mathbf{b}, b) \setminus \{0\}$,

$$|c(L, \mathbf{b}, b; z, F) - c(L, \mathbf{b}, b; z, G)| \leq \max\{\mathcal{E}_+(F - G), \mathcal{E}_-(F - G)\},$$

where for a Hamiltonian H , we set $\mathcal{E}_+(H) = \int_0^1 \max_M(H_t) dt$ and $\mathcal{E}_-(H) = \mathcal{E}_+(-H)$. Note that $\mathcal{E}_\pm(H) \leq \int_0^1 \max_M |H_t| dt$.

(5) *Normalization:* For each $H \in C^\infty([0, 1] \times M, \mathbb{R})$ and $b \in C^\infty([0, 1], \mathbb{R})$,

$$c(L, \mathbf{b}, b; z, H + b) = c(L, \mathbf{b}, b; z, H) + \int_0^1 b(t) dt.$$

(6) *Monotonicity:* if two Hamiltonians F, G satisfy $F_t \leq G_t$ for all $t \in [0, 1]$, then $c(L, \mathbf{b}, b; z, F) \leq c(L, \mathbf{b}, b; z, G)$ for each $z \in HF(L, \mathbf{b}, b)$.

(7) *Lagrangian control:* if $\Gamma(H, \mathbf{b}, b)$ is rational and $(H_t)|_L = c(t) \in \mathbb{R}$ for all $t \in [0, 1]$ then setting $c_+(H) = c(L, \mathbf{b}, b; 1_L, H)$ we have

$$c_+(H) = \int_0^1 c(t) dt,$$

hence for all $H \in C^\infty([0, 1] \times M, \mathbb{R})$, $\int_0^1 \min_L H_t dt \leq c_+(H) \leq \int_0^1 \max_L H_t dt$.

(8) *Homotopy invariance:* for H_t of mean-zero for all $t \in [0, 1]$, $c(L, \mathbf{b}, b; z, H)$ depends only on the class $\tilde{\phi}_H \in \widetilde{\text{Ham}}(M, \omega)$ of the Hamiltonian isotopy $\{\phi_H^t\}_{t \in [0, 1]}$.

(9) *Triangle inequality:* for each $z, w \in HF(L, \mathbf{b}, b)$, and Hamiltonians F, G ,

$$c(L, \mathbf{b}, b; m_2^{\mathbf{b}, b}(z, w), F \# G) \leq c(L, \mathbf{b}, b; z, F) + c(L, \mathbf{b}, b; w, G),$$

where $F \# G(t, x) = F(t, x) + G(t, (\phi_F^t)^{-1}x)$ generates the flow $\{\phi_F^t \phi_G^t\}_{t \in [0, 1]}$.

6.3 Orbifold setting

We shall consider only a very simple kind of a symplectic orbifold X . It is called a global quotient symplectic orbifold, and consists of the data of a closed symplectic manifold \widetilde{M} and an effective symplectic action of a finite group G on it.

In fact, we shall only consider $\widetilde{M} = M^k = M \times \dots \times M$ for a symplectic manifold M and the symmetric group $G = \text{Sym}_k$ acting on \widetilde{M} by permutations of the coordinates. The corresponding global quotient orbifold is called the symmetric power $X = \text{Sym}^k(M)$ of M .

The inertia orbifold IX of a global quotient orbifold X is itself a global quotient orbifold given by the action of G on the disjoint union $\bigsqcup_{g \in G} \widetilde{M}^g$, where for $g \in G$, \widetilde{M}^g is the fixed point submanifold of g , and $f \in G$ acts by $\widetilde{M}^g \rightarrow \widetilde{M}^{fgf^{-1}}$, $x \mapsto fx$. See [ALR07] for further details.

6.3.1 *Orbifold Hofer metric.* While on a symplectic orbifold one can define smooth functions and, hence, Hamiltonian diffeomorphisms, in the case of a global quotient orbifold X , the data of a smooth Hamiltonian $H \in C^\infty([0, 1] \times X, \mathbb{R})$ is equivalent to the data of a G -invariant Hamiltonian $\widetilde{H} \in C^\infty([0, 1] \times \widetilde{M}, \mathbb{R})$, that is $H(t, g \cdot x) = H(t, x)$ for all $t \in [0, 1], x \in \widetilde{M}, g \in G$. In our particular case $\widetilde{M} = M^k$ and our Hamiltonians satisfy $H(t, x_1, \dots, x_k) = H(t, x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$ for all $t \in [0, 1], x_1, \dots, x_k \in M$, and $\sigma \in \text{Sym}_k$. Given an orbifold

Hamiltonian diffeomorphism ϕ of X , which is equivalently a G -equivariant Hamiltonian diffeomorphism $\tilde{\phi}$ of \tilde{M} that is generated by a G -invariant Hamiltonian H on \tilde{M} , we define its *orbifold Hofer distance* to the identity by

$$d_{\text{Hofer}}(\phi, \text{id}) = \inf_{\phi_H^1 = \tilde{\phi}} \int_0^1 \max_{\tilde{M}} H(t, -) - \min_{\tilde{M}} H(t, -) dt,$$

where the infimum runs over all such G -invariant Hamiltonians generating $\tilde{\phi}$. Finally, we remark that orbifold Hamiltonian diffeomorphisms form a group $\text{Ham}(X)$, it is isomorphic to the identity component $\text{Ham}(\tilde{M})^G$ of the subgroup of $\text{Ham}(\tilde{M})$ consisting of G -equivariant Hamiltonian diffeomorphisms, and d_{Hofer} extends to a bi-invariant (non-degenerate) metric on $\text{Ham}(X)$. We denote by $\widetilde{\text{Ham}}(X)$ the universal cover of this group with basepoint at the identity. This is also a group.

6.3.2 Orbifold Lagrangian Floer homology with bulk and spectral invariants. It was explained in [CP14] and summarized in [MS21] that the above setup of Lagrangian Floer homology with weak bounding cochain and bulk deformation \mathbf{b}, b holds in the setting of a smooth Lagrangian submanifold L in the regular locus of a closed effective symplectic orbifold X (which for us will be a global quotient orbifold \tilde{M}/G , and whose regular locus is \tilde{M}^0/G where \tilde{M}^0 is the set of points with trivial stabilizer) with the additional feature that we may consider bulk deformations by classes in the homology $H_*(IX; \Lambda_+)$ of the inertia orbifold of X by counting holomorphic disks with possible orbifold singularities at the interior punctures. Furthermore, the above constructions of spectral invariants and arguments proving their properties go through in this situation.

6.3.3 Lagrangians in symmetric products and their spectral invariants. Consider S^2 with the symplectic form ω of total area 1. Under this normalization consider the Lagrangian configuration $L_{2,B}^{0,j}$, $j = 0, 1$, see (1). In [MS21] Mak and Smith show, as translated to our normalizations, that if

$$M = (S^2 \times S^2, \omega \oplus 2a\omega),$$

with $0 < a < 3B - 1 = B - C$, then the Lagrangian link $\mathbb{L}_{2,B} = \mathbb{L}_{2,B}^0 \times S^1$ in M is Hamiltonianly non-displaceable: for all $\phi \in \text{Ham}(X)$, $\phi(\mathbb{L}_{2,B}) \cap \mathbb{L}_{2,B} \neq \emptyset$.

The key observation of this paper is that, in fact, the proof of [MS21] gives strictly stronger information. We recall their approach. The two connected components of $\mathbb{L}_{2,B}$ are

$$L_{2,B}^j = L_{2,B}^{0,j} \times S,$$

where $S \subset S^2(2a)$ is the equator. Consider the product Lagrangian

$$\mathcal{L}'_{2,B} = L_{2,B}^0 \times L_{2,B}^1 \subset \tilde{M} = M \times M.$$

Since $\mathcal{L}'_{2,B}$ is disjoint from the diagonal $\Delta_M \subset M \times M$, it descends to a smooth Lagrangian submanifold $\mathcal{L}_{2,B}$ in the regular locus X^{reg} of the symplectic global quotient orbifold $X = (M \times M)/(\mathbb{Z}/2\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts by exchanging the factors. Similarly, in the situation of configurations for arbitrary values (k, B) of parameters, one defines

$$\mathcal{L}'_{k,B} = L_{k,B}^0 \times \cdots \times L_{k,B}^{k-1} \subset M^k,$$

and it descends to a Lagrangian $\mathcal{L}_{k,B}$ in the regular locus of X^{reg} the symplectic orbifold $X = \text{Sym}^k(M) = M^k/\text{Sym}_k$, where the symmetric group Sym_k acts on M^k by permutations of the coordinates.

In [MS21] the authors have proved for $k = 2$, and we show in § 6.3.5 that the same extends to arbitrary values of parameters (k, B) and $a < ((k + 1)B - 1)/(k - 1)$ the following statement.

THEOREM O. *There exists an integrable almost complex structure J_M on M such that with respect to the complex structure $J = \text{Sym}^k(J_M)$ on $X = \text{Sym}^k(M)$, the Lagrangian $\mathcal{L}_{k,B} \subset X$ has a well-defined Fukaya algebra in the sense of [FOOO09, CP14, FOOO19]. Moreover, this Fukaya algebra admits gapped orbifold bulk deformation \mathbf{b} and weak bounding cochain b with the bulk-deformed Floer homology*

$$HF((\mathcal{L}_{k,B}, \mathbf{b}, b), (\mathcal{L}_{k,B}, \mathbf{b}, b)) \cong HF(\mathcal{L}_{k,B}, \mathbf{b}, b)$$

of $(\mathcal{L}_{k,B}, \mathbf{b}, b)$ with itself well-defined and isomorphic to $H^*(\mathcal{L}_{k,B}; \Lambda)$ with coefficients in the Novikov field $\Lambda = \Lambda_\Gamma$, where $\Gamma = \Gamma(L, \mathbf{b}, b)$. Moreover, for $B \in (1/(k + 1), 1/2)$,

$$a \in (0, ((k + 1)B - 1)/(k - 1))$$

rational, \mathbf{b}, b can be chosen in such a way that Γ is rational. Furthermore, $HF(\mathcal{L}_{k,B}, \mathbf{b}, b)$ is an associative unital algebra, and we denote its unit by $1_{\mathcal{L}_{k,B}}$.

The proof of this theorem appears in § 6.3.5. It implies, in particular, that the unit $1_{\mathcal{L}_{k,B}} \in HF(\mathcal{L}_{k,B}, \mathbf{b}, b)$ does not vanish. This enables us to use spectral invariants associated to this element. For $H \in C^\infty([0, 1] \times X, \mathbb{R})$ we set

$$\sigma_{k,B}(H) = c(\mathcal{L}_{k,B}, \mathbf{b}, b; 1_{\mathcal{L}_{k,B}}, H).$$

Then $\sigma_{k,B}$ satisfies the properties of Lagrangian spectral invariants, including rational spectrality, from § 6.2.

- (1) *Spectrality:* $\Gamma(H, \mathbf{b}, b)$ being rational, the condition

$$\sigma_{k,B}(H) \in \text{Spec}(H, L, \mathbf{b}, b)$$

holds for each Hamiltonian H .

- (2) *Hofer–Lipschitz:*

$$|\sigma_{k,B}(F) - \sigma_{k,B}(G)| \leq \max\{\mathcal{E}_+(F - G), \mathcal{E}_-(F - G)\}.$$

- (3) *Normalization:* For each $H \in C^\infty([0, 1] \times M, \mathbb{R})$ and $b \in C^\infty([0, 1], \mathbb{R})$,

$$\sigma_{k,B}(H + b) = \sigma_{k,B}(H) + \int_0^1 b(t) dt.$$

- (4) *Monotonicity:* if two Hamiltonians F, G satisfy $F_t \leq G_t$ for all $t \in [0, 1]$, then $\sigma_{k,B}(F) \leq \sigma_{k,B}(G)$.

- (5) *Lagrangian control:* $\Gamma(H, \mathbf{b}, b)$ being rational, if $(H_t)|_{\mathcal{L}_{k,B}} = c(t) \in \mathbb{R}$ for all $t \in [0, 1]$, then we have

$$\sigma_{k,B}(H) = \int_0^1 c(t) dt,$$

hence for each Hamiltonian H , $\int_0^1 \min_{\mathcal{L}_{k,B}} H_t dt \leq \sigma_{k,B}(H) \leq \int_0^1 \max_{\mathcal{L}_{k,B}} H_t dt$.

- (6) *Homotopy invariance:* for H_t of mean-zero for all $t \in [0, 1]$, $\sigma_{k,B}(H)$ depends only on the class $\phi_H \in \widetilde{\text{Ham}}(X, \omega)$ of the Hamiltonian isotopy $\{\phi_H^t\}_{t \in [0,1]}$.

- (7) *Triangle inequality:* for each two Hamiltonians F, G ,

$$\sigma_{k,B}(F \# G) \leq \sigma_{k,B}(F) + \sigma_{k,B}(G).$$

6.3.4 *Proof of Theorems F and G.* We now prove Theorems F and G. For a Hamiltonian $F \in C^\infty(M_a, \mathbb{R})$ let $H \in C^\infty(X, \mathbb{R})$ be the Hamiltonian on X determined by the Sym_k -invariant Hamiltonian $\tilde{H} = F \oplus \cdots \oplus F$ in $C^\infty([0, 1] \times M^k, \mathbb{R})$, that is $\tilde{H}(t, x_1, \dots, x_k) = F(t, x_1) + \cdots + F(t, x_k)$. Observe that if F is mean-zero, then so is H . Furthermore, if $F_1 \mapsto H_1$, $F_2 \mapsto H_2$, then $F_1 \# F_2 \mapsto H_1 \# H_2$ and the map $F \mapsto H$ induces a map $\widehat{\text{Ham}}(M_a) \rightarrow \widehat{\text{Ham}}(X)$.

We set

$$c_{k,B}(F) = \frac{1}{k} \sigma_{k,B}(H),$$

$$\mu_{k,B}(F) = \lim_{m \rightarrow \infty} \frac{1}{m} c_{k,B}(F \#^m).$$

The limit exists by Fekete's lemma by the subadditivity property of $\sigma_{k,B}$. The Hofer–Lipschitz property of $c_{k,B}$ and $\mu_{k,B}$ follows from that of $\sigma_{k,B}$ by the subadditivity of Hofer's energy functional. The monotonicity, normalization, Lagrangian control, and independence of Hamiltonian properties are immediate consequences of those for $\sigma_{k,B}$. Note the factor $1/k$ in the definition of $c_{k,B}$: it serves for instance to obtain the Hofer–Lipschitz property with coefficient 1, as $\mathcal{E}_+(H) \leq k \mathcal{E}_+(F)$ for $F \mapsto H$. Similarly, if $F_t|_{L_{k,B}^j} \equiv c_j(t)$, then $H_t|_{\mathcal{L}_{k,B}} \equiv \sum_{0 \leq j < k} c_j(t)$.

Subadditivity for $c_{k,B}$ and commutative subadditivity for $\mu_{k,B}$ are direct consequences of the subadditivity of $\sigma_{k,B}$. Positive homogeneity of $\mu_{k,B}$ is immediate from the definition of $\mu_{k,B}$.

It remains to prove the conjugation invariance of $\mu_{k,B}$, controlled additivity of $c_{k,B}$ and the Calabi properties of $c_{k,B}$ and $\mu_{k,B}$.

To prove conjugation invariance of $\mu_{k,B}$, we note that by the subadditivity of $\sigma_{k,B}$ and, hence, of $c_{k,B}$, we have for all $m \in \mathbb{Z}_{>0}$,

$$c_{k,B}(\phi^m) - q_{k,B}(\psi) \leq c_{k,B}(\psi \phi^m \psi^{-1}) \leq c_{k,B}(\phi^m) + q_{k,B}(\psi), \quad (10)$$

where $q_{k,B}(\psi) = c_{k,B}(\psi) + c_{k,B}(\psi^{-1})$. Hence, dividing by m and taking the limit as $m \rightarrow \infty$, we obtain

$$\mu_{k,B}(\phi) = \mu_{k,B}(\psi \phi \psi^{-1})$$

as required.

Now we prove the Calabi property. The proofs for $c_{k,B}$ and $\mu_{k,B}$ are identical, so we focus on $\mu_{k,B}$ for example. Suppose that $U \subset M_a$ is disjoint from $\mathbb{L}_{k,B}$. Given a Hamiltonian $H \in C^\infty([0, 1] \times M, \mathbb{R})$ supported in $[0, 1] \times U$, consider the Hamiltonian

$$G(t, x) = H(t, x) + b(t) \quad (11)$$

such that for all $t \in [0, 1]$,

$$b(t) = -\frac{1}{\text{vol}(M_a)} \int_U H(t, x) \omega^2.$$

In that case $\mu_{k,B}(H) = 0$. Hence, in view of the normalization property of $\mu_{k,B}$ we obtain

$$\mu_{k,B}(G) = -\int_0^1 b(t) dt = -\frac{1}{\text{vol}(M_a)} \int_0^1 \int_U H(t, x) \omega^2 dt = -\frac{1}{\text{vol}(M_a)} \text{Cal}_U(\{\phi_H^t\}).$$

Finally, to prove controlled additivity it suffices to observe that for H supported in $[0, 1] \times U$ and each Hamiltonian $F \in C^\infty([0, 1] \times M_a, \mathbb{R})$, there is a bijective correspondence between the generators of the Lagrangian Floer complex of $F \# H$ and that of F , and, furthermore,

$$\text{Spec}(\mathcal{L}_{k,B}, \mathbf{b}, b; 1_{\mathcal{L}_{k,b}}, \widetilde{F \# H}) = \text{Spec}(\mathcal{L}_{k,B}, \mathbf{b}, b; 1_{\mathcal{L}_{k,b}}, \widetilde{F}).$$

The proof now proceeds by the spectrality and Hofer–Lipschitz axioms for the family $\widetilde{F\#sH}$ of Hamiltonians where $s \in [0, 1]$, as that of the Lagrangian control property. We obtain that $c_{k,B}(F\#sH)$ is constant in $s \in [0, 1]$ and, hence, $c_{k,B}(F\#H) = c_{k,B}(F)$. The proof is now concluded by the normalization axiom to pass to the mean-zero Hamiltonian G as in (11) instead of H . \square

6.3.5 *Proof of Theorem O.* We explain how the Floer-theoretical approach of Mak-Smith [MS21, Remark 1.10] applies to multiple level sets of the moment map $z : S^2 \rightarrow [-1/2, 1/2]$. This allows us to prove Theorem O.

Consider our Lagrangian configuration $\mathbb{L}_{k,B}^0 \subset S^2$ of k Lagrangians on S^2 . For simplicity let us shorten the notation: $L_{k,B}^{0,i} = L_i, \mathbb{L}_{k,B}^0 = \mathbb{L}, i \in I = \{0, \dots, k - 1\}$.

Now considering the manifolds $L'_i = L_i \times S$ in $M = M_a = S^2 \times S^2(2a)$ for $0 < a < B - C$, where $S \subset S^2(2a)$ is the equator, we obtain a Lagrangian submanifold $\mathcal{L}'_{k,B}$ of $\widetilde{M} = M^k$ and a Lagrangian submanifold $\mathcal{L}_{k,B}$ of $X = \text{Sym}^k(M)$. Recall that X is the global quotient orbifold $X = \widetilde{M}/\text{Sym}_k$ where Sym_k denotes the symmetric group on k elements. Set also $\mathbb{L}' = \sqcup_{i \in I} L'_i$.

We consider the orbifold Lagrangian Floer homology of $\mathcal{L}_{k,B}$ in X with bulk deformation and bounding cochain (\mathbf{b}, b) of a special form. As in [MS21], the bulk deformation will take the form

$$\mathbf{b} = \beta_{\text{orb}}[X_\gamma] + \mathbf{b}_{\text{smooth}}$$

for $\mathbf{b}_{\text{smooth}} \in H_{4k-2}(X; \Lambda_+), \beta_{\text{orb}} \in \Lambda_+, \gamma$ a conjugacy class in $G = \text{Sym}_k$, and $X_\gamma \cong \widetilde{M}^g/C(g)$, for any $g \in \gamma$ and $C(g)$ its centralizer, the connected component of the inertia orbifold IX of X corresponding to γ . Of course $[X_\gamma]$ is the fundamental class of X_γ .

We make the following choice of γ which is important for our particular situation: γ is the conjugacy class of transpositions in $G = \text{Sym}_k$, that is, the permutations of type $(1)^{k-2}(2)$. Note that each element of γ is of order 2. It is also easy to calculate that $X_\gamma \cong M \times \text{Sym}^{k-2} M$.

Recall, following [OS04, MS21] that a map $u : S \rightarrow (X, \mathcal{L}_{k,B})$, for a stable disk S , corresponds ‘tautologically’ to a map $\nu : \Sigma \rightarrow (M, \mathbb{L}_{k,B})$, where $\pi_\Sigma : \Sigma \rightarrow S$ is a degree k branched cover: u is recovered from ν by $u(z) = \nu(\pi_\Sigma^{-1}(z)) \in X = \text{Sym}^k(M)$. Vice versa, ν is obtained as follows: first consider the fiber product of u and the projection $p : M^k \rightarrow X$. This yields a Sym_k -equivariant branched cover $\widetilde{\Sigma} \rightarrow S$ and a Sym_k -equivariant holomorphic map $V : \widetilde{\Sigma} \rightarrow M^k$. Consider $V_1 = \pi_1 \circ V : \widetilde{\Sigma} \rightarrow M$, where $\pi_1 : M^k \rightarrow M$ is the projection to the first coordinate. This map is invariant under the action of the stabilizer $G_1 \cong \text{Sym}_{k-1}$ of $1 \in \{1, \dots, k\}$ under the action of Sym_k . We then define $\pi_\Sigma : \Sigma \rightarrow S$ to be the quotient $\widetilde{\Sigma}/G_1$ by the action of G_1 with the induced projection to S and the induced map $V : \Sigma \rightarrow M$. It is then easy to see that the boundary components correspond. Furthermore, the ω_X -area of u coincides with the ω_M -area of ν . In particular, following the argument of [MS21, Lemma 3.2], one can show the following.

LEMMA 26. *Let $\nu : (\Sigma, \partial\Sigma) \rightarrow (M, \mathbb{L}')$ be obtained by tautological correspondence from a stable disk $u : (S, \partial S) \rightarrow (X, \mathcal{L}_{k,B})$. Then $\partial\Sigma$ has k connected components. If Σ has k connected components, then it consists of k disks. If Σ has $k - 1$ connected components, then it consists of $k - 2$ disks and one curve with two boundary components.*

We remark that of course Σ could have less than $k - 1$ connected components. However, we shall not require knowledge of the topology of Σ in this case. Similarly, the curve with two

boundary components obtained in the second case of Lemma 26 might or might not be an annulus. However, only the annuli will contribute to the lowest order term of the superpotential. For more details about the tautological correspondence we refer to [OS04, MS21].

Following the dimension calculations in [MS21, Section 3.1] we obtain the following analogue of [MS21, Lemma 3.12].

LEMMA 27. *Suppose that for each $b = b_0 + b_+ \in H^1(\mathcal{L}_{k,B}; \Lambda_0)$, each non-constant J_X -holomorphic orbifold stable map u to $(X, \mathcal{L}_{k,B})$ that contributes to the sum*

$$\sum_k m_k^{\mathbf{b}, b_0} (b_+ \otimes \cdots \otimes b_+), \quad (12)$$

tautologically corresponds to a J_M -holomorphic curve $\nu : (\Sigma, \partial\Sigma) \rightarrow (M, \mathbb{L}')$ with non-zero Maslov index $\mu(\nu)$. Then

$$H^1(\mathcal{L}_{k,B}; \Lambda_0) / H^1(\mathcal{L}_{k,B}; 2\pi\sqrt{-1}\mathbb{Z})$$

consists of weak bounding cochains. Moreover, for each such stable map u , the corresponding ν satisfies $\mu(\nu) = 2$.

Dimension counting. For the reader's convenience, we briefly outline dimension counting and thus outline the proof of Lemma 27. It would be convenient to analyze a slightly more general case when we have a pseudo-holomorphic map $\nu : (\Sigma, \partial\Sigma) \rightarrow (N, Q)$, where $Q = Q_1 \sqcup \cdots \sqcup Q_k$ is an n -dimensional Lagrangian submanifold of a symplectic manifold N . As was explained above, this curve together with a k -fold branched cover $\Sigma \rightarrow S$, where S is a disk, corresponds to an orbifold disk $u : (S, \partial S) \rightarrow (X, \mathcal{K})$ in the k -fold symmetric product with the boundary on $\mathcal{K} := (Q_1 \times \cdots \times Q_k) / \text{Sym}_k$. (In our particular application, $N = M_a$, $Q = \mathbb{L}_{k,B}$ and $\mathcal{K} = \mathcal{L}_{k,B}$.)

Here Σ is a Riemann surface with k boundary components obtained as a degree k branched cover of the disk S with ℓ_2 branch points of order 2. Thus, the total space of Σ 's, automorphisms taken into account, has dimension $2\ell_2 - 3$, where we used that $\dim PSL(2, \mathbb{R}) = 3$. Let us emphasize that these ℓ_2 points automatically map by u to our component X_γ of the inertia orbifold, constrained by orbifold bulk of codimension 0.

The space of parameterized holomorphic maps ν of Σ with n -dimensional Lagrangian boundary has dimension $n\chi(\Sigma) + \mu$, where μ stands for the Maslov index of ν .

In addition, S is equipped with r boundary marked points which go to the bounding cochain in \mathcal{K} of codimension 1, and with ℓ_1 interior marked points which go to the smooth bulk of codimension 2 in X . Finally, we have an extra output boundary point, $w \in \partial\Sigma$.

Combining this, we get that the virtual dimension of moduli space \mathcal{M} of such curves u equals

$$\dim \mathcal{M} = n\chi(\Sigma) + \mu + (r + 1 + 2\ell_1) + (2\ell_2 - 3) - r - 2\ell_1,$$

where χ stands for the Euler characteristic. Furthermore, by the Riemann–Hurwitz formula, $\chi(\Sigma) = k - \ell_2$. Finally, $\dim \mathcal{K} = nk$. It follows that

$$\dim \mathcal{M} = \dim \mathcal{K} + (2 - n)(k - \chi(\Sigma)) + \mu - 2.$$

Let us compute the Maslov index of curves ν corresponding to u which contribute to the coefficient of the unit $1_{\mathcal{K}} \in H^*(\mathcal{K}; \Lambda_0)$ in the sum as in (12). We need the evaluation map $u \mapsto u(w)$ from \mathcal{M} to \mathcal{K} to have non-zero degree and, in particular, $\dim \mathcal{M} = \dim \mathcal{K}$. Thus, we get

$$(2 - n)(k - \chi(\Sigma)) + \mu = 2. \quad (13)$$

This finishes the general discussion of the dimension count. Now let us apply it in two special cases relevant to us. In both cases, we use a toric almost complex structure and [MS21,

Lemma 3.24] in order to show that the Maslov index is non-negative for the holomorphic curves in question.

When $N = S^2$, i.e. when we work directly on S^2 without any stabilization, we have $n = 1$ so (13) holds³ for an annulus Σ with Maslov 0. In fact, writing for a general curve u contributing to the sum that $\ell_2 = k - \chi(\Sigma)$, and noting that μ is non-negative and even in our situation, we see that the only two possibilities are $\ell_2 = 0$ and $\mu = 2$, which corresponds to a smooth disk, and $\ell_2 = 2$ and $\mu = 0$, which corresponds to an annulus.

When $N = S^2 \times S^2(2a)$, we have that $n = 2$ and the term involving Σ in formula (13) vanishes, giving $\mu = 2$. In fact, using the more general fact that if u contributes to the sum, we must have $\mu = (2 - n)(k - \chi(\Sigma)) + \mu \leq 2$, and μ being non-negative and even, implies that $\mu = 2$ or $\mu = 0$. If the last option is impossible, we obtain $\mu = 2$.

Furthermore, one can prove following [MS21, Lemma 3.25] that there exists an almost complex structure J_M on M such that the Maslov non-zero condition in Lemma 27 is satisfied. We now outline how this is achieved.

When setting the Lagrangian Floer theory with bulk, and calculating the superpotential, we perform the Fukaya trick to obtain the following situation. From the point of view of complex structures one counts holomorphic curves with boundaries on $\mathbb{K} = \sqcup_i K_i$ with $K_i = R_i \times S_i$ instead of $L'_i = L_i \times S$ where R_i and S_i are small perturbations of L_i and S of the form $z^{-1}(\gamma_i)$ and $z^{-1}(\delta_i)$ for $\delta_i \in \mathbb{R}$ small with $\gamma_i < \gamma_j, \delta_i < \delta_j$ whenever $i < j$. From the point of view of symplectic areas of these holomorphic curves, they are still counted with respect to the L'_i . Indeed, the correspondence between curves with boundary on \mathbb{L}' and those with boundary on \mathbb{K} is carried out by a suitable C^1 -small diffeomorphism of M_a carrying L'_i to K_i for each i .

Identifying $S^2 \times S^2$ with $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^1$ considered as the Riemann sphere $\mathbb{C} \cup \{\infty\}$, let R_i be given as $\{|z| = r_i\}$ and $S_i = \{|w| = \rho_i\}$. We require that the following *non-resonance condition* is satisfied: for all $0 \leq i < j \leq k - 1$, $r_i/r_j < \rho_i/\rho_j$ and there are no non-zero integer vectors $\xi = (a_1, \dots, a_k), \eta = (b_1, \dots, b_k) \in \mathbb{Z}^k$ such that $\sum a_i \log(r_i) = \sum b_i \log(\rho_i)$. In this situation, by following [Ahl78, Chapter 6, Section 5] and [MS21] it is easy to show the following.

LEMMA 28. *The non-constant curves $\nu : \Sigma \rightarrow M$ from Lemma 26 have non-zero Maslov index: $\mu(\nu) \neq 0$. The same holds for the restriction of ν to any connected component of Σ with at least two boundary components.*

Indeed, by positivity of intersections and the contrapositive Maslov 0 assumption, the analysis reduces to there being no simultaneous pair of holomorphic maps (ν_1, ν_2) from a compact Riemann surface C with $2 \leq l \leq k$ boundary components to \mathbb{C} sending the boundary components to the circles of radii whose logarithms are $\vec{r} = (\log(r_{i_1}), \dots, \log(r_{i_l}))$ and $\vec{\rho} = (\log(\rho_{i_1}), \dots, \log(\rho_{i_l}))$. For such a pair of maps, the period matrix P_l of periods of harmonic conjugates of the harmonic measures w_j corresponding to the boundary components of C satisfies

$$P_l \vec{r} = 2\pi\eta \in 2\pi\mathbb{Z}^l, \quad P_l \vec{\rho} = 2\pi\xi \in 2\pi\mathbb{Z}^l.$$

Let us show this for the first projection $\nu_1 : C \rightarrow \mathbb{C}$, as the case of the second one is similar. For such a pair of maps, the pull-back $f : C \rightarrow \mathbb{R}$ of the harmonic function $\log(|z|)$ to C by ν_1 is a harmonic function on C that is constant on the boundary components of C . Therefore, it decomposes as $f = \sum \log(r_j) w_j$. Furthermore, e^f extends to the holomorphic function $\nu_1 : C \rightarrow \mathbb{C}$. Hence, the harmonic conjugate of f must have periods in $2\pi\mathbb{Z}$, which is equivalent to $P_l \vec{r} \in 2\pi\mathbb{Z}^l$.

³ We thank Cheuk Yu Mak for this clarification.

Now note that P_l is symmetric and has a one-dimensional kernel corresponding to constants, which is spanned by $(1, \dots, 1)$. Hence, $\xi, \eta \in \mathbb{Z}^l \setminus \{0\}$, and yet

$$\langle \vec{r}, \xi \rangle = \frac{1}{2\pi} \langle \vec{r}, P_l \vec{\rho} \rangle = \frac{1}{2\pi} \langle P_l \vec{r}, \vec{\rho} \rangle = \langle \eta, \vec{\rho} \rangle,$$

which cannot hold under the non-resonance condition.

As a consequence, the conditions of Lemma 27 are satisfied. What remains is to show that for suitable choices of \mathbf{b} , the superpotential $W^{\mathbf{b}}$ has critical points.

Recall that $I = \{i \in \mathbb{Z} \mid 0 \leq i < k\}$. Consider the coordinates $\{p_i, q_i\}$ on

$$H^1(\mathcal{L}_{k,B}; \Lambda_0) / H^1(\mathcal{L}_{k,B}; 2\pi\sqrt{-1}\mathbb{Z}) \cong (\Lambda_0 \setminus \Lambda_+)^I$$

corresponding to the basis $\{e_i, f_i\}$ of $H^1(\mathcal{L}_{k,B}; \mathbb{C})$ given by the circles L_i, S_i oriented in the direction of the Hamiltonian flow of z on S^2 . More precisely, $p_i = \exp(x_i)$, $q_i = \exp(y_i)$ where $x_i, y_i \in \Lambda_0$ are the coordinates on $H^1(\mathcal{L}_{k,B}; \Lambda_0) = H^1(\mathcal{L}_{k,B}; \mathbb{C}) \otimes \Lambda_0$ corresponding to $\{e_i, f_i\}$.

For $i \in I$ set $n(i)$ to be the number of $j \in I$ such that $j > i$ and $s(i)$ to be the number of $j \in I$ such that $j < i$. Of course, $n(i) + s(i) = k - 1$ for all $i \in I$.

The part of the superpotential corresponding to smooth disks in X with boundary on \mathcal{L} is given by

$$W_{\text{smooth}} = T^a \sum_{i \in I} (q_i + q_i^{-1}) + \sum_{i \in I} (T^{n(i)C+B} p_i + T^{s(i)C+B} p_i^{-1}).$$

We let D_1 denote the divisor in X corresponding to $D_{(1)} \times M^{k-1}$ for the toric divisor $D_{(1)} = z^{-1}(\{\pm 1/2\}) \times S^2(2a) \subset M$.

Consider the bulk deformation $\mathbf{b} = \beta_{\text{orb}}[X_\gamma] + \beta D_1$ for $\beta_{\text{orb}}, \beta \in \Lambda_+ \cup \{0\}$. Then the smooth part of the superpotential $W^{\mathbf{b}}$ is

$$W_{\text{smooth}}^{\mathbf{b}} = T^a \sum_{i \in I} (q_i + q_i^{-1}) + e^\beta \sum_{i \in I} (T^{n(i)C+B} p_i + T^{s(i)C+B} p_i^{-1}).$$

It remains to calculate the leading order term of the orbifold disk part $W_{\text{orb}}^{\mathbf{b}}$ of the superpotential. By analyzing the area of each Maslov index 2 tautological curve $\nu : (\Sigma, \partial\Sigma) \rightarrow (M, \mathbb{K})$, considered as a curve with boundary on \mathbb{L}' , with Σ having $\leq k - 1$ connected components, we see that the area of its homology class $\nu_*[\Sigma, \partial\Sigma] \in H_2(M, \mathbb{L}')$ is:

- (1) at least $C + a$ if its projection to the first factor in $M = S^2 \times S^2(2a)$ does not pass through either pole; and
- (2) at least $B + C$ if its projection to the second factor does not pass through either pole.

Furthermore, in the first case the minimal area $C + a$ is achieved when Σ has $k - 1$ connected components, $k - 2$ disks and one curve with two boundary components, and ν is constant on each disk component. Indeed, if in the first case Σ has at most $k - 2$ connected components, then the area of ν will be at least $2C + a$: its projection to the first factor covers at least two annuli contributing at least $2C$ to the area, and by Lemma 28 its projection to the second factor must pass through a pole, contributing at least a to the area. Now on the curve with two boundary components, ν coincides with the curves analyzed in [MS21, Section 3.3], whereby it contributes to the leading order term of $W_{\text{orb}}^{\mathbf{b}}$ only if the curve is an annulus.

By choosing translation-invariant trivializations of the tangent bundles of K_i suitably,⁴ and selecting $\beta_{\text{orb}} \in \Lambda_+$ so that

$$\frac{\beta_{\text{orb}}^2}{2} T^{C+a} = T^B, \tag{14}$$

$W_{\text{orb}}^{\mathbf{b}}$ is now given by

$$W_{\text{orb}}^{\mathbf{b}} = T^B \sum_{i \in I \setminus \{k-1\}} \varepsilon_i \cdot p_{i+1}^{-1} p_i (q_{i+1} + q_i^{-1}) + o(T^B)$$

for signs $\varepsilon_i \in \{\pm 1\}$, where $o(T^B)$ denotes higher-order terms of valuations strictly greater than B .

The superpotential of interest is

$$W^{\mathbf{b}} = W_{\text{smooth}}^{\mathbf{b}} + W_{\text{orb}}^{\mathbf{b}}.$$

Note that

$$W_{\text{smooth}}^{\mathbf{b}} = T^a \sum_{i \in I} (q_i + q_i^{-1}) + T^B (p_{k-1} + p_0^{-1}) + o(T^B)$$

since $e^\beta = 1 + o(1) \in 1 + \Lambda_+$ and, hence,

$$W^{\mathbf{b}} = T^a \sum_{i \in I} (q_i + q_i^{-1}) + T^B \left(p_{k-1} + p_0^{-1} + \sum_{i \in I \setminus \{k-1\}} \varepsilon_i \cdot p_{i+1}^{-1} p_i (q_{i+1} + q_i^{-1}) \right) + o(T^B).$$

LEMMA 29. For each $\beta \in \Lambda_+ \cup \{0\}$ the superpotential $W^{\mathbf{b}}$ has critical points.

Proof of Lemma 29. We start by finding solutions to the leading order term of the equation $dW^{\mathbf{b}} = 0$.

We first consider derivatives in q_i :

$$\partial_{q_i} W^{\mathbf{b}} = T^a (1 - q_i^{-2}) + o(T^a), \tag{15}$$

where the higher-order terms start with valuation at least B . Hence, the solutions to the leading order of this equation are $q_i \in \{\pm 1\}$. Let us choose $q_i = 1$ for all $0 \leq i < k$.

Now we proceed to consider the derivatives in the p_i , for $0 < i < k - 1$:

$$\partial_{p_i} W^{\mathbf{b}} = 2T^B (-\varepsilon_{i-1} p_i^{-2} p_{i-1} + \varepsilon_i p_{i+1}^{-1}) + o(T^B), \tag{16}$$

for $i = 0$

$$\partial_{p_0} W^{\mathbf{b}} = T^B (2\varepsilon_0 p_1^{-1} - p_0^{-2}) + o(T^B) \tag{17}$$

and for $i = k - 1$

$$\partial_{p_{k-1}} W^{\mathbf{b}} = T^B (-2\varepsilon_{k-2} p_{k-1}^{-2} p_{k-2} + 1) + o(T^B). \tag{18}$$

The solutions to the leading order equations are, hence, $p_i \in \mathbb{C}$ satisfying

$$\begin{aligned} p_{i-1}^{-1} p_i^2 p_{i+1}^{-1} &= \sigma_i, & 0 < i < k - 1, \\ p_0^2 p_1^{-1} &= \sigma_0, & p_{k-2}^{-1} p_{k-1}^2 &= \sigma_{k-1} \end{aligned} \tag{19}$$

for certain numbers $\sigma_i \in \mathbb{R} \setminus \{0\}$. Let us search for solutions $p_i \in \mathbb{C} \setminus \{0\}$ of the form $p_i = \exp(P_i)$, for $P_i \in \mathbb{C}$. Choosing C_i such that $\sigma_i = \exp(C_i)$, we obtain that it is enough to find $P_i \in \mathbb{C}$ such

⁴ We trivialize the tangent bundle of L_i where i is even and of S_i where i is odd along the S^1 -action on S^2 , and that of L_i where i is odd and of S_i where i is even along the inverse S^1 -action. See [Cho04, MS21] for further details on trivializations and orientations.

that for $P = (P_0, \dots, P_{k-1})$, $C = (C_0, \dots, C_{k-1})$

$$AP = C,$$

where

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & & & & & \\ \vdots & & & & & \vdots \\ & & & & & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}$$

is the standard A_k Cartan matrix. Since this is a Toeplitz tridiagonal matrix, its eigenvalues are known to be

$$2 \left(1 + \cos \left(\frac{j\pi}{k+1} \right) \right), \quad 1 \leq j \leq k.$$

In particular, it is invertible. Hence $p_i = \xi_i := \exp(P_i)$ for $P = A^{-1}C$ is a solution to the lowest order term of the critical point equation.

In summary, $q_i = 1$, $p_i = \xi_i$ are solutions to the leading order term of $dW^{\mathbf{b}} = 0$. As in [MS21], following [FOOO11], we proceed to a solution to the full equation $dW^{\mathbf{b}} = 0$, for \mathbf{b} corresponding to β , inductively in the T -adic valuation. Rewrite the equation $dW^{\mathbf{b}} = 0$ in the form

$$\begin{aligned} q_i^2 &= 1 + F_{q,i}, \\ p_{i-1}^{-1} p_i^2 p_{i+1}^{-1} &= \sigma_i + F_{p,i}, \quad 0 < i < k-1, \\ p_0^2 p_1^{-1} &= \sigma_0 + F_{p,0}, \\ p_{k-2}^{-1} p_{k-1}^2 &= \sigma_{k-1} + F_{p,k-1}, \end{aligned}$$

where $F_{p,i}, F_{q,i} = o(1)$ are the higher-order terms. Since $q_i = 1$, $p_i = \xi_i$ are solutions to the zero order term of this equation, we search for solutions to the full equation in the form $q_i = \exp(Q_i)$, $p_i = \xi_i \exp(P_i)$ for Q_i, P_i of positive valuation.⁵ Observe that the matrix $2 \text{id} \oplus A$, where 2id corresponds to the derivative of the left-hand side of the equation in the q_i variables and A corresponds to that in the p_i variables, is invertible. Hence, proceeding order by order in Q_i, P_i we obtain the existence of genuine solutions.

Note that as β is an element in $\Lambda_+ \cup \{0\}$, it is gapped with respect to a discrete submonoid $G_\beta \subset \mathbb{R}_{\geq 0}$.

To make the iterative method work, we look at gapped elements of Λ_0 with exponents in a suitable discrete submonoid $G \subset \mathbb{R}_{\geq 0}$. Specifically, we start with the monoid $G_0 = G(\mathcal{L}_{k,B}, \omega, J)$ generated by areas of (orbifold) holomorphic disks with boundary on $\mathcal{L}_{k,B}$. By means of the tautological correspondence it is contained in the submonoid $G(M, \mathbb{L}_{k,B})$ generated by a, B, C . Now let G_1 be the submonoid generated by G_0 , G_β and $(B - C - a)/2$ coming from (14). Consider the subsets $G_1^a = \{g \in G_1 \mid g > a\}$, $G_1^B = \{g \in G_1 \mid g > B\}$. We have $G_1^a - a, G_1^B - B \subset \mathbb{R}_{>0}$. The gapped submonoid G that we work with is that generated by $G_1^a - a$ and $G_1^B - B$. Indeed the exponents of the coefficients of all $F_{q,i}, F_{p,i}$ will be contained in it: the coefficients of $F_{q,i}$ are contained in $G_1^a - a$ by (15), and the coefficients of $F_{p,i}$ are contained in $G_1^B - B$ by (16), (17),

⁵ Note that for $R = R_0 + R_+ \in \Lambda_0$ of valuation 0, where $R_0 \in \mathbb{C}, R_+ \in \Lambda_+$, $\exp(R) = \exp(R_0) \cdot \exp(R_+)$ by definition.

and (18). Since it is gapped we may enumerate its elements as $g_0 = 0 < g_1 < g_2 < \dots$, and work inductively assuming a solution modulo T^{g_j} and seeking a solution modulo $T^{g_{j+1}}$. \square

Remark 30. In principle, the choice $\beta = 0$ is sufficient for our purposes. However, different choices of β might lead to different spectral invariants. It would be interesting to explore this dependence further.

Proof of Theorem O. It is a direct consequence of Theorem N and Lemma 29. \square

7. Further directions

7.1 Other configurations

Consider a finite collection $\mathcal{C} := \{L_i\}$, $i = 1, \dots, k$ of pairwise disjoint embedded circles in S^2 . Each such collection defines a graph $\Gamma_{\mathcal{C}}$ whose vertices are the connected components of $S^2 \setminus \mathbb{L}$ with $\mathbb{L} = \bigsqcup L_j$, and a pair of components are joined by an edge if they have a common boundary circle. Note that $\Gamma_{\mathcal{C}}$ is a tree. In addition, the tree is vertex-weighted: the weight of a vertex is the area of the corresponding component. We denote the weighting by $w_{\mathcal{C}}$. An elementary inductive argument shows that the pair $(\Gamma_{\mathcal{C}}, w_{\mathcal{C}})$ determines \mathcal{C} up to a Hamiltonian isotopy.

For given collection \mathcal{C} , take $a > 0$, and denote by \mathcal{L} the image of $\prod (L_i \times S_{eq}^1)$ in the k th symmetric product of $S^2 \times S^2(2a)$.

Question 31. For which vertex-weighted tree $(\Gamma_{\mathcal{C}}, w_{\mathcal{C}})$ can one define non-trivial Lagrangian estimators coming from the Lagrangian orbifold Floer homology of \mathcal{L} , with an appropriate choice of $a > 0$?

An obvious necessary condition for non-vanishing of the Floer homology is that for every Hamiltonian diffeomorphism $\phi \in \text{Ham}(S^2)$, the symmetric product of

$$\phi \times \text{id} \in \text{Ham}(S^2 \times S^2(2a))$$

does not displace \mathcal{L} . This is equivalent to the following *matching property*: for every $\phi \in \text{Ham}(S^2)$ there exists a permutation σ such that

$$\phi(L_i) \cap L_{\sigma(i)} \neq \emptyset \quad \forall i \in \{1, \dots, d\}.$$

It would be interesting (and seems to be not totally trivial) to describe this matching property in terms of the vertex-weighted tree $(\Gamma_{\mathcal{C}}, w_{\mathcal{C}})$.

Note that in the present paper we dealt with the case of a linear graph with the weight $w(v) = B$ if the vertex v has degree 1, and $w(v) = C$ if the degree of v is 2. One readily checks that the inequality $B > C$ is equivalent to the matching property.

In addition, it would be interesting to explore the analogue of Question 31 on higher genus surfaces.

It is an intriguing and completely open question whether the methods of the present paper are applicable to Lagrangian configurations on more general symplectic manifolds. Symplectic toric manifolds provide a promising playground, in which case the simplest Lagrangian configuration is provided by (a suitable modification of) the collection of Bohr–Sommerfeld toric fibers.

Finally, we expect that generalizations of the quantitative methods used in this paper to more complex configurations of Lagrangians would yield further results on quantitative symplectic topology, including the Hofer metric and questions of C^0 symplectic topology. We hope to investigate them in a sequel.

7.2 Next destination: asymptotic cone of $\text{Ham}(S^2)$

Flats in $\text{Ham}(S^2)$ described in Theorem A give rise to infinite-dimensional abelian subgroups in the asymptotic cone (in the sense of Gromov [Gro07, 3.29]) of $\text{Ham}(S^2)$. Recall [CZ11] that the latter is a group equipped with a bi-invariant metric which, roughly speaking, reflects the large-scale geometry of $(\text{Ham}(S^2), d_{\text{Hofer}})$. For closed surfaces of genus ≥ 2 , the asymptotic cone of the group of Hamiltonian diffeomorphisms contains a free group with two generators [AKK⁺19, Cho22]. The construction is based on a chaotic dynamical system called *the eggbeater map* (see [PS16] for symplectic aspects of this map). For the torus and the sphere, existence of a free non-abelian subgroup in the asymptotic cone is still unknown. Furthermore, while there is a hope that a suitable modification of eggbeaters could work in the case of torus, in the case of the sphere egg-beaters fail to induce a non-abelian subgroup in the asymptotic cone (an observation of Michael Khanevsky). It would be interesting to explore the algebraic and geometric structure of the asymptotic cone of $\text{Ham}(S^2)$. As a first step, it would be natural to explore asymptotic growth (in the sense of Hofer's metric) of subgroups generated by a finite number of pairwise 'highly non-commuting' flats.

7.3 Comparison with periodic Floer Homology?

Instead of pulling back spectral estimators $c_{k,B}^0$ on S^2 from $S^2 \times S^2(2a)$ (see Theorem I), we could have run our construction omitting the factor $S^2(2a)$. In other words, we could have worked directly with orbifold Lagrangian spectral invariants on the symmetric products on S^2 . In this way, the spectral estimators become well-defined when $B = C = 1/(k + 1)$, which is the limiting case for the assumptions of Theorem I. Interestingly enough, in this limiting case for certain radially symmetric Hamiltonians, our invariants agree with the ones constructed in [CHS21] by means of periodic Floer homology. It would be interesting to find a conceptual explanation of this coincidence.

ACKNOWLEDGEMENTS

We thank Dan Cristofaro-Gardiner, Vincent Humilière, and Sobhan Seyfaddini for very useful communications regarding their work [CHS21]. While our original approach was based on the study of Lagrangian configurations of two circles (which suffices for proving the results on flats in Hofer's geometry), our interest in configurations of $k \geq 3$ circles was in part triggered by their work. We thank Mohammed Abouzaid, Michael Brandenbursky, Cheuk Yu Mak, and Ivan Smith for useful discussions, Vitaly Bergelson for bringing [Ber85, Theorem 1.2] to our attention, and Gerhard Knieper for a stimulating question leading to Corollary 1. LP was partially supported by the Israel Science Foundation grant 1102/20. ES was partially supported by an NSERC Discovery Grant, by the Fonds de recherche du Québec—Nature et technologies, and by the Fondation Courtois.

CONFLICTS OF INTEREST

None.

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Compositio Mathematica is owned by the Foundation Compositio Mathematica and published by the London Mathematical Society in partnership with Cambridge University Press. All surplus income from the publication of *Compositio Mathematica* is returned to mathematics and higher education through the charitable activities of the Foundation, the London Mathematical Society and Cambridge University Press.