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# FACTORIZATIONS OF OUTER FUNCTIONS AND EXTREMAL PROBLEMS

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The author has proved that an outer function in the Hardy space $H^1$ can be factored into a product in which one factor is strongly outer and the other is the sum of two inner functions. In an endeavor to understand better the latter factor, we introduce a class of functions containing sums of inner functions as a special case. Using it, we describe the solutions of extremal problems in the Hardy spaces $H^p$ for $1 \leqq p < \infty$.

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## 1. Introduction

$N$, $N_+$ and $H^p$ for $1 \leqq p < \infty$ denote the Nevanlinna class, the Smirnov class and the Hardy space, respectively on the open unit disc $U$ in the complex plane. A function $h$ in $N_+$ is called outer if it is not divisible in $N_+$ by a non-constant inner function. A function $g$ in $H^1$ is called strongly outer if the only functions $f$ in $H^1$ such that $f/g$ is non-negative are scalar multiples of $g$. If $g$ is not outer and so $g = qh$ for some inner $q$, then $f = (1+q)^2 h$ belongs to $H^1$ and $f/g = (1+q)^2/q$ is non-negative. A norm one function in $H^1$ is outer if and only if it is an extreme point of the unit ball of $H^1$[**2**]. On the other hand, a norm one function in $H^1$ is strongly outer if and only if it is an exposed point of the unit ball of $H^1$ (cf. [**2**, **12**]. Like outer functions, strongly outer functions appear in many important areas, for example, function theory, operator theory and prediction theory.

It is not difficult to give a characterization of a strongly outer function similar to the above definition of an outer function. If $g$ is divisible in $H^1$ by a sum of two inner functions $q_1$, $q_2$ where $q_1 + q_2$ is not constant and $Im\bar{q}_1 q_2 \leqq 0$ almost everywhere, then $f = -i(q_1 - q_2)g/(q_1 + q_2)$ is not a scalar multiple of $g$ and $f/g$ is non-negative because $-i(q_1 - q_2)/(q_1 + q_2) \geqq 0$ almost everywhere. Thus $g$ is not strongly outer. The converse is also true by the following factorization theorem [**12**].

**Theorem.** *If an outer function $h$ in $H^1$ is not strongly outer, then $h = (q_1 + q_2)g$ where both $q_1$ and $q_2$ are inner, $Im\bar{q}_1 q_2 \leqq 0$ almost everywhere, $(q_1 - q_2)^{-1}$ is summable and $g$ is strongly outer. If $q_1$ is a finite Blaschke product of degree $n$ then so is $q_2$.*

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The aim of this paper is to gain a better understanding of this theorem and of the sum of two inner functions. The sum of two inner functions appeared in H. Helson's papers [7] and [8]. D. Sarason [15] examined cases in which the sum of two non-constant inner functions is outer. In this paper, we introduce functions in $H^2$ which have the form; $k=s+q\bar{s}$ where $s$ is in $L^2$ and $q$ is inner. If $s=1$, then $k=1+q$. If $s=q_1$ and $q=q_1q_2$ where $q_1$ and $q_2$ are inner, then $k=q_1+q_2$. If $f$ is the square of $H^2$ function $s+q\bar{s}$, then put $q_1=$the inner part of $f+iq$ and $q_2=$the inner part of $f-iq$. Then $Im\bar{q}_1q_2\leqq 0$, $q_1+q_2$ is non-constant and $f$ is divisible in $H^1$ by $q_1+q_2$. By the remark above the Theorem, $f$ is not strongly outer. The following factorization theorem can be proved easily by a theorem of E. Hayashi ([5, 6]).

**Theorem.** *If an outer function $h$ in $H^1$ is not strongly outer, then $h=(s+q\bar{s})^2g$ where $q$ is a non-constant inner function, $s+q\bar{s}$ is in $H^2$ and $g$ is strongly outer.*

**Proof.** Suppose $h=k^2$ and $k$ is outer in $H^2$. By a theorem of E. Hayashi ([4, 5]),

$$H^2\cap(k/\bar{k})\bar{H}^2=g_0(H^2\ominus zqH^2)$$

and $k/\bar{k}=\bar{q}\bar{g}_0/g_0$ where $q$ is inner and $g_0^2$ is strongly outer. Hence $k=lg_0$ where $l\in H^2\ominus qzH^2$ and $\bar{q}l^2\geqq 0$. Put $s=l/2$, then $l=s+q\bar{s}$ and $h=l^2g_0^2$.

In this theorem, we should like to be able to choose $s+q\bar{s}=q_1+q_2$ for some inner functions $q_1$ and $q_2$. Unfortunately we could not do except in some special cases [12]. Note that by an example of J. Inoue [9], we cannot choose $s+q\bar{s}=1+q$.

## 2. Bad parts of outer functions

In this section we study a function in $H^2$ which has the form $s+q\bar{s}$ where $s$ is in $L^2$ and $q$ is an inner function. If $\prod_{j=1}^n(q_j+q_j')$ where $q_j$ and $q_j'$ are inner functions for $1\leqq j\leqq n$, then $\prod_{j=1}^n(q_j+q_j')=s+q\bar{s}$ for $q=\prod_{j=1}^n q_jq_j'$. Two natural questions are the following: (1) When is $s+q\bar{s}$ an outer function? (2) When can $s+q\bar{s}$ be divisible in $H^2$ by $1+q'$ where $q'$ denotes some nonconstant inner function? The question (1) is related with a paper of D. Sarason [15]. He studied it when $s+q\bar{s}$ is a sum of two inner functions. The question (2) is related with a paper of J. Inoue [9]. By the second theorem in the Introduction, Inoue's result is the following: There exists an outer function $f$ in $H^2$ which is not divisible in $H^2$ by any nonconstant $1+q'$ but is divisible in $H^2$ by some nonconstant $s+q\bar{s}$, where $q$ and $q'$ are inner functions. Because of the first theorem in the Introduction, we are also interested in nonconstant outer function $q_1+q_2$ such that both $q_1$ and $q_2$ are inner functions, $Im\bar{q}_1q_2\leqq 0$ almost everywhere and $(q_1-q_2)^{-1}$ is summable.

**Proposition 1.** *Suppose $s$ is a nonnegative function in $N_+$ and $s^{-1}$ is summable. If $i-s=q_1l$ where $q_1$ is an inner function and $l$ is an outer function, then $q_2=(i+s/i-s)q_1$ is an inner function, $q_1+q_2$ is an outer function, $Im\bar{q}_1q_2\leqq 0$ almost everywhere and*

*$(q_1-q_2)^{-1}$ is summable. If $s$ is a rational function, then both $q_1$ and $q_2$ are finite Blaschke products of the same degree.*

**Proof.** Since $|q_2|=1$ a.e. on $\partial U$ and $q_2=(i+s)/l$, $q_2$ is inner. Since $q_1+q_2=2il$, $q_1+q_2$ is outer. By a simple calculation,

$$\frac{-Im\bar{q}_1 q_2}{|q_1+q_2|^2}=\frac{-i(q_1-q_2)}{q_1+q_2}=s\geqq 0 \quad \text{a.e.}$$

and so $Im\bar{q}_1q_2\leqq 0$ a.e. on $\partial U$. Since $(q_1-q_2)^{-1}=(i-s)/(-2s)$ and $s^{-1}$ is summable, $(q_1-q_2)^{-1}$ is summable. If $s$ is a rational function, by [7] the number of zeros of $s-i$ and that of $s+i$ are equal. Hence $q_1$ and $q_2$ are finite Blaschke products of the same degree.

In Proposition 1, if $s=-z/(1-z)^2$, then $q_1$ and $q_2$ have degree one. However even if $q_1$ and $q_2$ have degree one and $q_1+q_2$ is outer, $Im\bar{q}_1q_2$ is not necessarily non-negative. In fact, suppose $|a|<1$ and $|\rho|=1$. Then, $\rho z+\bar{\rho}(z-a/1-\bar{a}z)$ is outer if and only if $|Re\rho|\leqq|a|$, [15]. However $Im\bar{z}(z-a/1-\bar{a}z)$ is not non-negative on $\partial U$.

**Proposition 2.** *Suppose $s+q\bar{s}$ is in $H^2$, where $q$ is an inner function and $s$ is in $L^2$. Then $s+q\bar{s}$ is an outer function if and only if there exists a function $t$ in $L^2$ such that $s+(t-q\bar{t})$ is an outer function.*

**Proof.** If $l=s+(t-q\bar{t})$ is outer, then $s+q\bar{s}=l+q\bar{l}\in H^2$ and $q\bar{l}\in H^2$. Hence $q\bar{l}=q_0l$ for some inner function $q_0$. Then $s+q\bar{s}=l(1+q(\frac{\bar{l}}{l}))=l(1+q_0)$ and hence $s+q\bar{s}$ is outer. Conversely if $s+q\bar{s}=2l$ is outer, then $q\bar{l}=l$ and hence $s+q\bar{s}=l+q\bar{l}$. Let $k=l-s$, then $k+q\bar{k}=0$ and so $k=t-q\bar{t}$, where $t=k/2$. Thus $l=s+(t-q\bar{t})$ is outer.

**Corollary 1.** *Suppose $s+q\bar{s}$ is in $H^2$, where $q$ is an inner function and $s$ is in $L^2$. If $s$ and $q$ satisfy one of the following (1) ~ (3), then $s+q\bar{s}$ is an outer function.*

*(1) $s$ is an outer function.*
*(2) $q=q_1q_2$ and $s=q_1h$ where $q_1$ and $q_2$ are inner functions, $h$ is an outer function and $q_2\bar{h}=\alpha h$ for some complex number $\alpha$.*
*(3) $q=q_1q_2$ and $s=q_1h$ where $\{q_j\}_{j=1,2,3}$ are inner functions, $h$ is an outer function, $q_2\bar{h}=q_3h$, and $q_1+q_3$ is an outer function.*

**Proof.** (1) is clear by Proposition 2 and (2) is a special case of (3). For (3), let $t=(q_3-q_1)h/4$, then

$$q\bar{t}=\tfrac{1}{4}q(\bar{q}_2h-\bar{q}_1\bar{h})=\tfrac{1}{4}(q_1h-q_2\bar{h})=\tfrac{1}{4}(q_1-q_3)h$$

because $q_2\bar{h}=q_3h$. Hence $t-q\bar{t}=(q_3-q_1)h/4$ and so $s+(t-q\bar{t})=(q_3+q_1)h/2$. This implies (3) because $q_1+q_3$ is outer.

**Proposition 3.** *Suppose $q_1$ is an inner function and $s+q\bar{s}$ is a non-zero function in $H^2$, where $q$ is an inner function and $s$ is in $L^2$. Then $s+q\bar{s}$ is divisible in $H^2$ by $1+q_1$ if and only if there exists a function $t$ in $L^2$ such that $q\{\bar{s}+(\bar{t}-q\bar{t})\}=q_1\{s+(t-q\bar{t})\}$. In particular, if $q\bar{s}=q_1 s$ then $s+q\bar{s}$ is divisible by $1+q_1$.*

**Proof.** If there exists a function $t$ in $L^2$ such that $q\{\bar{s}+(\bar{t}-\bar{q}t)\}=q_1\{s+(t-q\bar{t})\}$, then $s+q\bar{s}=s+t-q\bar{t}+q(\bar{s}+\bar{t}-\bar{q}t)=s+t-q\bar{t}+q_1(s+t-q\bar{t})=(s+t-q\bar{t})(1+q_1)$ and hence $s+q\bar{s}$ is divisible in $H^2$ by $1+q_1$. Conversely if $l=(s+q\bar{s})/(1+q_1)$ is in $H^2$, then

$$\bar{q}=\frac{\bar{s}+\bar{q}s}{s+q\bar{s}}=\frac{\bar{l}}{l}\frac{1+\bar{q}_1}{1+q_1}=\frac{\bar{l}}{l}\bar{q}_1$$

and hence $q\bar{l}=q_1 l$. If $k=l-s$, then $k=t-q\bar{t}$ for some $t\in L^2$ and hence $l=s+(t-q\bar{t})$. This implies that $q\{\bar{s}+(\bar{t}-\bar{q}t)\}=q_1\{s+(t-q\bar{t})\}$.

**Corollary 2.** *Suppose $s+q\bar{s}$ is in $H^2$, where $q$ is an inner function and $s$ is in $L^2$.*

(1) *If $s$ is an outer function and $q\bar{s}\neq\alpha s$ for any $\alpha$ in $C$ with $|\alpha|=1$, then there exists a non-constant inner function $q_1$ such that $s+q\bar{s}$ is divisible in $H^2$ by $1+q_1$.*
(2) *If $h$ is an outer function, $q\bar{h}=q_1q_2^2h$ and $s=q_2h$ where $q_1$ and $q_2$ are inner functions, then $s+q\bar{s}$ is divisible in $H^2$ by $1+q_1$.*
(3) *If $q$ is a finite Blaschke product, then there exists a non-constant finite Blaschke product $q_1$ such that $s+q\bar{s}$ is divisible in $H^2$ by $1+q_1$, or $s+q\bar{s}$ is not an outer function.*

**Proof.** (1) Since $s$ is outer, $q\bar{s}=q_1 s$ for some inner function $q_1$. By the hypothesis, $q_1$ is non-constant and hence Proposition 3 implies (1).
(2) $q\bar{s}=q_1q_2h=q_1 s$ implies (2) by Proposition 3.
(3) Since $\bar{q}(s+q\bar{s})^2\geqq 0$ a.e. on $\partial U$ and $q$ is a finite Blaschke product, $(s+q\bar{s})^2=\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_jz)l^2$, where $|a_j|\leqq 1(1\leqq j\leqq n)$ and $l$ is outer in $H^2$(**[2, 11]**). Therefore if $s+q\bar{s}$ is outer, then $s+q\bar{s}=\prod_{j=1}^{n}(-\bar{a}_j)^{1/2}(z-a_j)l$ and $|a_j|=1$. Thus $s+q\bar{s}$ is divisible in $H^2$ by $z-a_j$.

When $q_1$ and $q_2$ are inner functions, we write $q_1\prec q_2$ if there exists a nonzero function $f$ in $H^1$ such that $\bar{q}_1q_2=f/|f|$. If both $q_1$ and $q_2$ are finite Blaschke product, then $q_1\prec q_2$ is equivalent to (degree of $q_1$)$\leqq$(degree of $q_2$). For each $g$ in $H^1$, *sing* $g$ denotes the set of the unit circle on which $g$ cannot be analytically extended.

**Proposition 4.** *If $q_1$ and $q_2$ are inner functions and the inner part of $q_1+q_2$ is $q$, then $q\prec q_1$ and $q\prec q_2$.*

**Proof.** Let $q_1+q_2=qh$, then $|\bar{q}q_1-h|=|\bar{q}q_2-h|=1$. By a theorem of P. Koosis (cf. **[4**, Chapter 4, Lemma 5.4**]**), $q\prec q_1$ and $q\prec q_2$.

**Corollary 3.** *Suppose $q_1$ and $q_2$ are inner functions and $q_1+q_2=qh$ where $q$ is an inner function and $h$ is an outer function.*
*(1) If $q_1$ is a finite Blaschke product, then $q$ is also a finite Blaschke product and (degree of $q$)$\leqq$(degree of $q_1$).*
*(2) If (sing $q_1$)$\cap$(sing $q_2$) is empty, then $q$ is a finite Blaschke product.*
*(3) Suppose $q_1=\exp(-(a+z)/(a-z))$ and $q_2=-\alpha\exp(-(b+z)/(b-z))$, where $|a|=|b|=1$, $b=-\bar{a}$ and $|\alpha|=1$. If $\alpha=1$, then $q=z$ or $q$ is constant. If $\alpha\neq 1$, then $q$ is always constant, that is, $q_1+q_2$ is an outer function.*

**Proof.** (1) By Proposition 4, $\bar{q}q_1=f/|f|$ for some function $f\in H^1$ and hence $\bar{q}_1(qf)\geqq 0$ a.e. on $\partial U$. If $q_1$ is a finite Blaschke product of degree $m$, $qf=\prod_{j=1}^{n}(z-a_j)(1-\bar{a}_j z)l$ and $n\leqq m$ where $|a_j|\leqq 1(1\leqq j\leqq n)$ and $l$ is strongly outer. Hence $q$ is a finite Blaschke product of degree $k$ and $k\leqq n$.
(2) By Proposition 4, $\bar{q}q_1=f/|f|$ for some function $f\in H^1$ and hence $\bar{q}q_1=g/\bar{g}$ for some outer function $g\in H^2$. Therefore $\bar{q}_1 qg=\bar{g}$ and so $qg\in H^2\ominus q_1 zH^2$. Hence *sing* $q_1\supseteq$ *sing* $qg$ and by [**10**, Lemma 4], *sing* $q_1\supseteq$ *sing* $q$. Similarly *sing* $q_2\supseteq$ *sing* $q$ and by the hypothesis $q$ is a finite Blaschke product.
(3) By (2), $q$ is a finite Blaschke product. If $q(x)=0$ for some point $x\in U$, then $\exp(-(a+x)/(a-x))=\alpha\exp(-(b+x)/(b-x))$ and hence

$$-\frac{a+x}{a-x}=-\frac{b+x}{b-x}+i\rho \text{ and } \rho=t+2n\pi$$

where $n$ is some integer and $\alpha=e^{it}$. If $\rho=0$ then $q=z$ because $a\neq b$. Suppose $\rho\neq 0$. Then

$$x^2-\left\{\left(1-\frac{2i}{\rho}\right)b+\left(1+\frac{2i}{\rho}\right)a\right\}x+ab=0.$$

If $A$ and $B$ are the solutions of the above quadratic equation, then $AB=ab=-1$ and

$$A+B=\left(1+\frac{2i}{\rho}\right)a-\overline{\left(1+\frac{2i}{\rho}\right)a}.$$

This implies $|A|=|B|=1$ and contradicts $|x|<1$.

(1) of Corollary 3 was proved by D. Sarason [**15**, Proposition 3]. Our proof is different from his.

### 3. Projection

For each inner function $q$, we define two operators on $L^2$

$$L_q(s)=\frac{s+q\bar{s}}{2} \quad \text{and} \quad L'_q(s)=\frac{s-q\bar{s}}{2}.$$

*If* $q=1$, then $L_q(s)$ is the real part of $s$ and $L'_q(s)$ is the imaginary part of $s$. In general, $|L_q(s)|\leqq|s|$ and $|L'_q(s)|\leqq|s|$. Hence $L_q$ and $L'_q$ are contractive. $L_q$ and $L'_q$ commute with multiplication operators by real valued functions in $L^\infty$. Moreover on $L^2$, we have $L_qL_q=L_q$ and $L_qL'_q=0$ and $L_q+L'_q$ is the identity operator. By results of the last section, we are interested in a function $s$ such that $L_q(s)$ belongs to $H^2$. Since $q=(1+q)^2/|1+q|^2$, we define $q^{1/2}=(1+q)/|1+q|$. Put

$$6\mathscr{A}_q=\left\{g\in H^2\colon \frac{g}{1+q} \text{ is a real valued function}\right\}.$$

**Theorem 5.** *Let $q$ be a non-constant inner function. Then*

$$\{s\in L^2\colon L_q(s)\in H^2\}=\mathscr{A}_q+iq^{1/2}L^2_{\mathbf{R}},$$

*where $L^2_{\mathbf{R}}=\{s\in L^2\colon s$ is a real valued function$\}$. In particular, if $s+q\bar{s}$ belongs to $H^2$ for some $s$ in $L^2$, then $s+q\bar{s}=t+q\bar{t}$ for some $t$ in $H^2$.*

**Proof.** If $g\in\mathscr{A}_q$ then $u=g/(1+q)$ is real and $g=u(1+q)$. Hence $q\bar{g}=g$ and so $L_q(g)=g\in H^2$. If $s=iq^{1/2}u$ and $u\in L^2_{\mathbf{R}}$ then $L_q(s)=0$. This implies that $\{s\in L^2\colon L_q(s)\in H^2\}$ $\supseteq\mathscr{A}_q+iq^{1/2}L^2_{\mathbf{R}}$. Conversely, suppose $g=L_q(s)\in H^2$. If $g=0$, then $s=-q\bar{s}$ and $s^2=-q|s|^2$. Hence $(i\bar{q}^{1/2}s)^2=-\bar{q}s^2=|s|^2\geqq0$ and so $i\bar{q}^{1/2}s=-u$ is real. Thus $s=iq^{1/2}u$ and $u\in L^2_{\mathbf{R}}$. If $g\neq0$, $s+q\bar{s}=2g$ and

$$\frac{s}{g}+\overline{\left(\frac{s}{g}\right)}=2,$$

Put $t=s/g-1$, then $t+\bar{t}=0$ and so $t=iv$ for some $v\in L^2_{\mathbf{R}}$. Hence $s=g+ivg$ and $vg=q^{1/2}u$, where $u=v\bar{q}^{1/2}g$ is in $L^2_{\mathbf{R}}$. Thus $s=g+iq^{1/2}u$. This completes the proof of the theorem.

**Corollary 4.** *Let $q$ be a non-constant inner function. Then*

$$\{s\in H^2\colon L_q(s)\in H^2\}=\mathscr{A}_q+i\mathscr{A}_q$$

*and hence $H^2\ominus qzH^2=\mathscr{A}_q+i\mathscr{A}_q$. $L_q$ is the projection from $H^2\ominus qzH^2$ onto $\mathscr{A}_q$ and has kernel $i\mathscr{A}_q$.*

**Proof.** If $g\in\mathscr{A}_q$, then $g=v(1+q)$ for some real valued function $v$ and so $g=q^{1/2}u$

where $u=v|1+q|$. Hence $\mathscr{A}_q\subset q^{1/2}L_R^2$ and $(q^{1/2}L_R^2)\cap H^2=\mathscr{A}_q$. Now Theorem 5 implies the corollary.

The proof of Theorem 5 is related to that of [**14**, Theorem 3]. The equality in Corollary 4, that is, $H^2\ominus qzH^2=\mathscr{A}_q+i\mathscr{A}_q$ is known by [**12**, (1) of Theorem 3].

**Corollary 5.** *Let $q$ be an inner function.*
(1) *If $q=z^n$, then $\mathscr{A}_q=\{\sum_{j=0}^n b_jz^j\colon b_j=\bar{b}_{n-j}\}$.*
(2) *If $q=\prod_{l=1}^\infty(-\bar{a}_l/|a_l|)(z-a_l/1-\bar{a}_lz)$ and $\sum_{l=1}^\infty(1-|a_l|)<\infty$, then*

$$\mathscr{A}_q=\left\{\sum_{j=0}^\infty \frac{c_jB_j+\bar{c}_jzB_j'}{1-\bar{a}_jz}\colon \sum_{j=0}^\infty \frac{|c_j|^2}{(1-|a_j|)^2(1+|a_j|)}<\infty\right\}$$

*where* $B_j=\prod_{l=1}^{j-1}(-\bar{a}_j/|a_j|)(z-a_j/1-\bar{a}_jz)$, $B_j'=\prod_{l=j}^\infty(-\bar{a}_j/|a_j|)(z-a_j/1-\bar{a}_jz)$, $a_0=0$, $B_0=1$ *and* $B_0'=q$.

**Proof.** (1) If $s\in H^2\ominus qzH^2$, then $s=\sum_{j=0}^n a_jz^j$ and hence $s+q\bar{s}=\sum_{j=0}^n(a_j+\bar{a}_{n-j})z^j$. Now corollary 4 implies (1).
(2) If $s\in H^2\ominus qzH^2$, then by [**1**]

$$s=\sum_{j=0}^\infty c_j(1+|a_j|)^{1/2}B_j(1-\bar{a}_jz)^{-1}(1-|a_j|)$$

and $\sum_{j=0}^\infty |c_j|^2<\infty$. Hence

$$s+q\bar{s}=\sum_{j=0}^\infty\left(c_j\frac{B_j}{1-\bar{a}_jz}+\bar{c}_j\frac{q\bar{B}_j}{1-\bar{a}_jz}\right)(1+|a_j|)^{1/2}(1-|a_j|)$$

$$=\sum_{j=0}^\infty\left(\frac{c_jB_j+\bar{c}_jzB_{j+1}'}{1-\bar{a}_jz}\right)(1+|a_j|)^{1/2}(1-|a_j|).$$

Now Corollary 4 implies (2).

A theorem of P. R. Ahern and D. N. Clark [**1**, Theorem 3.1], lets one describe $\mathscr{A}_q$ for arbitrary inner function $q$.

## 4. Extremal problems

Let $1\leqq q\leqq\infty$ and $1/p+1/l=1$. If $\phi\in L^l$, we denote by $T_\phi^p$ the continuous functional defined on the Hardy space $H^p$ by

$$T_\phi^p(f)=\int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})d\theta/2\pi.$$

A function $f$ in $H^p$, which satisfies $T_\phi^p(f)=\|T_\phi^p\|$ and $\|f\|_p\leqq 1$, is called an extremal function. A function $\phi$ in $L^l$ is called an extremal kernel when $\|\phi\|_l=\|T_\phi^p\|$. The existence and uniqueness of extremal functions and extremal kernels is known for $1<p\leqq\infty$ (cf. [3, Theorem 8.1]). For $p=1$, the situation is very different. An extremal function may not exist, the dual extremal kernel always exists and is unique if an extremal function exists (cf. [3, Theorem 8.1]). For $p=1$, the set $S_\phi$ of all extremal functions is defined by

$$S_\phi=\{f\in H^1:\ T_\phi^1(f)=\|T_\phi^1\| \text{ and } \|f\|_1=1\}.$$

$S_\phi$ has been described in general by E. Hayashi [5, 6]. In this section, we describe $S_\phi$ completely in ways different from that of E. Hayashi. Moreover using the result we describe extremal kernels and extremal functions for $1<p<\infty$.

**Theorem 6.** *Suppose $p=1$ and $S_\phi$ is nonempty. Then there exist an inner function $q$ and a strong outer function $g$ which satisfy the following (1)~(4).*
*(1) The unique extremal kernel of $T_\phi^1$ is $\bar q|g|/g$.*
*(2) $f$ is a member of $S_\phi$ if and only if*

$$f=\gamma q_0\left(\frac{s+q\bar s}{1+q_0}\right)^2 g,$$

*where $\gamma$ is a positive constant, $\|f\|_1=1$, $q_0$ is an inner function, $s$ is in $H^2\ominus qzH^2$ and $(s+q\bar s)/(1+q_0)$ is an outer function in $H^2$.*
*(3) $f$ is a member of $S_\phi$ if and only if*

$$f=\gamma q_0(t+q\overline{q_0 t})^2 g,$$

*where $\gamma$ is a positive constant, $\|f\|_1=1$, $q_0$ is an inner function, $t$ is in $H^2\ominus qzH^2$ and $t+q\overline{q_0t}$ is an outer function in $H^2\ominus qzH^2$.*
*(4) $f$ is a member of $S_\phi$ if and only if*

$$f=\gamma\{(s+q\bar s)^2+(t+q\bar t)^2\}g,$$

*where $\gamma$ is a positive constant, $\|f\|_1=1$, and $s$ and $t$ are in $H^2\ominus qzH^2$.*

**Proof.** (1) is known from [5].
(2) If $f=\gamma q_0(s+q\bar s/1+q_0)^2 g$, then

$$\frac{|f|}{f}=q_0\frac{|1+q_0|^2}{(1+q_0)^2}\frac{|s+q\bar{s}|^2}{(s+q\bar{s})^2}\frac{|g|}{g}=\bar{q}\frac{|g|}{g},$$

and hence $f\in S_\phi$. Conversely, if $f\in S_\phi$ and $f=q_0h^2$, where $q_0$ is inner and $h$ is outer, then $\gamma_1(1+q_0)^2h^2\in S_\phi$ for some positive constant $\gamma_1$. Since $(1+q_0)h$ is outer in $H^2$, by a theorem of E. Hayashi ([**5**, **6**]),

$$H^2\cap q_0(h/\bar{h})\bar{H}^2=q_0(H^2\ominus qzH^2)$$

and $q_0(h/\bar{h})=\bar{q}\bar{q}_0/q_0$, where $q$ is inner and $g=g_0^2$, is strongly outer. Hence $(1+q_0)h=kg_0$ where $k\in H^2\ominus qzH^2$ and $\bar{q}k^2\geqq 0$. Since $k\in\mathscr{A}_q$, by Corollary 4, $k=s+q\bar{s}$ for some function $s\in H^2\ominus qzH^2$. Now $q_0h$ belongs to $g_0(H^2\ominus qzH^2)$ because $q_0h=q_0(k/\bar{h})\bar{h}$. Therefore $q_0h/g_0$ belongs to $H^2\ominus qzH^2$ and hence $h/g_0=(s+q\bar{s})/(1+q_0)$ belongs to $N_+\cap L^2=H^2$. This implies (2).
(3) Put $(s+q\bar{s})/(1+q_0)=l$ in (2); then

$$\frac{\bar{l}}{l}=\frac{\bar{s}+\bar{q}s}{1+\bar{q}_0}\frac{1+q_0}{s+q\bar{s}}=q_0\bar{q}.$$

Hence $l=q\overline{q_0 l}$ and so $l=t+q\overline{q_0 t}$, where $t=l/2\in H^2$. This implies (3).
(4) By (2), the 'if' part is clear. Conversely if $f\in S_\phi$, then by (2) $f=q_0k^2g$, where $k=\gamma^{1/2}(s+q\bar{s})/(1+q_0)$. Since $\bar{q}q_0k^2=|k|^2$, $q\bar{k}=q_0k$ and hence $k\in H^2\ominus qzH^2$. By Corollary 4, $k=l+im$ for some functions $l$, $m\in\mathscr{A}_q$ and hence $q_1k=l-im$ for some inner function $q_1$. Thus $q_1k^2=l^2+m^2$ and hence $\bar{q}q_1k^2=|k|^2$. Therefore $q_1=q_0$. Corollary 4 implies (4) because $f=\gamma\{l^2+m^2\}g$.

If $(s+q\bar{s})/(1+q_0)$ belongs to $H^2$, then $q_0\prec q$ and $(s+q\bar{s})/(1+q_0)$ belongs to $H^2\ominus qzH^2$. In fact, if $l=(s+q\bar{s})/(1+q_0)$, then by the proof of (3) of Theorem 6, $q\bar{l}=q_0l$. Hence $l$ belongs to $H^2\ominus qzH^2$ and $q_0\prec q$ because $\bar{q}_0q=l^2/|l|^2$. Theorem 7 and Theorem 1 in [**13**] describe extremal kernels and extremal functions in case $1<p<\infty$.

**Theorem 7.** *Suppose $1<p<\infty$ and $1/p+1/l=1$. Then $\phi$ is the unique extremal kernel and $f$ is the unique extremal function of $T_\phi^p$ if and only if there exist an inner function $q$ and a strong outer function $g$ which satisfy the following:*

$$\phi=\|T_\phi^p\|\bar{q}\frac{|g|}{g}\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/l}g^{1/l}$$

*and*

$$f=q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/p}g^{1/p},$$

*where $q_0$ is an inner function, $\|f\|_p=1$, $\|\phi\|_l=\|T_\phi^p\|$, $s\in H^2\ominus qzH^2$ and $(s+q\bar{s})/(1+q_0)$ is an outer function in $H^2$.*

**Proof.** If $\phi$ is the unique extremal kernel and $f$ is the unique extremal function of $T^p_\phi$, then by [**13**, Theorem 1]

$$\phi=\phi_0 h,\ f=\|T^l_\phi\|^{-l/p}Qh^{l/p}$$

and

$$\|T^l_\phi\|^{-l}Qh^l\in S_{\phi_0},\ \phi_0=\bar{Q}|h|^l h^{-l},$$

where $h$ is outer with $|\phi|=|h|$ and $Q$ is the inner part of $f$. By Theorem 6,

$$\|T^l_\phi\|^{-l}Qh^l=q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^2 g,$$

where $q$ and $q_0$ are inner, $g$ is strongly outer, $\|q_o(s+q\bar{s}/1+q_0)^2 g\|_1=1$, $s\in H^2\ominus qzH^2$ and $(s+q\bar{s})/(1+q_0)$ is outer in $H^2$. Hence $Q=q_0$, $h=\|T^l_\phi\|\,(s+q\bar{s}/1+q_0)^{2/l}g^{1/l}$ and $\phi_0=\bar{q}_0(|h|^l/h^l)=\bar{q}(|g|/g)$.
Thus

$$\phi=\bar{q}\frac{|g|}{g}\|T^l_\phi\|\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/l}g^{1/l}$$

and

$$f=\|T^l_\phi\|^{-l/p}q_0h^{l/p}=q_0\left(\frac{s+q\bar{s}}{1+q_0}\right)^{2/p}g^{1/p}.$$

Theorem 6 is a generalization of [**11**, Theorem 2]. Theorem 7 is a generalization of [**13**, Theorem 2]. But the descriptions are different from the previous ones. In those descriptions, the bad part $q_0(s+q\bar{s}/1+q_0)^{2/l}$ is important. If $f$ is an inner function, then it is clear that $\|f+\bar{z}\bar{H}^l\|=\|f\|_l$ for $1\leqq l\leqq\infty$. If $f=q_0(s+q\bar{s}/1+q_0)^{2/l}$, then, by Theorem 8, $\|f+\bar{z}\bar{H}^l\|=\|f\|_l$ for $1\leqq l\leqq\infty$. Theorem 8 also shows [**13**, Corollary 3]. To prove Theorem 8 we need the following lemma.

**Lemma.** *Suppose* $1\leqq l\leqq\infty$ *and* $f=qh$ *is in* $H^l$, *where* $q$ *is an inner function and* $h$ *is an outer function. Then* $\|f+\bar{z}\bar{H}^l\|=\|f\|_l$ *if and only if* $qh^{2-l}/|h|^{2-l}$ *is an inner function.*

**Proof.** For $l\neq 1$ the lemma is known [**13**, Corollary 2]. Suppose $l=1$. By [**3**, p. 133], if $\|f+\bar{z}\bar{H}^1\|=\|f\|_1$, then there exists an extremal function $Q\in H^\infty$ and $|Q|=1$ a.e. on $\{\theta;\ f(e^{i\theta})\neq 0\}$ and $Qf\geqq 0$ a.e. on $\partial U$. Hence $Q$ is inner and so $f/|f|$ is inner. The converse is clear.

**Theorem 8.** *Suppose* $1\leqq l\leqq\infty$ *and* $f$ *is a nonzero function in* $H^l$.
(1) $\|f+\bar{z}\bar{H}^2\|=\|f\|_2$ *for an arbitrary function* $f$ *in* $H^2$.
(2) *For* $2<l<\infty$, $\|f+\bar{z}\bar{H}^l\|=\|f\|_l$ *if and only if*

$$f=q\left(\frac{s+q\bar{s}}{1+Q}\right)^{2/l-2},$$

*where $q$ and $Q$ are inner functions with $Q \prec q$.*
(3) *For $1 \leqq l < 2$, $\|f+\bar{z}\bar{H}^l\| = \|f\|_l$ if and only if*

$$f=q\left(\frac{s+Q\bar{s}}{1+q}\right)^{2/2-l},$$

*where $q$ and $Q$ are inner functions with $q \prec Q$.*
(4) *Suppose $l=\infty$ and $S_{\bar{f}}$ is nonempty. Then $\|f+\bar{z}\bar{H}^\infty\| = \|f\|_\infty$ if and only if $f$ is an inner function.*

**Proof.** (1) is clear because $f$ is orthogonal to $\bar{z}\bar{H}^2$. Suppose $f=qh$ where $q$ is inner and $h$ is outer.
(2) If $\|f+\bar{z}\bar{H}^l\| = \|f\|_l$, then by Lemma $qh^{2-l}/|h|^{2-l}=Q$ is inner. Hence $\bar{q}Qh^{l-2}=|h^{l-2}|$. If $1<t<l/l-2$, then $h^{l-2}\in H^t$ and so $h^{l-2}\in H^1$. Now Theorem 6 implies that

$$Qh^{l-2}=Q\left(\frac{s+q\bar{s}}{1+Q}\right)^2 \text{ and } Q \prec q.$$

Hence $h=(s+q\bar{s}/1+Q)^{2/l-2}$ and so $f=q(s+q\bar{s}/1+Q)^{2/l-2}$. Conversely if $f=q(s+q\bar{s}/1+Q)^{2/l-2}$, then $h=(s+q\bar{s}/1+Q)^{2/l-2}$ and hence

$$\bar{q}\,\frac{h^{l-2}}{|h|^{l-2}}=\bar{q}\frac{(s+q\bar{s})^2}{(1+Q)^2}\,\frac{|1+Q|^2}{|s+q\bar{s}|^2}=Q.$$

The lemma implies $\|f+\bar{z}\bar{H}^l\| = \|f\|_l$.
(3) If $\|f+\bar{z}\bar{H}^l\| = \|f\|_l$, then by the lemma $qh^{2-l}/|h|^{2-l}=Q$ is inner. Hence $\bar{Q}qh^{2-l}=|h|^{2-l}$ and $h^{2-l}\in H^1$ because $h^l\in H^1$ and $l>2-l>0$. Again by Theorem 6

$$qh^{2-l}=q\left(\frac{s+Q\bar{s}}{1+q}\right)^2 \text{ and } q \prec Q.$$

Hence $h=(s+Q\bar{s}/1+q)^{2/2-l}$ and so $f=q(s+Q\bar{s}/1+q)^{2/2-l}$. Conversely if $f=q(s+Q\bar{s}/1+q)^{2/2-l}$, then

$$\bar{q}\,\frac{h^{l-2}}{|h|^{l-2}}=\bar{q}\frac{(s+Q\bar{s})^2}{(1+q)^2}\,\frac{|1+q|^2}{|s+Q\bar{s}|^2}=Q.$$

The lemma implies $\|f+\bar{z}\bar{H}^l\| = \|f\|_l$.

(4) If $S_{\bar{f}}$ is nonempty and $\|f+\bar{z}\bar{H}^{\infty}\| = \|f\|_{\infty}$, then $f$ is inner by Theorem 6.

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