ON APPROXIMATION BY FEJÉR MEANS TO PERIODIC FUNCTIONS SATISFYING A LIPSCHITZ CONDITION.[1]

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S. M. Nikolski [4, Theorem 1; cf. 3, esp. pp. 144 and 148] considered the remainder term in the approximation by the n-th Fejér mean, $\sigma_n(x)$, to a function, $f(x)$, of period 2π satisfying a Lipschitz condition of order α, $0 < \alpha \leq 1$. In this connection, he introduced the quantity

$$(1) \qquad \Delta_n(\alpha) = \sup_f \{ \max_x |\sigma_n(x) - f(x)| \},$$

where the maximum is taken over all x and the supremum is taken over all functions of period 2π, bounded by 1 (a notational convenience only) and satisfying a Lipschitz condition of order α.

He observed that

$$(2) \qquad \Delta_n(\alpha) = 2^{1+\alpha} \pi^{-1} n^{-1} \int_0^{\frac{1}{2}\pi} t^{\alpha} (\sin^2 nt)(\sin t)^{-2}\, dt$$

and proved that[2]

$$(3) \qquad \Delta_n(1) = 2\pi^{-1} n^{-1} \log n + \underline{O}(n^{-1}),$$

$$(4) \qquad \Delta_n(\alpha) = 2\,\Gamma(\alpha)(\sin \tfrac{1}{2}\alpha\pi)\pi^{-1}(1-\alpha)^{-1} n^{-\alpha} + \underline{o}(n^{-\alpha}),\ 0 < \alpha < 1.$$

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[2] All $\underline{O}$- and $\underline{o}$-terms are taken as n becomes infinite.

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In this note, these estimates will be refined somewhat by applying the method and results of [1].

Thus, it will be shown that

$$\Delta_n(1) = (2/\pi)n^{-1} \log n + A_1 n^{-1} + \underline{o}(n^{-2}), \tag{5}$$

where

$$A_1 = (2/\pi)\log \tfrac{1}{2}\pi + (4/\pi)\int_0^1 t^{-1} \sin^2 t \, dt$$

$$+ (4/\pi)\int_1^\infty t^{-1}\{\sin^2 t - \tfrac{1}{2}\}\, dt \tag{6}$$

$$+ \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} t\left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right) dt;$$

and

$$\Delta_n(\alpha) = \frac{2\Gamma(\alpha)\sin\frac{1}{2}\alpha\pi}{(1-\alpha)\pi\, n^\alpha} + A_\alpha n^{-1} + \underline{O}(n^{-2}), \qquad 0<\alpha<1, \tag{7}$$

where

$$A_\alpha = \frac{2^\alpha}{\pi}\int_0^{\frac{1}{2}\pi} t^\alpha\left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right) dt - \frac{2}{(1-\alpha)\pi^{2-\alpha}}. \tag{8}$$

First, for $0 < \alpha \leq 1$,

$$n\Delta_n(\alpha) = \frac{2^{1+\alpha}}{\pi}\int_0^{\frac{1}{2}\pi} t^{\alpha-2}\sin^2 nt\, dt + \frac{2^{1+\alpha}}{\pi}\int_0^{\frac{1}{2}\pi} t^\alpha\left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right)\sin^2 nt\, dt$$

$$= \frac{2^{1+\alpha}}{\pi}\int_0^{\frac{1}{2}\pi} t^{\alpha-2}\sin^2 nt\, dt + \frac{2^\alpha}{\pi}\int_0^{\frac{1}{2}\pi} t^\alpha\left(\frac{1}{\sin^2 t} - \frac{1}{t^2}\right)dt + \underline{o}(n^{-1}),$$

as a consequence of [1,(2.7),p.91], since the mean-value of $\sin^2 t$ over a full period is $\frac{1}{2}$.

Denoting the first integral in the last member by $\delta_n(\alpha)$, it is convenient to separate the cases $\alpha = 1$ and $0 < \alpha < 1$.

The case $\alpha = 1$. Proof of (5):

$$\delta_n(1) = (4/\pi)\int_0^{\frac{1}{2}\pi} t^{-1}\sin^2 nt\, dt = (4/\pi)\int_0^{\frac{1}{2}\pi n} x^{-1} \sin^2 x\, dx$$

$$= (4/\pi)\int_0^1 x^{-1} \sin^2 x\, dx + (4/\pi)\int_1^{\frac{1}{2}\pi n} x^{-1} \sin^2 x\, dx$$

$$= (4/\pi)\int_0^1 x^{-1} \sin^2 x\, dx + (4/\pi)\int_1^{\frac{1}{2}\pi n} x^{-1} \{\sin^2 x - \tfrac{1}{2}\}\, dx$$

$$+ (2/\pi)\int_1^{\frac{1}{2}\pi n} x^{-1}\, dx$$

$$= (4/\pi)\int_0^1 x^{-1} \sin^2 x\, dx + (4/\pi)\int_1^{\infty} x^{-1} \{\sin^2 x - \tfrac{1}{2}\}\, dx$$

$$- (4/\pi)\int_{\frac{1}{2}\pi n}^{\infty} x^{-1} \{\sin^2 x - \tfrac{1}{2}\}\, dx + (2/\pi)\log(\tfrac{1}{2}\pi n)$$

$$= (2/\pi)\log(\tfrac{1}{2}\pi n) + (4/\pi)\int_0^1 x^{-1} \sin^2 x\, dx$$

$$+ (4/\pi)\int_1^{\infty} x^{-1} \{\sin^2 x - \tfrac{1}{2}\}\, dx + \underline{O}(n^{-2}),$$

where the indicated estimate $\underline{O}(n^{-2})$ is justified by [1, Theorem 2.4, p. 93].

This proves (5).

The case $0 < \alpha < 1$. Proof of (7). Here

$$\delta_n(\alpha) = (2^{1+\alpha}/\pi)n^{1-\alpha}\int_0^{\frac{1}{2}\pi n} x^{\alpha-2} \sin^2 x\, dx$$

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$$= (2^{1+\alpha}/\pi) n^{1-\alpha} \left[\int_0^\infty - \int_{\frac{1}{2}\pi n}^\infty \right]$$

$$= \frac{2\Gamma(\alpha)\sin\frac{1}{2}\alpha\pi}{(1-\alpha)\pi} n^{1-\alpha} - 2^{1+\alpha} \pi^{-1} n^{1-\alpha} \int_{\frac{1}{2}\pi n}^\infty x^{\alpha-2} \sin^2 x \, dx.$$

Now, writing $2 \sin^2 x = 1 - \cos 2x$ and $y = 2x$, we have

(9) $$2^{1+\alpha} \pi^{-1} n^{1-\alpha} \int_{\frac{1}{2}\pi n}^\infty x^{\alpha-2} \sin^2 x \, dx$$

$$= 2\pi^{\alpha-2}(1-\alpha)^{-1} - 2\pi^{-1} n^{1-\alpha} \int_{\pi n}^\infty y^{\alpha-2} \cos y \, dy.$$

To estimate the last integral, we observe that the areas bounded by successive arches of the graph of the integrand alternate in sign and decrease steadily in magnitude. (By an "arch" is meant a portion of the curve joining consecutive zeros, $(k + \frac{1}{2})\pi$, and $(k + \frac{3}{2})\pi$, $k = n, n+1, \ldots$.)

This remark can be applied once the integral is decomposed and the arches "paired":

$$\int_{\pi n}^\infty y^{\alpha-2} \cos y \, dy = \int_{\pi n}^{\pi(n+\frac{1}{2})} + \int_{\pi(n+\frac{1}{2})}^\infty$$

$$= \int_{\pi n}^{\pi(n+\frac{1}{2})} + \int_{\pi(n+\frac{1}{2})}^{\pi(n+\frac{3}{2})} + \left\{ \int_{\pi(n+\frac{3}{2})}^{\pi(n+\frac{5}{2})} + \int_{\pi(n+\frac{5}{2})}^{\pi(n+\frac{7}{2})} \right\} + \ldots .$$

Now, from the above remark,

$$\left| \int_{\pi(n+\frac{1}{2})}^\infty y^{\alpha-2} \cos y \, dy \right| < \left| \int_{\pi(n+\frac{1}{2})}^{\pi(n+\frac{3}{2})} y^{\alpha-2} \cos y \, dy \right|$$

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$$< \int_{\pi(n+\frac{1}{2})}^{\pi(n+\frac{3}{2})} y^{\alpha-2}\, dy = \pi^{\alpha-1}(1-\alpha)^{-1}\ \{(n+\tfrac{1}{2})^{\alpha-1} - (n+\tfrac{3}{2})^{\alpha-1}\}$$

$$= \underline{O}(n^{\alpha-2}),$$

where the inequality yielding the $\underline{O}$-term is obtained by applying the mean-value theorem of the differential calculus to the function $x^{\alpha-1}$, using $n + \frac{1}{2}$ and $n + \frac{3}{2}$ as the end-points of the interval.

Likewise,

$$\int_{\pi n}^{\pi(n+\frac{1}{2})} y^{\alpha-2} \cos y\, dy = \underline{O}(n^{\alpha-2}),$$

so that

$$(10) \qquad n^{1-\alpha} \int_{\pi n}^{\infty} y^{\alpha-2} \cos y\, dy = \underline{O}(n^{-1}).$$

Combining this with (9) provides the desired estimate of $\delta_n(\alpha)$, $0 < \alpha < 1$, and thereby completes the proof of (7).

Remarks. 1. An alternative proof of (5) can be obtained by relating $n\triangle_n(1)$ to constants introduced by L. Féjer analogous to the Lebesgue constants, using, say, [1, (4.4), p. 97]. Thus,

$$n\triangle_n(1)$$

$$= \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\sin^2 nt}{\sin t}\, dt + \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} \left(\frac{t}{\sin^2 t} - \frac{1}{\sin t}\right) \sin^2 nt\, dt$$

$$= \frac{2}{\pi} \log n + \frac{2}{\pi} \log 2 + \frac{4}{\pi}\int_0^1 \frac{\sin^2 t}{t}\, dt$$

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$$+ \frac{4}{\pi} \int_{1}^{\infty} \frac{1}{t} \{\sin^2 t - \tfrac{1}{2}\} \, dt + \frac{2}{\pi} \int_{0}^{\frac{1}{2}\pi} \left(\frac{t}{\sin^2 t} - \frac{1}{\sin t} \right) dt$$

$$+ \underline{o}\,(1/n).$$

This gives a different representation for A_1 from (6).

2. Similarly, Nikolski's formula (3), above, can be derived quite easily by using [2, (7)], and noting the boundedness of

$$\int_{0}^{\frac{1}{2}\pi} \left(\frac{t}{\sin^2 t} - \frac{1}{\sin t} \right) \sin^2 nt \, dt.$$

3. B. Sz.-Nagy [5, p. 72] has given a different treatment of $\Delta_n(1)$, which he calls $\rho_{1n}^{(1)}$, and obtained a complete asymptotic expansion. From his work, it follows that $2/\pi \leq A_1 \leq 6/\pi$.

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