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NON-FREE GROUPS GENERATED BY TWO 2×2 MATRICES

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1. Introduction. Let m be any real or complex number, and let G_m be the group generated by the 2×2 matrices $A = (1, m; 0, 1)$ and $B = (1, 0; m, 1)$, where we use the notation $(c_{11}, c_{12}; c_{21}, c_{22})$ to denote (by rows) the elements of a 2×2 matrix C. Thus, G_m is the set of all finite products (or *words*) of the form

$$\ldots A^{h(3)}B^{h(2)}A^{h(1)},$$

where the $h(i)$ are nonzero integers with $h(1)$ possibly zero. If a non-trivial word of this form equals the identity $I = (1, 0; 0, 1)$, then G_m is *non-free*; otherwise, G_m is *free*. Sanov [**6**] showed that G_m is free for $m = 2$, and Brenner [**1**] that G_m is free for $|m| \geqq 2$. As a consequence, algebraic numbers m such that G_m is free are dense in the complex plane (since G_m is free if m is an algebraic number whose algebraic conjugate m^* satisfies $|m^*| \geqq 2$). As noted in [**3**], G_m is free for all transcendental m. Chang, Jennings, and Ree [**2**] and Lyndon and Ullman [**4**] provided successive weakening of the condition $|m| \geqq 2$. On the other hand, Ree [**5**] showed that the m for which G_m is non-free are dense in various regions in the complex plane, including the unit disc.

Our attention in this paper will be directed to the question of whether or not G_m is free for rational numbers m such that $-2 < m < 2$. No such m is known for which G_m is free, and it may be that none exists. Among other results, we show that G_m is non-free for $m = a/b$ and $a = 1, 2, 3$ or 4, provided $|m| < 2$. However, obtaining further results along this line by similar methods would be costly, as we show in Section 3.

We note that the group generated by the matrices $(1, 2; 0, 1)$ and $(1, 0; \lambda, 1)$ is isomorphic to G_m if $m^2 = 2\lambda$. This notation is used in [**2**] and [**5**], and we shall use it in Section 3.

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2. Some general results. Let W be a word in G_m with k exponents, each of which is nonzero. If W does not reduce, then the relation $W = I$ will be called *a relation with k terms*. Note that if G_m is non-free, then, for some even

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integer k, there exists a word in the *canonical form*

$$W = A^{h(k)}B^{h(k-1)} \dots A^{h(2)}B^{h(1)}$$

such that $W = I$, where all $h(i)$ are nonzero.

THEOREM 2.1. *G_m cannot have a relation with fewer than 6 terms.*

Proof. Clearly $A^x \neq I$ and $B^x \neq I$ if $x \neq 0$. Regarding A^xB^y and $A^wB^xA^yB^z$, the (1, 1)-element of the former and the (2, 2)-element of the latter are $1 + xym^2 \neq 1$.

However, G_m may indeed have a six-term relation: $AB^yA^{-1}BA^yB^{-1} = I$ when $m = 1$ for any integer y.

LEMMA 2.2.1. *If $A^wB^xA^yB^zA^tB^u = (s_{11}, s_{12}; s_{21}, s_{22})$, then*

$$s_{11} = a + m^2u(at + b), \quad s_{12} = m(at + b),$$
$$s_{21} = m(cu + d), \quad s_{22} = c,$$

where

$$a = 1 + m^2(wx + wz + yz) + wxyzm^4, \quad b = w + y + m^2wxy,$$
$$c = 1 + m^2(xt + zt + xy) + xyztm^4, \quad d = x + z + m^2xyz.$$

Proof. This is clear.

LEMMA 2.2.2. *$A^wB^xA^yB^zA^tB^u = I$ with $wxyztu \neq 0$ if and only if*

(1) $$wx = zt,$$

(2) $$x + z + u + m^2xyz = 0$$

and

(3) $$wu = yz.$$

Proof. This is clear.

THEOREM 2.2. *If $A^wB^xA^yB^zA^tB^u = I$, with $wxyztu \neq 0$, then*
(i) $|m| \leqq \sqrt{3}$;
(ii) *if $m^2 > 1$, then $m^2 = (n + 2)/n$ for some integer $n \geqq 1$.*

Proof. The integer solutions of (1) can be parameterized by

$$w = w_1w_2, \quad x = x_1x_2, \quad z = w_1x_2, \quad t = w_2x_1.$$

Then, if $wu = yz$, it follows that $w_2u = x_2y$. All integer solutions to this are given by

$$w_2 = w_3w_4, \quad u = u_1u_2, \quad x_2 = w_3u_1, \quad y = w_4u_2.$$

Equation (2) may now be re-written as

$$m^2 = -\frac{1}{u_1w_3w_4}\left[\frac{1}{w_1u_2} + \frac{1}{x_1u_2} + \frac{1}{x_1w_1w_3}\right].$$

Thus, m^2 has a maximum value of 3. If one of w_1, x_1, u_2 (say u_2) is not ± 1, then m^2 is maximized by $(u_2 + 2)/u_2$. If one of u_1, w_3 or w_4 is not ± 1, then m^2 is at least halved. A consideration of a small number of cases (possible integer values of the six variables) completes part (ii).

We now leave our study of relations with a fixed number of terms. Following [**5**], we call m *free* (*non-free*) if G_m is free (non-free). The following theorem indicates that certain real rational numbers are not free.

THEOREM 2.3. *Let a, b, r and r' be integers.*
(i) *1 is not free.*
(ii) *If m is not free, then m/b is not free; $1/b$ is not free.*
(iii) *If $b = r(a^2 - 1) + 1$, then $m = a/b$ is not free.*
(iv) *If $b = r'(a^2 - 1) - 1$, then $m = a/b$ is not free.*

Proof. (i) If $m = 1$, then $A^{-1}BAB^{-1}AB = I$.
(ii) To obtain A^b and B^b, replace m by bm. Thus, if $W = I$ is a relation in A and B for m, by multiplying the exponents in W by b we obtain a relation in $G_{m/b}$. The second part follows from (i).
(iii) $B^bA^{-b}B^r(A^b)^bB^{-1}A^{rb} = I$.
(iv) Replace r by $-r$ in (iii) and use the fact that if m is not free, then $-m$ is not free.

Our next theorem, which contains a necessary and sufficient condition for a group to be non-free, is the basis for a procedure given in Section 3 for obtaining a relation for non-free rational values of m. We note that the condition does not involve matrix multiplications.

THEOREM 2.4 (See [**4**]). *The group G_m is not free if and only if nonzero integers $h(k)$ can be found such that the recursion*

$$(4) \quad x(n+2) = x(n) + mh(n+1)x(n+1)$$

eventually produces the value $x(k+1) = 0$ from the starting values $x(0) = 0$ and $x(1) = 1$, where k is odd.

Proof. Let k be an odd integer. A simple induction argument shows that the matrix $W_k = A^{h(k)}B^{h(k-1)} \ldots B^{h(2)}A^{h(1)}$ has the form $(y(k), x(k+1); y(k-1), x(k))$ and the matrix $W_{k+1} = B^{h(k+1)}A^{h(k)} \ldots B^{h(2)}A^{h(1)}$ has the form $(y(k), x(k+1); y(k+1), x(k+2))$, where $x(0) = 0$, $x(1) = 1$ and $x(n+2) = x(n) + mh(n+1)x(n+1)$ and $y(0) = 0$, $y(1) = 1$ and $y(n+2) = y(n) + mh(n+2)y(n+1)$. If $x(k+1) = 0$, then either $W_kBW_k^{-1}$ or $W_{k+1}BW_{k+1}^{-1}$ is commutative with B, and thus the group is not free.

Interchanging the roles of A and B, the matrix $\hat{W}_k = B^{h(k)}A^{h(k-1)} \ldots A^{h(2)}B^{h(1)}$ has the form $(x(k), y(k-1); x(k+1), y(k))$, while $\hat{W}_{k+1} = A^{h(k+1)}B^{h(k)} \ldots A^{h(2)}B^{h(1)} = (x(k+2), y(k+1); x(k+1), y(k))$; the commuting elements are, respectively, $\hat{W}_kA\hat{W}_k^{-1}$ and A, and $\hat{W}_{k+1}A\hat{W}_{k+1}^{-1}$ and A.

COROLLARY 2.4.1. *If $m = a/b$, the group G_m is not free if and only if nonzero* *integers $h(k)$ can be found such that the recursion*

$$(5) \quad z(n+2) = b^2 z(n) + ah(n+1)z(n+1)$$

eventually produces the value $z(k+1) = 0$ from the starting values $z(0) = 0$ and $z(1) = 1$, where k is odd.

Proof. Put $x(n) = z(n)/b^{n-1}$ in Theorem 2.4.

The difference between Corollary 2.4.1 and Theorem 2.4 is that only integers appear at each stage of the recursion (5). This is a decided advantage when using a computer to show that a group G_m is non-free.

Remark. The "if" part of Theorem 2.4 remains true for k even. Thus, if the recursion (5) produces a value $z(n+2)$ equal to 0 for any $n \geqq 0$, then the group G_m is non-free.

COROLLARY 2.4.2. *If a and r are integers and*

$$b = \begin{cases} \frac{1}{2}ra^2 \pm 1, & \text{if } a \text{ is even} \\ ra^2 \pm 1, & \text{in any case,} \end{cases}$$

then $m = a/b$ is not free.

Proof. For $b = ra^2 \pm 1$, take $h(1) = 1$, $h(2) = -(a^2r^2 \pm 2r)$, $h(3) = -(a^2r \pm 1)^2$. Then $z(2) = a$, $z(3) = 1$, and $z(4) = 0$. For a even, take $h(1) = 1$, $h(2) = -(\frac{1}{4}a^2r^2 \pm r)$, $h(3) = -(\frac{1}{2}a^2r \pm 1)^2$. Then again $z(2) = a$, $z(3) = 1$, and $z(4) = 0$.

We note that this corollary generalizes and corrects a proof of a result in [**4**], and is used later to prove Theorem 3.1.

COROLLARY 2.4.3. *G_m is non-free for $m = 2/b$ and $b \geqq 2$.*

Proof. Let $b = 2r + 1$ and use Corollary 2.4.2 with $a = 2$. If b is even, the result follows from Theorem 2.3(ii).

Two sufficient conditions for a group G_m to be non-free are contained in the next theorem.

THEOREM 2.5. *Let $\{h_1(i)\}$ and $\{h_2(i)\}$ be two sequences of nonzero integers which determine sequences $\{x_1(i)\}$ and $\{x_2(i)\}$, respectively, satisfying the recursion (4), and for which $x_1(0) = x_2(0) = 0$ and $x_1(1) = x_2(1) = 1$.*

(a) *If there exist positive integers j and k, both of which are either odd or even, such that*

$$(6) \quad x_1(j)x_2(k+1) - x_1(j+1)x_2(k) = 0$$

and, in case $j = k$, there also exists an integer $n \leqq j$ such that $h_1(n) \neq h_2(n)$, then the group G_m is non-free.

(b) *If there exist integers $j \geqq 0$ and $k \geqq 1$, both of which are either odd or even,* *such that*

(7) $x_1(j+1)x_2(k) - x_1(j+2)x_2(k+1) = 0,$

then the group G_m *is non-free.*

Proof. (a) If j and k are both odd, let

$$W_1 = A^{h_1(j)}B^{h_1(j-1)}\ldots A^{h_1(1)} \quad \text{and} \quad W_2 = A^{h_2(k)}B^{h_2(k-1)}\ldots A^{h_2(1)}.$$

Then the (1, 2)-element of $W_1^{-1}W_2$ is

$$\frac{x_1(j)x_2(k+1) - x_1(j+1)x_2(k)}{\det(W_1)},$$

which is 0 by (6). Thus, the group G_m is non-free by Theorem 2.4. (The requirement that there exist $n \leqq j$ such that $h_1(n) \neq h_2(n)$ in case $j = k$ ensures that $W_1^{-1}W_2$ does not reduce to the trivial word I.)

The case that j and k are both even is similar. Let

$$W_1 = B^{h_1(j)}A^{h_1(j-1)}\ldots A^{h_1(1)} \quad \text{and} \quad W_2 = B^{h_2(k)}A^{h_2(k-1)}\ldots A^{h_2(1)}.$$

(b) The proof of this part is similar to that of part (a). If j and k are both odd, let

$$W_1 = B^{h_1(j+1)}A^{h_1(j)}\ldots A^{h_1(1)} \quad \text{and} \quad W_2 = A^{h_2(k)}B^{h_2(k-1)}\ldots A^{h_2(1)}.$$

If j and k are both even, let

$$W_1 = A^{h_1(j+1)}B^{h_1(j)}\ldots A^{h_1(1)} \quad \text{and} \quad W_2 = B^{h_2(k)}A^{h_2(k-1)}\ldots A^{h_2(1)}.$$

Note that part (a) of this theorem remains true with the sequences $\{x_1(i)\}$ and $\{x_2(i)\}$ replaced by sequences $\{z_1(i)\}$ and $\{z_2(i)\}$, respectively, which satisfy the recursion (5) and for which $z_1(i) = b^{i-1}x_1(i)$ and $z_2(i) = b^{i-1}x_2(i)$, $i \geqq 2$. In part (b), however, (7) must be replaced by

$$b^2z_1(j+1)z_2(k) - z_1(j+2)z_2(k+1) = 0.$$

The following theorem says that, as soon as we know that G_m is not free for $m = a/b$, we know it is not free for an infinite set $m = a/b'$, where b' is any integer in an arithmetic progression $kr \pm b$. This, as we show in Section 3, raises the results of a computer study from the status of a tabulation to that of a theorem.

THEOREM 2.6. *Let* $m = a/b$. *Suppose* $\{h(i)\}$ *is a sequence of nonzero integers such that the recursion* (5) *leads to an* $l > 0$ *such that* $z(l) = 0$. *Then not only is* G_m *not free, but* $G_{m'}$ *is not free for* $m' = a/(Mr \pm b)$, *where*

$$M = \begin{cases} \frac{1}{2}aL, & \text{if } 2|a \text{ or } 4|L \\ aL, & \text{otherwise,} \end{cases}$$

and L *is the least common multiple of* $z(1), z(2), \ldots, z(l-1)$.

Proof. Let $\{h(i)\}$ be a sequence which produces by (5) a sequence $\{z(j)\}$ such that $z(l) = 0$ for $m = a/b$ and $l > 0$. Then the new sequence $\{h'(i)\}$ defined by

$$h'(1) = h(1),$$
$$h'(n+1) = -\left[\frac{k^2 z(n)}{az(n+1)} r^2 \pm \frac{2kbz(n)}{az(n+1)} r - h(n+1)\right],$$
$$n \geqq 1, k \text{ arbitrary}, r \geqq 0,$$

yields precisely the same sequence $\{z(j)\}$ for $m = a/(kr \pm b)$ (and hence, if a/b is non-free, so is $a/(kr \pm b)$). Calling this new sequence $\{z'(j)\}$, we have from (5) that

$$z'(n+2) = (kr \pm b)^2 z(n) + ah'(n+1)z'(n+1),$$
$$z'(0) = 0, z'(1) = 1.$$

If by induction $z'(j) = z(j)$ for $j = 1, 2, \ldots, n+1$, we have

$$\begin{aligned} z'(n+2) &= k^2 r^2 z'(n) \pm 2kbrz'(n) + b^2 z'(n) \\ &\quad + az'(n+1)(-1)\left[\frac{k^2 z(n)}{az(n+1)} r^2 \pm \frac{2kbz(n)}{az(n+1)} r - h(n+1)\right] \\ &= b^2 z(n) + az(n+1)h(n+1) \\ &= z(n+2). \end{aligned}$$

Thus, since we require integers, we are left with the requirements

(8) $az(n+1) | k^2 z(n), \quad n = 1, 2, \ldots, l-2$

and

(9) $az(n+1) | 2kbz(n), \quad n = 1, 2, \ldots, l-2.$

Since a and b are relatively prime, and $z(n)$ and $z(n+1)$ may also be relatively prime, we replace (9) by the stronger condition

(10) $az(n+1) | 2k, \quad n = 1, 2, \ldots, l-2.$

Condition (10) is satisfied if $k = aL$, where L is the least common multiple of the $z(i)$, and $k = aL$ would also satisfy (8). However, $k = \frac{1}{2}aL$ would also satisfy (10); for k to be an integer, this would require a or L to be even. If a is even, $k = \frac{1}{2}aL$ satisfies (8) also, but if a is odd, (8) becomes

$$az(n+1) | \tfrac{1}{4} a^2 L^2 z(n)$$

or

$$4 | aL \frac{L}{z(n+1)} z(n),$$

and this is guaranteed if $4|L$.

COROLLARY 2.6.1. *G_m is not free for $m = 3/b$ and $b \geqq 2$.*

Proof. Let $r \geqq 1$. By Theorem 2.3 (ii), if $m = 3/(kr \pm l)$ is not free for $(k, l) = 1$, then it is not free for any k, l. By Corollary 2.4.2, $m = 3/(9r \pm 1)$ is not free. Working mod 18, this leaves the residue classes ± 5 and ± 7. To prove that 3/5 is not free, the sequence $h(i) = 1, -3, 12, 6, -6, -11, -25$ yields the sequence $z(i) = 0, 1, 3, -2, 3, 4, 3, 1, 0$. Hence, using Theorem 2.6, we have $L = -12$, whence $M = -18$ and $3/(18r \pm 5)$ is not free. For 3/7, the sequence $h(i) = 1, -5, -12, -22, 24, 11, -49$ yields $z(i) = 0, 1, 3, 4, 3, -2, 3, 1, 0$, so that $3/(18r \pm 7)$ is not free.

3. Numerical results. Based on Corollary 2.4.1, we have written a computer program using integer multiple precision arithmetic to search for a sequence of nonzero integers $h(i)$ such that the recursion (5) eventually produces a value $z(n + 2) = 0$. We have closely followed the suggestion in [**4**] that the numbers $h(i)$ be chosen so as to minimize $|z(n + 2)|$ at each step. Thus, $h(n + 1)$ is chosen to be the nonzero integer closest to

$$k(n+1) = \frac{-b^2 z(n)}{az(n+1)}.$$

More precisely, if

$$\operatorname{sgn}(x) = \begin{cases} 1, & x \geqq 0 \\ -1, & x < 0, \end{cases}$$

then $h(n + 1)$ is the integer part of

$$k(n+1) + \tfrac{1}{2} \operatorname{sgn}(k(n+1))$$

unless $|k(n + 1)| < \frac{1}{2}$, in which case $h(n + 1)$ is set to $\operatorname{sgn}(k(n + 1))$.

In particular, $k(1) = 0$ so that $h(1) = 1$. However, for certain values of m for which G_m is non-free this procedure does not yield a relation. Our modification is to let $h(1)$ assume various starting values, with the remaining values $h(i)$ chosen according to the above procedure.

Our main numerical result is contained in the following theorem, which was obtained by using our computer program to determine a relation for a finite number of rational numbers of the form $4/b$ and then applying Theorem 2.6. Only an outline of the proof is given as the numerical details are lengthy.

THEOREM 3.1. *G_m is not free for $m = 4/b$ and $b \geqq 3$.*

Outline of the proof. By Corollary 2.4.3, $4/b$ is non-free if b is even. By Corollary 2.4.2, $4/b$ is non-free if $b = 8r \pm 1$, $r \geqq 1$, and thus it follows from Theorem 2.3 (ii) that $4/b$ is non-free for $b = 24r \pm 3$.

Using our program with various values of $h(1)$ between 1 and 10 and applying Theorem 2.6, we were able to show that all numbers of the form $4/b$ are non-free except possibly those in the residue classes ± 19, ± 59, ± 163, ± 275, ± 283, ± 347, ± 397, ± 467, ± 499, ± 541, ± 571, ± 611, ± 653, ± 845, ± 877 and ± 989 (mod 2016).

The next stage was to examine these residue classes mod 10,080. If b is an odd multiple of 5, then it is either of the form $\pm 5 \pmod{40}$ or $\pm 15 \pmod{40}$, and it follows from Theorem 2.3 (ii) and Corollary 2.4.2, respectively, that $4/b$ is non-free. On applying Theorem 2.6 for various numbers of the form $4/b$, we found that all numbers $4/b$ are non-free with the possible exception of those in the residue classes ± 59, ± 499, ± 571, ± 1139, ± 1669, ± 1741, ± 2291, ± 2299, ± 2669, ± 3379, ± 3461, ± 3749, ± 4091, ± 4379, ± 4531 and ± 4909 (mod 10,080).

On examining these residue classes mod 50,400 and applying Theorem 2.6, all residue classes were accounted for and thus the theorem was proven.

We cannot decide whether there exists a rational number m such that $|m| < 2$ and G_m is free.

The following theorem contains a negative result regarding the utility of any procedure similar to the one we have used to prove that a group G_m is non-free. Simply stated, it says that if there exist non-free rational numbers arbitrarily close to 2, then either the number of terms in a non-trivial relation is unbounded as $m \to 2^-$, or else the magnitude of the exponents in such a relation is unbounded. Stated another way, if there exist non-free rational numbers arbitrarily close to 2, then given any fixed finite time t, any procedure which searches for a relation will require time greater than t if m is a non-free rational number sufficiently close to 2.

Note that the following lemmas and theorem use the notation mentioned in the introduction, where $m^2 = 2\lambda$.

LEMMA 3.2.1. *Let* $A = (1, 2; 0, 1)$, $B = (1, 0; \lambda, 1)$ *and*

$$W = A^{h(n)}B^{h(n-1)} \ldots A^{h(2)}B^{h(1)},$$

where all $h(i)$ *are nonzero integers. Let* $w(\lambda) = \sum_{i=1}^{n} |h(i)|$ *and let* $q(\lambda) = \sum_{i=0}^{m} a_i\lambda^i$ *denote the* (1, 2)*-element of* W, *where* $m = (n-2)/2$. *Let* $N = m + \sum_{i=0}^{m} |a_i|$.

If n *and* $w(\lambda)$ *are bounded, then* N *is bounded.*

Proof. This is obvious.

LEMMA 3.2.2. *Let* $q(\lambda)$ *and* N *be defined as in Lemma* 3.2.1. *If* $q(\hat{\lambda}) = 0$, *then there exists a disc* $\{\lambda \mid |\lambda - \hat{\lambda}| < r\}$ *in which non-free numbers are dense. Moreover,* r *is a function of* N, *and if* N *is bounded, then* r *is bounded away from* 0.

Proof.

$$\begin{aligned}\lambda q(\lambda) &= \lambda(q(\lambda) - q(\hat{\lambda}))\\ &= \lambda\{a_0(\lambda - \hat{\lambda}) + a_1(\lambda^2 - \hat{\lambda}^2) + \ldots + a_m(\lambda^{m+1} - \hat{\lambda}^{m+1})\}\end{aligned}$$

so that $|\lambda q(\lambda)| = |\lambda|\,|\lambda - \hat{\lambda}|R$, where

$$R = |a_0 + a_1(\lambda + \hat{\lambda}) + \ldots + a_m(\lambda^m + \ldots + \hat{\lambda}^m)|.$$

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Thus, if

$$|\lambda - \hat{\lambda}| < \frac{1}{|\lambda|R}, \quad \text{then} \quad |\lambda q(\lambda)| < 1.$$

By a theorem of Ree [**5**], non-free numbers are dense in the region $\{\lambda | \, |\lambda q(\lambda)| < 1\}$. Clearly, then, any number λ for which $|\lambda - \hat{\lambda}| < 1/|\lambda|R$ must be such that $|\lambda| < 2$. If $|\lambda| < 2$, then $1/|\lambda|R > r$, where

$$r = \frac{1}{2\{|a_0| + 4|a_1| + 12|a_2| + \ldots + (m+1)2^m|a_m|\}},$$

so that if $|\lambda - \hat{\lambda}| < r$, then $|\lambda - \hat{\lambda}| < 1/|\lambda|R$. Thus it follows from Ree's theorem that non-free numbers are dense in the disc $\{\lambda | \, |\lambda - \hat{\lambda}| < r\}$. Clearly r is bounded away from 0 if N is bounded.

THEOREM 3.2. *If there exists a sequence of non-free numbers* $\{\lambda_i\}$ *such that*

$$\lim_{i\to\infty} \lambda_i = 2$$

and if $W_i = I$ *is a non-trivial* n_i*-term relation in* $A = (1, 2; 0, 1)$ *and* $B_i = (1, 0; \lambda_i, 1)$, *then at least one of the following is true:*

$$\lim_{i\to\infty} n_i = \infty \quad \text{or} \quad \lim_{i\to\infty} w(\lambda_i) = \infty.$$

Proof. Suppose that $\lim_{i\to\infty} n_i$ and $\lim_{i\to\infty} w(\lambda_i)$ are both bounded. If, for each value of i, $q(\lambda_i) = \sum_{j=0}^{m_i} a_j^{(i)}\lambda_i^j$ denotes the (1, 2)-element of W_i, then $q(\lambda_i) = 0$ and it follows from Lemma 3.2.1 that $N_i = m_i + \sum_{j=0}^{m_i} |a_j^{(i)}|$ is bounded. From Lemma 3.2.2 it follows that there exists a disc $\{\lambda | \, |\lambda - \lambda_i| < r_i\}$, for each value of i, in which non-free numbers are dense, and since all N_i are bounded, there exists a uniform bound ρ such that $|r_i| \geqq \rho > 0$. This gives rise to a contradiction, since all non-free numbers are known to be less than 2 in modulus.

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