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Approximation and Interpolation by Entire Functions of Several Variables

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Abstract. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be C^∞ and let $h\colon \mathbb{R}^n \to \mathbb{R}$ be positive and continuous. For any unbounded nondecreasing sequence $\{c_k\}$ of nonnegative real numbers and for any sequence without accumulation points $\{x_m\}$ in $\mathbb{R}^n$, there exists an entire function $g\colon \mathbb{C}^n \to \mathbb{C}$ taking real values on $\mathbb{R}^n$ such that

$$|g^{(\alpha)}(x) - f^{(\alpha)}(x)| < h(x), \quad |x| \geq c_k, |\alpha| \leq k, k = 0, 1, 2, \ldots,$$

$$g^{(\alpha)}(x_m) = f^{(\alpha)}(x_m), \quad |x_m| \geq c_k, |\alpha| \leq k, m, k = 0, 1, 2, \ldots.$$

This is a version for functions of several variables of the case $n = 1$ due to L. Hoischen.

1 Introduction

A theorem of Carleman [Ca], extending the well-known theorem of Weierstrass on approximation by polynomials of continuous functions on compact intervals, states that for every continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and every positive continuous function $h\colon \mathbb{R} \to \mathbb{R}$, there is a function $g\colon \mathbb{R} \to \mathbb{R}$ which is the restriction to $\mathbb{R}$ of an entire function and satisfies $|g(x) - f(x)| < h(x)$ for all $x \in \mathbb{R}$. L. Hoischen proved the following generalization which allows approximation of both f and its derivatives. Here and in the rest of the paper, t is a fixed positive integer.

Theorem 1.1 ([H1], see also [FG]) *Let $f\colon \mathbb{R}^t \to \mathbb{C}$ and let $N \in \mathbb{N}$.*

(a) *If f is a C^N function, then for each positive continuous $h\colon \mathbb{R}^t \to \mathbb{R}$ there is an entire function $g\colon \mathbb{C}^t \to \mathbb{C}$ such that $|f^{(\alpha)}(x) - g^{(\alpha)}(x)| < h(x)$ when $|\alpha| \leq N$.*
(b) *If f is a C^∞ function, then for each positive continuous function $h\colon \mathbb{R}^t \to \mathbb{R}$, and each sequence $\{c_n\}$ of real numbers with $0 \leq c_n \leq c_{n+1}$ $(n = 0, 1, 2, \ldots)$ and $\lim_{n\to\infty} c_n = \infty$, there is an entire function $g\colon \mathbb{C}^t \to \mathbb{C}$ such that for all $n = 0, 1, 2, \ldots$ $|f^{(\alpha)}(x) - g^{(\alpha)}(x)| < h(x)$ when $|x| \geq c_n$ and $|\alpha| \leq n$.*

In both of these statements, if f takes real values on $\mathbb{R}^t$, then we may require the same property for g.

For the case $t = 1$, this theorem is improved in [H2] to give approximation as well as interpolation on a closed discrete set. We prove the following theorem, which extends this result to functions of several variables.

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Theorem 1.2 *Let $f\colon \mathbb{R}^t \to \mathbb{C}$ and let $N \in \mathbb{N}$.*

(a) *If f is a C^N function, then for each positive continuous $h\colon \mathbb{R}^t \to \mathbb{R}$, and for any sequence $\{x_m\}_{m=0}^{\infty}$ in $\mathbb{R}^t$ without accumulation points, there exists an entire function g such that*

$$|g^{(\alpha)}(x) - f^{(\alpha)}(x)| < h(x), \qquad (x \in \mathbb{R}^t,\ |\alpha| \le N)$$

and

$$g^{(\alpha)}(x_m) = f^{(\alpha)}(x_m), \qquad (|\alpha| \le N,\ m = 0, 1, 2, \dots).$$

(b) *If f is a C^∞ function, then for each positive continuous $h\colon \mathbb{R}^t \to \mathbb{R}$, each sequence $\{c_n\}$ of real numbers with $0 \le c_n \le c_{n+1}$ and $\lim_{n\to\infty} c_n = \infty$, and for any sequence $\{x_m\}_{m=0}^{\infty}$ in $\mathbb{R}^t$ without accumulation points, there exists an entire function g such that for all $n = 0, 1, 2, \dots$*

$$|g^{(\alpha)}(x) - f^{(\alpha)}(x)| < h(x), \qquad (|x| \ge c_n,\ |\alpha| \le n)$$

and

$$g^{(\alpha)}(x_m) = f^{(\alpha)}(x_m), \qquad (|x_m| \ge c_n,\ |\alpha| \le n,\ m = 0, 1, 2, \dots).$$

In both of these statements, if f takes real values on $\mathbb{R}^t$, then we may require the same property for g.

Our proof differs from Hoischen's in the technical details but follows the same outline. We need the following classical interpolation result corresponding to the single variable analog used in [H2]. P. M. Gauthier has pointed out to the author that a relatively simple deduction relying on the Oka–Weil theorem can be found, in more general form, in [GP].

Lemma 1.3 *Let $z_m \in \mathbb{C}^t$, $m = 0, 1, 2, \dots$ be distinct and without accumulation points. Let $k_m \ge 0$ be integers $m = 0, 1, 2, \dots$. Let $w_{m,\alpha}$ be any complex numbers, $|\alpha| \le k_m$, $m = 0, 1, 2, \dots$ There exists an entire function $\phi\colon \mathbb{C}^t \to \mathbb{C}$ such that $\phi^{(\alpha)}(z_m) = w_{m,\alpha}$ whenever $|\alpha| \le k_m$, $m = 0, 1, 2, \dots$.*

Note that if $z_m \in \mathbb{R}^t$ and $w_{m,\alpha} \in \mathbb{R}$ for $|\alpha| \le k_m$, $m = 0, 1, 2, \dots$, then we may ask that ϕ take real values on $\mathbb{R}^t$. (Write $\phi = \phi_1 + i\phi_2$, where ϕ_1, ϕ_2 are entire functions taking real values on $\mathbb{R}^t$, and replace ϕ with ϕ_1.)

We shall make use of the following fact about continuous functions on metric spaces.

Lemma 1.4 *Let (X, σ_X), (Y, σ_Y), and (Z, σ_Z) be metric spaces. Let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous. Let $\varepsilon\colon X \to \mathbb{R}$ be a positive continuous function. Then there is a positive continuous function $\delta\colon X \to \mathbb{R}$ such that for all $x \in X$ and $y \in Y$,*

$$\sigma_Y(f(x), y) < \delta(x) \ \Rightarrow\ \sigma_Z(gf(x), g(y)) < \varepsilon(x).$$

Proof Let $a \in X$. By continuity of g, there is $\delta_1(a) > 0$ such that for all $y \in Y$,

$$(1.1) \qquad \sigma_Y(f(a), y) < 2\delta_1(a) \Rightarrow \sigma_Z(gf(a), g(y)) < \tfrac{1}{4}\varepsilon(a).$$

By continuity of ε, f, and g, there is an open ball $B(a)$ centered on a such that for all $x \in B(a)$,

(i) $\varepsilon(x) > \frac{1}{2}\varepsilon(a)$,
(ii) $\sigma_Y(f(a), f(x)) < \delta_1(a)$,
(iii) $\sigma_Z(gf(a), gf(x)) < \frac{1}{4}\varepsilon(a)$.

Applying the triangle inequality and (ii), (1.1), (iii), (i) in that order, we see that for $x \in B(a)$ and $y \in Y$ we have

$$(1.2) \qquad \sigma_Y(f(x), y) < \delta_1(a) \Rightarrow \sigma_Z(gf(x), g(y)) < \varepsilon(x).$$

Let Φ be a partition of unity for X subordinate to $\{B(x) : x \in X\}$ [En, 4.4.1, 5.1.9]. For each $\varphi \in \Phi$, let x_φ be such that the support of φ is contained in $B(x_\varphi)$. Define $\delta\colon X \to \mathbb{R}$ by the formula $\delta(x) = \sum_{\varphi\in\Phi} \delta_1(x_\varphi)\varphi(x)$. Now suppose $x \in X$, $y \in Y$, and $\sigma_Y(f(x), y) < \delta(x)$. Among the finitely many values of $\varphi \in \Phi$ such that $\varphi(x) > 0$, choose one for which $\delta_1(x_\varphi)$ is maximal. Note that $x \in B(x_\varphi)$ and $\delta(x) \le \delta_1(x_\varphi)$. From (1.2), with $a = x_\varphi$, we get the desired conclusion. ■

We use standard multi-index notation. If $\alpha = (\alpha_1, \dots, \alpha_t)$ and $\beta = (\beta_1, \dots, \beta_t)$ are t-tuples of nonnegative integers and $z = (z_1, \dots, z_t)$ is a t-tuple of complex numbers, then we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_t, \quad f^{(\alpha)} = \partial^{\alpha_1+\cdots+\alpha_t} f/(\partial^{\alpha_1} z_1 \dots \partial^{\alpha_t} z_t),$$
$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_t}{\beta_t}, \quad z^\alpha = z_1^{\alpha_1} \dots z_t^{\alpha_t}, \text{ and } \sum_{\beta=0}^{\alpha} = \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_t=0}^{\alpha_t}.$$

Recall the formula $(fg)^{(\alpha)} = \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha-\beta)}$ for the derivative of a product. For $z \in \mathbb{C}^t$, let $|z| = (|z_1|^2 + \cdots + |z_t|^2)^{1/2}$. We refer the reader to [GF] for the basic properties of functions of several complex variables.

2 Proof of Theorem 1.2

The first lemma is a straightforward adaptation of the corresponding fact from [H2] to the present setting. We give the proof for completeness.

Lemma 2.1 (Cf. [H2, Lemma (b)]) *Suppose* $0 \le c_n \le c_{n+1}$ *for* $n = 0, 1, 2, \dots$ *and* $\lim_{n\to\infty} c_n = \infty$. *Let* $z_m \in \mathbb{C}^t$, $m = 0, 1, 2, \dots$, *be distinct and without accumulation points. Let* $k_m \ge 0$ *be integers and let* $B_m > 0$. *There is a continuous positive function* $D\colon \mathbb{C}^t \to \mathbb{R}$ *such that for all* $w_{m,\alpha} \in \mathbb{C}$, *where* $|\alpha| \le k_m$, $m = 0, 1, 2, \dots$, *with* $|w_{m,\alpha}| < B_m$, *there exists an entire function* ϕ *such that, for all* $m, n = 0, 1, 2, \dots$,

$$\phi^{(\alpha)}(z_m) = w_{m,\alpha}, \qquad |\alpha| \le k_m,$$

and

$$|\phi^{(\alpha)}(z)| < D(z), \qquad |z| \in [c_n, c_{n+1}],\ |\alpha| \le k_n.$$

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Proof For each $|\beta| \leq k_i$, there is, by Lemma 1.3, an entire function $b_{i,\beta}$, such that $b_{i,\beta}^{(\alpha)}(z_m) = 1$ when $(m,\alpha) = (i,\beta)$ and $b_{i,\beta}^{(\alpha)}(z_m) = 0$ for other values of (m,α), $|\alpha| \leq k_m$. Fix $\varepsilon_i > 0$ so that for each multi-index ζ, $\sum_{i=0}^{\infty} \varepsilon_i B_i \sum_{|\beta|\leq k_i} |b_{i,\beta}^{(\zeta)}(z)|$ converges uniformly on compact sets. By Lemma 1.3, there exists an entire function E, such that for $|\alpha| \leq k_m$, $m = 0, 1, 2, \ldots$, we have $E^{(\alpha)}(z_m) = 1/\varepsilon_m$ if $\alpha = 0$ and $E^{(\alpha)}(z_m) = 0$ otherwise. Fix $D\colon \mathbb{C}^t \to \mathbb{R}$ positive and continuous such that for all $n = 0, 1, 2, \ldots$, and all z such that $|z| \in [c_n, c_{n+1}]$, we have $D(z) > K_n(z)$, where

$$K_n(z) = 2^{k_n t} \sum_{|\gamma|\leq k_n} |E^{(\gamma)}(z)| \sum_{|\zeta|\leq k_n} \Bigl(\sum_{i=0}^{\infty} \varepsilon_i B_i \sum_{|\beta|\leq k_i} |b_{i,\beta}^{(\zeta)}(z)| \Bigr).$$

The infinite series converges to a continuous function by the choice of the coefficients ε_i.

Fix $w_{m,\alpha}$, $|\alpha| \leq k_m$, $m = 0, 1, 2, \ldots$ with $|w_{m,\alpha}| < B_m$. Define $\phi(z) = E(z)Q(z)$, where $Q(z) = \sum_{i=0}^{\infty} \varepsilon_i \sum_{|\beta|\leq k_i} w_{i,\beta} b_{i,\beta}(z)$. We check that ϕ is as desired. Fix m and α such that $|\alpha| \leq k_m$. The definition of E gives $\phi^{(\alpha)}(z_m) = (1/\varepsilon_m)Q^{(\alpha)}(z_m)$, and the definition of $b_{i,\beta}$ gives $Q^{(\alpha)}(z_m) = \varepsilon_m w_{m,\alpha}$. Hence $\phi^{(\alpha)}(z_m) = w_{m,\alpha}$. Noting that for $|\alpha| \leq k_n$ we have $\binom{\alpha}{\beta} = \prod_{i=1}^{t} \binom{\alpha_i}{\beta_i} \leq 2^{k_n t}$, we get for $|z| \in [c_n, c_{n+1}]$, $|\alpha| \leq k_n$ that $|\phi^{(\alpha)}(z)| \leq K_n(z) < D(z)$. ■

Lemma 2.2 (Cf. [H2, Lemma (c)]) *Assume* $0 \leq c_n \leq c_{n+1}$, $n = 0, 1, 2, \ldots$, *with* $\lim_{n\to\infty} c_n = \infty$, *and* $d_n > 0$, $n = 0, 1, 2, \ldots$. *Let* $h\colon \mathbb{R}^t \to \mathbb{R}$ *be a positive continuous function. There exists an entire function* $w\colon \mathbb{C}^t \to \mathbb{C}$, *positive on* $\mathbb{R}^t$, *such that* $u = (1/w)|_{\mathbb{R}^t}$ *(the restriction of* $1/w$ *to* $\mathbb{R}^t$*) satisfies*

$$\Bigl|\frac{u^{(\alpha)}(x)}{u(x)}\Bigr| \frac{1}{[u(x)]^{d_n}} < h(x), \qquad |x| \geq c_n,\ |\alpha| \leq n, \tag{2.1}$$

for each $n = 0, 1, 2, \ldots$.

Proof We begin with the following reduction.

Claim 2.3 It suffices to find a positive C^∞ function $u\colon \mathbb{R}^t \to \mathbb{R}$ satisfying (2.1).

Proof The point is that $b = 1/u$ is a positive C^∞ function which satisfies

$$\Bigl|\Bigl[\frac{1}{b(x)}\Bigr]^{(\alpha)}\Bigr| [b(x)]^{1+d_n} < h(x), \qquad |x| \geq c_n,\ |\alpha| \leq n, \tag{2.2}$$

As we will explain shortly, it then follows that there is an entire function w which is positive on $\mathbb{R}^t$ and satisfies (2.2). Then $u = (1/w)|_{\mathbb{R}^t}$ satisfies (2.1). To get w, first note that by induction on $|\alpha|$ we have $[1/b]^{(\alpha)} = P_\alpha(1/b,\ (b^{(\beta)})_{\beta\leq\alpha})$ for some polynomial $P_\alpha(z,\ (y_\beta)_{\beta\leq\alpha})$. Let $\bar{h}\colon \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function satisfying

$$\bar{h}(x) < h(x) - \Bigl|\Bigl[\frac{1}{b(x)}\Bigr]^{(\alpha)}\Bigr| [b(x)]^{1+d_n}, \qquad |x| \geq c_n,\ |\alpha| \leq n,\ n = 0, 1, 2, \ldots.$$

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Let $T_{\alpha,n}\colon \mathbb{R}^{>0} \times \mathbb{R}^{\{\beta:\beta\leq\alpha\}} \to \mathbb{R}$ be given by

$$T_{\alpha,n}(z,\ (y_\beta)_{\beta\leq\alpha}) = |P_\alpha(1/z,\ (y_\beta)_{\beta\leq\alpha})|z^{1+d_n}, \qquad |\alpha| \leq n,\ n = 0, 1, 2, \dots,$$

and let $U\colon \mathbb{R}^t \to \mathbb{R}^{>0} \times \mathbb{R}^{\{\beta:\beta\leq\alpha\}}$ be given by $U(x) = (b(x),\ (b^{(\beta)}(x))_{\beta\leq\alpha})$. The left hand side of (2.2) is $T_{\alpha,n}(U(x))$. Using Lemma 1.4, get positive continuous functions $\delta_{\alpha,n}\colon \mathbb{R}^t \to \mathbb{R}$ such that

$$\|U(x) - y\|_\infty < \delta_{\alpha,n}(x) \ \Rightarrow\ |T_{\alpha,n}(U(x)) - T_{\alpha,n}(y)| < \bar{h}(x), \qquad \left(x \in \mathbb{R}^t,\ y \in \mathbb{R}^{>0} \times \mathbb{R}^{\{\beta:\beta\leq\alpha\}}\right).$$

Let $\delta\colon \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function such that

$$\delta(x) < \delta_{\alpha,n}(x), \qquad |x| \geq c_n,\ |\alpha| \leq n,\ n = 0, 1, 2, \dots.$$

Theorem 1.1 gives an entire function w, real-valued on $\mathbb{R}^t$, so that

$$|b^{(\alpha)}(x) - w^{(\alpha)}(x)| < \delta(x), \qquad |x| \geq c_n,\ |\alpha| \leq n.$$

Suppose $|x| \geq c_n$, $|\alpha| \leq n$. Let $y = (w(x),\ (w^{(\beta)}(x))_{\beta\leq\alpha})$. We have

$$\|U(x) - y\|_\infty < \delta(x) < \delta_{\alpha,n}(x),$$

and hence $|T_{\alpha,n}(U(x)) - T_{\alpha,n}(y)| < \bar{h}(x)$ which gives

$$T_{\alpha,n}(y) < T_{\alpha,n}(U(x)) + \bar{h}(x) < h(x).$$

Thus, w satisfies (2.2) and Claim 2.3 is established. ■

We now aim to find a positive C^∞ function $u\colon \mathbb{R}^t \to \mathbb{R}$ satisfying (2.1). It will be enough to show that for all choices of $d_n > 0$ and nonnegative integers k_n, $n = 0, 1, 2, \dots$, and for every positive continuous function $h\colon \mathbb{R}^t \to \mathbb{R}$, there exists a C^∞ function $u\colon \mathbb{R}^t \to \mathbb{R}$ such that $u(x) > 1$ for every $x \in \mathbb{R}^t$ and for all $n = 0, 1, 2, \dots$

$$\Big|\frac{u^{(\alpha)}(x)}{u(x)}\Big| \frac{1}{[u(x)]^{d_n}} < h(x), \qquad |x| \in [n, n+1],\ |\alpha| \leq k_n.$$

Claim 2.4 Each term $u^{(\alpha)}/u^{1+q}$, $q > 0$ can be written as a linear combination

$$\sum_{\ell=1}^{m} \lambda_\ell \prod_{i=1}^{p_\ell} \Big[\frac{1}{u^{q_{\ell,i}}}\Big]^{(\beta_{\ell,i})},$$

where $m = m(\alpha) \in \mathbb{N}$ depends on α and for $\ell = 1, \dots, m$,

- $\lambda_\ell = \lambda_\ell(\alpha, q) \in \mathbb{R}$ depends on α and q
- $p_\ell = p_\ell(\alpha) \in \mathbb{N}$ and $\beta_{\ell,i} = \beta_{\ell,i}(\alpha) \leq \alpha$ (for $i = 1, \dots, p_\ell$) depend on α
- $q_{\ell,i} = q_{\ell,i}(\alpha, q) > 0$ depends on α and q for $i = 1, \dots, p_\ell$

Proof Build the desired expressions by recursion on $|\alpha|$ using the fact that when $\alpha_j > 0$,

$$\frac{u^{(\alpha)}}{u^{1+q}} = D_j\Big(\frac{u^{(\beta)}}{u^{1+q}}\Big) + (1+q)\frac{u^{(\beta)}}{u^{1+q/2}}\frac{u^{(\delta)}}{u^{1+q/2}},$$

where $\beta = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots, \alpha_n)$, $\delta = (0, \dots, 0, 1, 0, \dots, 0)$ (the 1 is the j-th coordinate), and D_j is the partial derivative with respect to the j-th variable. ■

We now make a further reduction.

Claim 2.5 It suffices to show that given nonnegative integers k_n, r_n, numbers $d_{n,i} > 0$, $0 \le i \le r_n$, $n = 0, 1, 2, \dots$, and a positive continuous function $h\colon \mathbb{R}^t \to \mathbb{R}$, there exists a C^∞ function $u\colon \mathbb{R}^t \to \mathbb{R}$ such that $u > 1$, and for all $n = 0, 1, 2, \dots$,

$$\Big|\Big[\frac{1}{[u(x)]^{d_{n,i}}}\Big]^{(\alpha)}\Big| < h(x), \qquad |x| \in [n, n+1];\ |\alpha| \le k_n;\ 0 \le i \le r_n. \tag{2.3}$$

Proof From Claim 2.4, given $d_n > 0$, nonnegative integers k_n, $n = 0, 1, 2, \dots$, and a positive continuous function $h\colon \mathbb{R}^t \to \mathbb{R}$, we get

$$\frac{u^{(\alpha)}}{u^{1+d_n}} = \sum_{\ell=1}^{m(\alpha)} \lambda_\ell(\alpha, d_n) \prod_{j=1}^{p_\ell(\alpha)} \Big[\frac{1}{u^{q_{\ell,j}(\alpha,d_n)}}\Big]^{(\beta_{\ell,j}(\alpha))}$$

for each multi-index α and each $n = 0, 1, 2, \dots$. Let $d_{n,i}$, $i = 0, \dots, r_n$, list the values of $q_{\ell,j}(\alpha, d_n)$ for $j = 1, \dots, p_\ell(\alpha)$, $\ell = 1, \dots, m(\alpha)$, $|\alpha| \le k_n$. Define

$$M_n = \max\{m(\alpha) : |\alpha| \le k_n\},$$
$$L_n = 1 + \max\{|\lambda_\ell(\alpha, d_n)| : \ell = 1, \dots, m(\alpha),\ |\alpha| \le k_n\}.$$

Let $\bar{h}\colon \mathbb{R}^t \to \mathbb{R}$ be a positive continuous function such that

$$\bar{h}(x) < 1 \quad \text{and} \quad \bar{h}(x) < \frac{1}{M_n L_n} h(x), \qquad |x| \in [n, n+1].$$

Let $u\colon \mathbb{R}^t \to \mathbb{R}$ be a C^∞ function such that $u > 1$, and for all $n = 0, 1, 2, \dots$,

$$\Big|\Big[\frac{1}{[u(x)]^{d_{n,i}}}\Big]^{(\beta)}\Big| < \bar{h}(x), \qquad |x| \in [n, n+1], \quad |\beta| \le k_n, \quad 0 \le i \le r_n.$$

Fix n as well as x, α such that $|x| \in [n, n+1]$, $|\alpha| \le k_n$. We have, using the fact that $\bar{h}(x)^p < \bar{h}(x)$ for each $p \in \mathbb{N}$ and $|\beta_{\ell,j}(\alpha)| \le |\alpha| \le k_n$ for $j = 1, \dots, p_\ell(\alpha)$, $\ell = 1, \dots, m(\alpha)$,

$$\begin{aligned}\Big|\frac{u^{(\alpha)}(x)}{u(x)^{1+d_n}}\Big| &\le \sum_{\ell=1}^{m(\alpha)} |\lambda_\ell(\alpha, d_n)| \prod_{j=1}^{p_\ell(\alpha)} \Big|\Big[\frac{1}{u(x)^{q_{\ell,j}(\alpha,d_n)}}\Big]^{(\beta_{\ell,j}(\alpha))}\Big| \\ &\le \sum_{\ell=1}^{m(\alpha)} |\lambda_\ell(\alpha, d_n)| \bar{h}(x) \le \sum_{\ell=1}^{m(\alpha)} |\lambda_\ell(\alpha, d_n)| \frac{1}{M_n L_n} h(x) \le h(x).\end{aligned}$$

This proves Claim 2.5. ■

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The proof of Lemma 2.2 is now completed essentially as in [H2]. Define a C^∞ function $\zeta\colon \mathbb{R} \to \mathbb{R}$, by $\zeta(x) = (1/c)\int_0^x p(t)dt$, where $p(t) = e^{-[t(1-t)]^{-1}}$ for $t \in (0,1)$, $p(t) = 0$ for other values of t, and $c = \int_0^1 p(t)dt$. Set

(2.4)
$$\frac{1}{u(x)} = \begin{cases} \varepsilon_m^{\gamma_m} & |x| \in [2m, 2m+1], \\ \bigl\{ \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m} + (\varepsilon_m - \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m})\zeta(2m+2-|x|)\bigr\}^{\gamma_m} & |x| \in [2m+1, 2m+2], \end{cases}$$

where $m = 0, 1, 2, \dots$ and $\gamma_m > 0$ are chosen so that $1 < \gamma_0 < \gamma_1 < \gamma_2 < \cdots$ and

$$\gamma_m d_{2m+1,i} - k_{2m+1} > 0, \qquad 0 \le i \le r_{2m+1}. \tag{2.5}$$

The ε_m are chosen so that $\varepsilon_m > 0$, $\varepsilon_{m+1} < \varepsilon_m < 1/2$. We will show that u is as desired if the ε_m are small enough.

(a) $u(x) > 1$.
Note that $\varepsilon_m^{\gamma_m} < \varepsilon_m^0 = 1$ since $0 < \varepsilon_m < 1$ and $\gamma_m > 0$. Also, $0 < \gamma_m < \gamma_{m+1}$ therefore, $\gamma_{m+1}/\gamma_m > 1$, and so $\varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m} < (\frac{1}{2})^{\gamma_{m+1}/\gamma_m} < \frac{1}{2}$. We have also $\varepsilon_m - \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m} > \varepsilon_m - \varepsilon_{m+1} > 0$. Thus, the second term of the second clause of (2.4) is ≥ 0 and both terms are $< 1/2$, giving $1/u(x) < 1$ and hence $u(x) > 1$.

(b) $u \in C^\infty$.
For each m, $\zeta(2m+2-|x|)$ is C^∞ since $|x|$ is C^∞ on $\mathbb{R}^t \setminus \{0\}$. The second clause of (2.4) agrees with $\varepsilon_m^{\gamma_m}$ not only at $|x| = 2m+1$, but for all $|x| \in [2m, 2m+1]$; similarly for $|x| \in [2m+2, 2m+3]$. The reciprocal of a positive C^∞ function is a C^∞ function.

(c) (2.3) holds.
On $[2m, 2m+1]$ (2.3) holds if ε_m is small enough (independently of γ_m as long as $\gamma_m > 1$ so that $\varepsilon_m^{\gamma_m} < \varepsilon_m$). For $|x| \in [2m+1, 2m+2]$, $|\alpha| \le k_{2m+1}$, $0 \le i \le r_{2m+1}$

$$\Bigl[\frac{1}{[u(x)]^{d_{2m+1,i}}}\Bigr]^{(\alpha)} = \Bigl(\bigl\{ \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m} + \bigl(\varepsilon_m - \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m}\bigr)\zeta(2m+2-|x|)\bigr\}^{\gamma_m d_{2m+1,i}}\Bigr)^{(\alpha)},$$

which equals a linear combination (the form of which depends only on α) of products of constants not depending on ε_m or ε_{m+1}, powers of $(\varepsilon_m - \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m})$, derivatives of $x \mapsto \zeta(2m+2-|x|)$ (which are bounded on $[2m, 2m+2]$) and expressions

$$\bigl(\varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m} + (\varepsilon_m - \varepsilon_{m+1}^{\gamma_{m+1}/\gamma_m})\zeta(2m+2-|x|)\bigr)^{\gamma_m d_{2m+1,i} - j} \qquad 0 \le j \le k_{2m+1}.$$

By (2.5) the exponents are > 0, therefore taking the ε_m's small enough gives (2.3).

Proof of Theorem 1.2 We prove only (b), as (a) follows by a similar and slightly easier argument. We may assume that the sequence $\{x_m\}$ contains no repetitions. Fix until further notice a C^∞ function u such that $u > 1$ and $1/u$ is the restriction to $\mathbb{R}^t$ of an entire function. Let f be the given C^∞ function. From Theorem 1.1, we get an entire function ψ_u such that

$$|(uf)^{(\alpha)}(x) - \psi_u^{(\alpha)}(x)| < 1 \qquad |x| \in [c_n, c_{n+1}],\ |\alpha| \le n,\ n = 0, 1, 2, \dots. \tag{2.6}$$

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In Lemma 2.1, take $B_m = 1$ and choose $k_m \geq m$ larger than the largest n such that $|x_m| \in [c_n, c_{n+1}]$ to get a positive continuous function $D\colon \mathbb{C}^t \to \mathbb{R}$ (independent of u) as in the statement of the lemma. Take for $|\alpha| \leq k_m$

$$w_{m,\alpha} = \begin{cases} (uf)^{(\alpha)}(x_m) - \psi_u^{(\alpha)}(x_m) & \text{if there exists an } n \text{ such that } |\alpha| \leq n \text{ and} \\ & \quad |x_m| \in [c_n, c_{n+1}] \text{ (and therefore } n \leq k_m), \\ 0 & \text{otherwise} \end{cases}$$

By (2.6) we have $|w_{m,\alpha}| < 1 = B_m$, $|\alpha| \leq k_m$. By the choice of D, there exists an entire function ϕ_u satisfying

$$\phi_u^{(\alpha)}(x_m) = w_{m,\alpha}, \qquad |\alpha| \leq k_m,$$

and

$$|\phi_u^{(\alpha)}(z)| < D(z), \qquad |z| \in [c_n, c_{n+1}],\ |\alpha| \leq k_n.$$

Therefore, for all $n = 0, 1, 2, \dots,$

$$\phi_u^{(\alpha)}(x_m) = (uf)^{(\alpha)}(x_m) - \psi_u^{(\alpha)}(x_m), \qquad |x_m| \in [c_n, c_{n+1}],\ |\alpha| \leq n, \tag{2.7}$$

$$|\phi_u^{(\alpha)}(x)| < D(x), \qquad |x| \in [c_n, c_{n+1}],\ |\alpha| \leq n \qquad (\text{since } k_n \geq n). \tag{2.8}$$

There is an entire function g_u defined on $\mathbb{R}^t$ by

$$g_u(x) = \frac{\phi_u(x)}{u(x)} + \frac{\psi_u(x)}{u(x)} \tag{2.9}$$

($1/u$ is entire and so are ϕ_u, ψ_u). For $|x_m| \in [c_n, c_{n+1}]$, $|\alpha| \leq n$, we have, using (2.9) and (2.7),

$$(ug_u)^{(\alpha)}(x_m) = \phi_u^{(\alpha)}(x_m) + \psi_u^{(\alpha)}(x_m) = (uf)^{(\alpha)}(x_m),$$

from which the desired interpolation property for the functions g_u, namely that $g_u^{(\alpha)}(x_m) = f^{(\alpha)}(x_m)$ if $|x_m| \in [c_n, c_{n+1}]$ and $|\alpha| \leq n$, follows by a straightforward induction on $|\alpha|$.

For $|x| \in [c_n, c_{n+1}]$, $|\alpha| \leq n$, (2.9), (2.6), and (2.8) give

$$\begin{aligned} |(uf)^{(\alpha)}(x) - (ug_u)^{(\alpha)}(x)| &= |(uf)^{(\alpha)}(x) - \phi_u^{(\alpha)}(x) - \psi_u^{(\alpha)}(x)| \\ &< |\phi_u^{(\alpha)}(x)| + 1 < D(x) + 1. \end{aligned}$$

Using

$$\begin{aligned} |u(x) f^{(\alpha)}(x) - u(x) g_u^{(\alpha)}(x)| \leq{}& |(uf)^{(\alpha)}(x) - (ug_u)^{(\alpha)}(x)| \\ &+ \sum_{\substack{\beta=0 \\ \beta\neq\alpha}}^{\alpha} \binom{\alpha}{\beta} \Big| u^{(\alpha-\beta)}(x) \Big(f^{(\beta)}(x) - g_u^{(\beta)}(x) \Big) \Big|, \end{aligned}$$

we get

$$(2.10)\quad |f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| < \frac{D(x)+1}{u(x)} + A_\alpha \sum_{\substack{\beta\le\alpha\\ \beta\ne\alpha}} |f^{(\beta)}(x) - g_u^{(\beta)}(x)| \frac{|u^{(\alpha-\beta)}(x)|}{u(x)},$$

$$|x| \in [c_n, c_{n+1}],\ |\alpha| \le n,$$

where $A_\alpha = \max\{\binom{\alpha}{\beta} : \beta \le \alpha\}$.

Claim 2.6 For $|x| \in [c_n, c_{n+1}]$, $|\alpha| \le n$, we have

$$(2.11)\quad |f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| < B_\alpha [D(x)+1] \sum_{j=1}^{|\alpha|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\le\mu_\beta<j (\beta\le\alpha)\\ \sum_{\beta\le\alpha}\mu_\beta<j}} \prod_{\beta\le\alpha} |u^{(\beta)}(x)|^{\mu_\beta},$$

where B_α is a positive constant depending only on α.

Proof We proceed by induction on $|\alpha|$. For $\alpha = 0$, (2.10) gives

$$|f(x) - g_u(x)| < \frac{D(x)+1}{u(x)},$$

which has the form of (2.11) for $\alpha = 0$ with $B_0 = 1$.

For the induction step, apply (2.11) for $\beta \le \alpha$, $\beta \ne \alpha$, to the terms in (2.10) to get that for $|x| \in [c_n, c_{n+1}]$,

$$\begin{aligned}(2.12)\quad |f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| &< \frac{D(x)+1}{u(x)}\\ &+ A_\alpha \sum_{\substack{\beta\le\alpha\\ \beta\ne\alpha}} B_\beta [D(x)+1] \frac{|u^{(\alpha-\beta)}(x)|}{u(x)} \sum_{j=1}^{|\beta|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\le\mu_\gamma<j (\gamma\le\beta)\\ \sum_{\gamma\le\beta}\mu_\gamma<j}} \prod_{\gamma\le\beta} |u^{(\gamma)}(x)|^{\mu_\gamma}\\ &< \frac{D(x)+1}{u(x)} \Big[1 + K_\alpha \sum_{\substack{\beta\le\alpha\\ \beta\ne\alpha}} |u^{(\alpha-\beta)}(x)| \sum_{j=1}^{|\beta|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\le\mu_\gamma<j (\gamma\le\beta)\\ \sum_{\gamma\le\beta}\mu_\gamma<j}} \prod_{\gamma\le\beta} |u^{(\gamma)}(x)|^{\mu_\gamma}\Big],\end{aligned}$$

where $K_\alpha = A_\alpha \cdot \max\{B_\beta : \beta \le \alpha\}$. Notice that if $\beta \le \alpha$, $\beta \ne \alpha$, $0 \le \mu_\gamma < j$ for $\gamma \le \beta$ and $\sum_{\gamma\le\beta}\mu_\gamma < j$, then

$$|u^{(\alpha-\beta)}(x)| \cdot \prod_{\gamma\le\beta} |u^{(\gamma)}(x)|^{\mu_\gamma} = \prod_{\gamma\le\alpha} |u^{(\gamma)}(x)|^{\mu'_\gamma},$$

where for $\gamma \neq \alpha - \beta$ we define $\mu'_\gamma = \mu_\gamma$ if $\gamma \leq \beta$ and $\mu'_\gamma = 0$ if $\gamma \not\leq \beta$. Then we define $\mu'_{\alpha-\beta} = \mu_{\alpha-\beta} + 1$ if $\alpha - \beta \leq \beta$ and $\mu'_{\alpha-\beta} = 1$ if $\alpha - \beta \not\leq \beta$. Hence

$$|u^{(\alpha-\beta)}(x)| \sum_{\substack{0\leq\mu_\gamma<j(\gamma\leq\beta)\\ \sum_{\gamma\leq\beta}\mu_\gamma<j}} \prod_{\gamma\leq\beta} |u^{(\gamma)}(x)|^{\mu_\gamma} \leq \sum_{\substack{0\leq\mu_\gamma<j+1(\gamma\leq\alpha)\\ \sum_{\gamma\leq\alpha}\mu_\gamma<j+1}} \prod_{\gamma\leq\alpha} |u^{(\gamma)}(x)|^{\mu_\gamma}.$$

Write M_α for the number of multi-indices β such that $\beta \leq \alpha$. From (2.12) we get

$$\begin{aligned}
&|f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| \\
&< \frac{D(x)+1}{u(x)}\Big[1 + K_\alpha \sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}} \sum_{j=1}^{|\beta|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\leq\mu_\gamma<j+1(\gamma\leq\alpha)\\ \sum_{\gamma\leq\alpha}\mu_\gamma<j+1}} \prod_{\gamma\leq\alpha} |u^{(\gamma)}(x)|^{\mu_\gamma}\Big] \\
&\leq \frac{D(x)+1}{u(x)}(1+K_\alpha)\Big[1 + \sum_{\substack{\beta\leq\alpha\\ \beta\neq\alpha}} \sum_{j=1}^{|\beta|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\leq\mu_\gamma<j+1(\gamma\leq\alpha)\\ \sum_{\gamma\leq\alpha}\mu_\gamma<j+1}} \prod_{\gamma\leq\alpha} |u^{(\gamma)}(x)|^{\mu_\gamma}\Big] \\
&\leq (D(x)+1)(1+K_\alpha)\Big[\frac{1}{u(x)} + M_\alpha \sum_{j=2}^{|\alpha|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\leq\mu_\gamma<j(\gamma\leq\alpha)\\ \sum_{\gamma\leq\alpha}\mu_\gamma<j}} \prod_{\gamma\leq\alpha} |u^{(\gamma)}(x)|^{\mu_\gamma}\Big] \\
&\leq (D(x)+1)(1+K_\alpha)M_\alpha\Big[\sum_{j=1}^{|\alpha|+1} \frac{1}{[u(x)]^j} \sum_{\substack{0\leq\mu_\gamma<j(\gamma\leq\alpha)\\ \sum_{\gamma\leq\alpha}\mu_\gamma<j}} \prod_{\gamma\leq\alpha} |u^{(\gamma)}(x)|^{\mu_\gamma}\Big].
\end{aligned}$$

This proves Claim 2.6. ∎

In (2.11) we have

$$\begin{aligned}
\frac{1}{[u(x)]^j} \sum_{\substack{0\leq\mu_\beta<j(\beta\leq\alpha)\\ \sum_{\beta\leq\alpha}\mu_\beta<j}} \prod_{\beta\leq\alpha} |u^{(\beta)}(x)|^{\mu_\beta} &\leq \sum_{\substack{0\leq\mu_\beta<j(\beta\leq\alpha)\\ \sum_{\beta\leq\alpha}\mu_\beta<j}} \Big(\prod_{\beta\leq\alpha} \frac{|u^{(\beta)}(x)|^{\mu_\beta}}{[u(x)]^{\mu_\beta}}\Big) \cdot \frac{1}{[u(x)]} \\
&\qquad \text{(since } u(x) > 1 \text{ and } (\sum_{\beta\leq\alpha}\mu_\beta) + 1 \leq j) \\
&\leq \sum_{\substack{0\leq\mu_\beta<j(|\beta|\leq n)\\ \sum_{|\beta|\leq n}\mu_\beta<j}} \prod_{|\beta|\leq n} \Big\{ \Big| \frac{u^{(\beta)}(x)}{u(x)} \Big|^{\mu_\beta} \cdot \frac{1}{u(x)^{1/L_n}} \Big\},
\end{aligned}$$

where L_n is the number of multi-indices β such that $|\beta| \le n$. This yields

(2.13)
$$|f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| < C_n[D(x)+1] \sum_{\substack{0\le\mu_\beta<n+1(|\beta|\le n)\\ \sum_{|\beta|\le n}\mu_\beta<n+1}} \prod_{|\beta|\le n} \Big\{ \Big|\frac{u^{(\beta)}(x)}{u(x)}\Big|^{\mu_\beta} \frac{1}{u(x)^{1/L_n}} \Big\},$$
$$|x| \in [c_n, c_{n+1}],\ |\alpha| \le n,$$

where $C_n = (n+1)\cdot\max\{B_\alpha : |\alpha| \le n\}$. Define $D_n = \sup_{|x|\in[c_n,c_{n+1}]} D(x)+1$. Choose a positive continuous function $H\colon \mathbb{R}^t \to \mathbb{R}$ so that $0 < H(x) < 1$ and for each $n = 0, 1, 2, \dots$ we have $C_nD_n(n+1)^{L_n}H(x) < h(x)$ when $|x| \in [c_n, c_{n+1}]$. Lemma 2.2 (taking $c_0 = 0$, $d_0 = 1$) gives an entire function w such that $u = (1/w)|_{\mathbb{R}^t}$ satisfies

(2.14)
$$\Big|\frac{u^{(\beta)}(x)}{u(x)}\Big| \frac{1}{u(x)^{1/[nL_n]}} < H(x), \quad |x| \ge c_n,\ |\beta| \le n,\ n \ge 1,$$
$$\frac{1}{u(x)} < H(x), \quad x \in \mathbb{R}^t.$$

Note that (2.14) gives $u(x) > 1$.

Fix $|x| \in [c_n, c_{n+1}]$, $|\alpha| \le n$.

Claim 2.7 $\prod_{|\beta|\le n}\Big\{ |u^{(\beta)}(x)/u(x)|^{\mu_\beta}\,(1/u(x)^{1/L_n})\Big\} \le H(x)$.

Proof For $\vec{\mu} = (\mu_\beta : |\beta| \le n)$, write $R(\vec{\mu})$ for the number of β such that $\mu_\beta > 0$ and assume first that this is not zero. We have

(2.15)
$$\prod_{|\beta|\le n} \Big\{ \Big|\frac{u^{(\beta)}(x)}{u(x)}\Big|^{\mu_\beta} \frac{1}{u(x)^{1/L_n}} \Big\} = \prod_{|\beta|\le n,\ \mu_\beta>0} \Big\{ \Big|\frac{u^{(\beta)}(x)}{u(x)}\Big| \frac{1}{u(x)^{1/[R(\vec{\mu})\mu_\beta]}} \Big\}^{\mu_\beta}.$$

For β such that $\mu_\beta > 0$, we have $1 \le R(\vec{\mu}) \le L_n$, $1 \le \mu_\beta \le n$ which gives $1 \le R(\vec{\mu})\mu_\beta \le nL_n$ and so, since $u(x) > 1$,

$$\Big(\Big|\frac{u^{(\beta)}(x)}{u(x)}\Big| \frac{1}{u(x)^{1/[R(\vec{\mu})\mu_\beta]}}\Big)^{\mu_\beta} \le \Big(\Big|\frac{u^{(\beta)}(x)}{u(x)}\Big| \frac{1}{u(x)^{1/[nL_n]}}\Big)^{\mu_\beta} \le H(x),$$

and the product of such factors in (2.15) is $\le H(x)$.

If $\mu_\beta = 0$ for all β, then the left hand side of (2.15) is equal to $1/u(x)$ and hence is $\le H(x)$ by (2.14). ■

In (2.13), each μ_β belongs to $\{0, 1, \dots, n\}$, therefore there are $\le (n+1)^{L_n}$ indices for the sum. We now get from (2.13) that

$$|f^{(\alpha)}(x) - g_u^{(\alpha)}(x)| \le C_nD_n(n+1)^{L_n}H(x) < h(x).$$

The last part of the theorem follows by repeating the proof while making use of the possibility in Lemma 1.3 and Theorem 1.1 of choosing functions which are real-valued on $\mathbb{R}^t$ when these results are applied. ■

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