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Generalized D-symmetric Operators II

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Abstract. Let H be a separable, infinite-dimensional, complex Hilbert space and let $A, B \in \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the algebra of all bounded linear operators on H. Let $\delta_{AB} \colon \mathcal{L}(H) \to \mathcal{L}(H)$ denote the generalized derivation $\delta_{AB}(X) = AX - XB$. This note will initiate a study on the class of pairs (A, B) such that $\overline{\mathcal{R}(\delta_{AB})} = \overline{\mathcal{R}(\delta_{A^*B^*})}$.

1 Introduction

Let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on an infinite dimensional complex Hilbert space H. For an operator A in $\mathcal{L}(H)$, the inner derivation on A, δ_A, is defined on $\mathcal{L}(H)$ by $\delta_A(X) = AX - XA$ for each X in $\mathcal{L}(H)$.The generalized derivation operator δ_{AB} associated with (A, B), defined on $\mathcal{L}(H)$ by $\delta_{AB}(X) = AX - XB$ has been much studied, and many of its spectral and metric properties are known (see [2, 6, 7, 9]).

J. G. Stampfli [8], J. H. Anderson, J. W. Bunce, J. A. Deddens, and J. P. Williams [1], and S. Bouali and J. Charles [4, 5] gave some properties and characterizations of D-symmetric operators, the class of operators that induce derivations for which the norm closures of their ranges are self-adjoint. In order to generalize these results, we initiate the study of a more general class of D-symmetric operators, in other words, the class of pairs of operators $A, B \in \mathcal{L}(H)$ that have $\overline{\mathcal{R}(\delta_{AB})} = \overline{\mathcal{R}(\delta_{A^*B^*})}$, where $\overline{\mathcal{R}(\delta_{AB})}$ is the norm closure of the range of δ_{AB}. We call such pairs D*-symmetric.

Notations

(i) For $A \in \mathcal{L}(H)$, $\sigma(A)$ is the spectrum of A.
(ii) Let $\mathcal{K}(H)$ be the ideal of all compact operators. For $A \in \mathcal{L}(H)$, let $[A]$ denote the coset of A in the Calkin algebra $\mathcal{C}(H) = \mathcal{L}(H)/\mathcal{K}(H)$.
(iii) $\mathcal{C}_1(H)$ is the ideal of trace class operators.
(iv) For $A, B \in \mathcal{L}(H)$, $\overline{\mathcal{R}(\delta_{AB})}^U$ denotes the ultraweak closure of $\mathcal{R}(\delta_{AB})$, and $\mathcal{L}(H)'^U$ denotes the continuous linear forms in the ultraweak topology.
(v) Let M be a subspace of $\mathcal{L}(H)$. We denote the orthogonal of M in the dual space of $\mathcal{L}(H)$, $\mathcal{L}(H)'$, by M^o.
(vi) For g and ω two vectors in H, we define $g \otimes \omega \in \mathcal{L}(H)$ as follows:

$$g \otimes \omega(x) = \langle x, \omega \rangle g \text{ for all } x \in H.$$

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2 D*-symmetric Pairs

Definition 2.1 Let $A, B \in \mathcal{L}(H)$. If $\overline{\mathcal{R}(\delta_{AB})} = \overline{\mathcal{R}(\delta_{A^*B^*})}$, we say that (A, B) is D*-symmetric.

Theorem 2.2 *Let $A, B \in \mathcal{L}(H)$. If A and B are D-symmetric operators with disjoint spectra, then (A, B) is D*-symmetric.*

Proof Let $X \in \overline{\mathcal{R}(\delta_{AB})}$. There exists a sequence $(X_n)_n \subset \mathcal{L}(H)$ such that $\|\delta_{AB}(X_n) - X\| \to 0$. Consider the operators on $H \oplus H$

$$M = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad M_n = \begin{pmatrix} 0 & X_n \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

It follows that

$$\delta_T(M_n) = \begin{pmatrix} 0 & \delta_{AB}(X_n) \\ 0 & 0 \end{pmatrix} \xrightarrow{\|\cdot\|} \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = M.$$

Thus $M \in \overline{\mathcal{R}(\delta_T)}$. Since A and B are D-symmetric operators with disjoint spectra, then T is D-symmetric by J. G. Stampfli [8, p. 260]. Hence there exists a sequence $(N_n)_n \subset \mathcal{L}(H \oplus H)$ such that $\delta_{T^*}(N_n) \xrightarrow{\|\cdot\|} M$. A simple calculation proves that there exists a sequence $(Y_n)_n \subset \mathcal{L}(H)$, such that $\delta_{A^*B^*}(Y_n) \xrightarrow{\|\cdot\|} X$. Thus $\overline{\mathcal{R}(\delta_{AB})} \subset \overline{\mathcal{R}(\delta_{A^*B^*})}$. We obtain the reverse inclusion in the same way. ■

Remark 2.3 Let A and B be two cyclic subnormal operators with disjoint spectra. A and B are D-symmetric operators by [4, Thm. 2.5]. Since $\sigma(A) \cap \sigma(B) = \varnothing$, Theorem 2.2 implies that (A, B) is D*-symmetric.

Theorem 2.4 *For A, B in $\mathcal{L}(H)$ the following are equivalent:*

(i) *(A, B) is D*-symmetric;*
(ii) *$\delta_{A^*}(A)\mathcal{L}(H) + \mathcal{L}(H)\delta_{B^*}(B) \subseteq \overline{\mathcal{R}(\delta_{AB})} \cap \overline{\mathcal{R}(\delta_{A^*B^*})}$;*
(iii) *$A^*\mathcal{R}(\delta_{AB}) + \mathcal{R}(\delta_{AB})B^* \subseteq \overline{\mathcal{R}(\delta_{AB})}$ and $A\mathcal{R}(\delta_{A^*B^*}) + \mathcal{R}(\delta_{A^*B^*})B \subseteq \overline{\mathcal{R}(\delta_{A^*B^*})}$.*

Proof (i) ⇒ (ii). For all $X \in \mathcal{L}(H)$ we have

$$\delta_{A^*}(A)X = \delta_{A^*B^*}(AX) - A\delta_{A^*B^*}(X) \quad \text{and} \quad X\delta_{B^*}(B) = \delta_{AB}(X)B^* - \delta_{AB}(XB^*).$$

Since $A\mathcal{R}(\delta_{A^*B^*}) \subseteq A\overline{\mathcal{R}(\delta_{AB})} \subseteq \overline{\mathcal{R}(\delta_{AB})}$ and $\mathcal{R}(\delta_{AB})B^* \subseteq \overline{\mathcal{R}(\delta_{A^*B^*})}B^* \subseteq \overline{\mathcal{R}(\delta_{AB})}$, it follows that

$$\delta_{A^*}(A)\mathcal{L}(H) + \mathcal{L}(H)\delta_{B^*}(B) \subseteq \overline{\mathcal{R}(\delta_{AB})}.$$

The implication (ii) ⇒ (iii) is a consequence of the following identities. For all X and Y in $\mathcal{L}(H)$,

$$A^*\delta_{AB}(X) + \delta_{AB}(Y)B^* = \delta_{AB}(A^*X + YB^*) + \delta_{A^*}(A)X + Y\delta_{B^*}(B)$$

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and

$$A\delta_{A^*B^*}(X) + \delta_{A^*B^*}(Y)B = \delta_{A^*B^*}(AX + YB) - \delta_{A^*}(A)X - Y\delta_{B^*}(B).$$

(iii) ⇒ (i). Suppose that (iii) holds. Then $A^{*n}\mathcal{R}(\delta_{AB}) \subseteq \overline{\mathcal{R}(\delta_{AB})}$ for each n in $\mathbb{N}$. We always have the inclusion $A^m\mathcal{R}(\delta_{AB}) \subseteq \overline{\mathcal{R}(\delta_{AB})}$ for each m in $\mathbb{N}$.

We shall prove that $\mathcal{R}(\delta_{AB})^o = \mathcal{R}(\delta_{A^*B^*})^o$. Let $f \in \mathcal{R}(\delta_{AB})^o$ and $X \in \mathcal{L}(H)$. Observe that

$$A^{*n}AX - AA^{*n}X = A^{*n}\delta_{AB}(X) - \delta_{AB}(A^{*n}X)$$

for each n in $\mathbb{N}$. Hence $A^{*n}AX - AA^{*n}X \in \overline{\mathcal{R}(\delta_{AB})}$ for each n in $\mathbb{N}$. A similar argument using mathematical induction on m shows that $A^{*n}A^mX - A^mA^{*n}X \in \overline{\mathcal{R}(\delta_{AB})}$ for each n and m in $\mathbb{N}$. Thus $f(A^{*n}A^mX) = f(A^mA^{*n}X)$ for each n and m in $\mathbb{N}$. It follows that $f(e^{\alpha A}e^{\beta A^*}X) = f(e^{\beta A^*}e^{\alpha A}X)$ for all complex numbers α and β.

An induction argument shows that

$$f\big((\alpha A + \beta A^*)^n X\big) = \sum_{k=0}^{n} \binom{n}{k} f\big((\alpha A)^k(\beta A^*)^{n-k}X\big)$$

for each n in $\mathbb{N}$ and for all complex numbers α and β. Hence

$$f(e^{\alpha A+\beta A^*}X) = f(e^{\alpha A}e^{\beta A^*}X) = f(e^{\beta A^*}e^{\alpha A}X)$$

for each X in $\mathcal{L}(H)$ and for all complex numbers α and β. A similar argument using $\mathcal{R}(\delta_{AB})B^* \subseteq \overline{\mathcal{R}(\delta_{AB})}$ shows that

$$f(Xe^{\alpha B+\beta B^*}) = f(Xe^{\alpha B}e^{\beta B^*}) = f(Xe^{\beta B^*}e^{\alpha B})$$

for each X in $\mathcal{L}(H)$ and for all complex numbers α and β.

Since $f(AX) = f(XB)$, it follows by induction that $f(A^nX) = f(XB^n)$ for all $n \in \mathbb{N}$, and hence $f(e^{\alpha A}X) = f(Xe^{\alpha B})$ or $f(e^{\alpha A}Xe^{-\alpha B}) = f(X)$ for all $\alpha \in \mathbb{C}$ and $X \in \mathcal{L}(H)$. These relations yield, for all $\lambda \in \mathbb{C}$, the equations

$$\begin{aligned} f(e^{\imath\lambda A^*}Xe^{-\imath\lambda B^*}) &= f(e^{\imath\overline{\lambda}A}e^{\imath\lambda A^*}Xe^{-\imath\lambda B^*}e^{-\imath\overline{\lambda}B}) \\ &= f(e^{\imath(\overline{\lambda}A+\lambda A^*)}Xe^{-\imath(\lambda B^*+\overline{\lambda}B)}). \end{aligned}$$

Define the function g on $\mathbb{C}$ as follows:

$$g(\lambda) = f(e^{\imath\lambda A^*}Xe^{-\imath\lambda B^*}).$$

Since $\overline{\lambda}A + \lambda A^*$ and $\lambda B^* + \overline{\lambda}B$ are self-adjoint operators, then $e^{\imath(\overline{\lambda}A+\lambda A^*)}$ and $e^{-\imath(\lambda B^*+\overline{\lambda}B)}$ are unitary operators . Thus for all $\lambda \in \mathbb{C}$,

$$|g(\lambda)| \le \|f\|\|X\|.$$

By Liouville's theorem, the entire function g side must be constant. In particular, the derivative vanishes at $\lambda = 0$. This gives $f(A^*X - XB^*) = 0$ for all $X \in \mathcal{L}(H)$. Thus $\mathcal{R}(\delta_{AB})^o \subseteq \mathcal{R}(\delta_{A^*B^*})^o$. We obtain the reverse inclusion in the same way. ■

Corollary 2.5 *If A and B are normal operators, then (A, B) is D^*-symmetric.*

Corollary 2.6 *Let U and V two isometries, then (U, V) is D^*-symmetric.*

Proof Let $P = I - UU^*$. Then for all $X \in \mathcal{L}(H)$,

$$\delta_{U^*V^*}(X) = \delta_{UV}(-U^*XV^*) - PXV^*.$$

Hence, to prove that $\mathcal{R}(\delta_{U^*V^*}) \subseteq \overline{\mathcal{R}(\delta_{UV})}$, it suffices to show that $PX \in \overline{\mathcal{R}(\delta_{UV})}$ for all $X \in \mathcal{L}(H)$. Let

$$T_n = \sum_{k=0}^{n-1} \Big(\frac{k}{n} - 1\Big) U^k PXV^{*k+1}, \quad n \in \mathbb{N}^*,$$

where $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. A simple calculation shows that

$$\delta_{UV}(T_n) - PX = -\frac{1}{n} \sum_{k=1}^{n} U^k PXV^{*k}.$$

Since $\langle U^j Px, U^k Py \rangle = 0$ for $j \neq k$ and x, y in H, then

$$\Big\| \sum_{k=1}^{n} U^k PXV^{*k} x \Big\|^2 = \sum_{k=1}^{n} \|U^k PXV^{*k} x\|^2 \leq n\|PX\|^2 \|x\|^2. \tag{2.1}$$

Thus $\|\delta_{UV}(T_n) - PX\| \leq n^{-\frac{1}{2}} \|PX\|$, that is, $PX \in \overline{\mathcal{R}(\delta_{UV})}$.

For the reverse inclusion, first prove that if $Q = I - VV^*$, then $PX \in \overline{\mathcal{R}(\delta_{U^*V^*})}$ and $XQ \in \overline{\mathcal{R}(\delta_{U^*V^*})}$ for all $X \in \mathcal{L}(H)$. Let

$$S_n = \sum_{k=0}^{n-1} \Big(\frac{k}{n} - 1\Big) U^{k+1} PXV^{*k}, \quad n \in \mathbb{N}^*.$$

A simple calculation shows that

$$\delta_{U^*V^*}(S_n) + PX = \frac{1}{n} \sum_{k=1}^{n} U^k PXV^{*k}.$$

It follows from (2.1) that $\|\delta_{U^*V^*}(S_n) + PX\| \leq n^{-\frac{1}{2}} \|PX\|$. Thus $PX \in \overline{\mathcal{R}(\delta_{U^*V^*})}$. Consider

$$R_n = \sum_{k=0}^{n-1} \Big(\frac{k}{n} - 1\Big) U^{k+1} XQV^{*k}, \quad n \in \mathbb{N}^*.$$

Then

$$\delta_{U^*V^*}(R_n) + XQ = \frac{1}{n} \sum_{k=1}^{n} U^k XQV^{*k}.$$

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Hence

$$(\delta_{U^*V^*}(R_n) + XQ)^* = \frac{1}{n}\sum_{k=1}^{n} V^k Q X^* U^{*k}.$$

Thus $\|\delta_{U^*V^*}(R_n) + XQ\| \le n^{-\frac{1}{2}}\|QX^*\|$, and so $XQ \in \overline{\mathcal{R}(\delta_{U^*V^*})}$. Since

$$U\delta_{U^*V^*}(X) = \delta_{U^*V^*}(UX) - PX \quad \text{and} \quad \delta_{U^*V^*}(X)V = \delta_{U^*V^*}(XV) - XQ,$$

then

$$U\mathcal{R}(\delta_{U^*V^*}) + \mathcal{R}(\delta_{U^*V^*})V \subseteq \overline{\mathcal{R}(\delta_{U^*V^*})}.$$

It follows from the proof of Theorem 2.4 that $\overline{\mathcal{R}(\delta_{UV})} \subseteq \overline{\mathcal{R}(\delta_{U^*V^*})}$. Thus (U, V) is D*-symmetric. ■

Definition 2.7 ([3]) Let A, B be in $\mathcal{L}(H)$ and $\mathcal{J}$ be a two sided ideal of $\mathcal{L}(H)$. The pair (A, B) is said to possess the Fuglede–Putnam property $(F, P)_{\mathcal{J}}$ if $AT = TB$ and $T \in \mathcal{J}$ implies $A^*T = TB^*$.

Theorem 2.8 *For $A, B \in \mathcal{L}(H)$, the following are equivalent:*

(i) *(A, B) is D*-symmetric;*
(ii) (a) *$([A], [B])$ is D*-symmetric in $\mathcal{C}(H)$, and*
(b) *(A, B) and (B, A) have the property $(F, P)_{\mathcal{C}_1}$;*
(iii) (a) *$([A], [B])$ is D*-symmetric in $\mathcal{C}(H)$, and*
(b) $\overline{\mathcal{R}(\delta_{AB})}^U = \overline{\mathcal{R}(\delta_{A^*B^*})}^U$.

Proof Note that $\overline{\mathcal{R}(\delta_{AB})}^U = \overline{\mathcal{R}(\delta_{A^*B^*})}^U$ if and only if

$$\mathcal{R}(\delta_{AB})^o \cap \big(\mathcal{L}(H)\big)^{\prime U} = \mathcal{R}(\delta_{A^*B^*})^o \cap \big(\mathcal{L}(H)\big)^{\prime U}.$$

On the other hand

$$\mathcal{R}(\delta_{AB})^o \simeq \mathcal{R}(\delta_{AB})^o \cap \mathcal{K}(H)^o \oplus \ker(\delta_{BA}) \cap \mathcal{C}_1(H), \tag{2.2}$$

[10, Thm. 3]. In particular,

$$\mathcal{R}(\delta_{AB})^0 \cap \mathcal{L}(H)^{\prime U} \simeq \ker(\delta_{BA}) \cap \mathcal{C}_1(H).$$

This proves that $\overline{\mathcal{R}(\delta_{AB})}^U = \overline{\mathcal{R}(\delta_{A^*B^*})}^U$ if and only if

$$\ker(\delta_{BA}) \cap \mathcal{C}_1(H) = \ker(\delta_{B^*A^*}) \cap \mathcal{C}_1(H).$$

Thus (ii) $\Leftrightarrow$ (iii).

Clearly the above shows that (i) $\Rightarrow$ (iii). Suppose that (iii) holds. Let $f \in \mathcal{R}(\delta_{AB})^0$. Then by (2.2), we have $f = f_0 + f_T$ such that $f_0 \in \mathcal{R}(\delta_{AB})^o \cap \mathcal{K}(H)^o$ and $T \in \ker(\delta_{BA}) \cap \mathcal{C}_1(H)$ (where $f_T(X) = tr(XT)$ for each X in $\mathcal{L}(H)$). Since $\overline{\mathcal{R}(\delta_{AB})}^U = \overline{\mathcal{R}(\delta_{A^*B^*})}^U$, it follows that $T \in \ker(\delta_{B^*A^*}) \cap \mathcal{C}_1(H)$. Let $Z \in \mathcal{R}(\delta_{A^*B^*})$. Then

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$[Z] \in \mathcal{R}(\delta_{[A^*][B^*]})$. Since $([A],[B])$ is D*-symmetric in $\mathcal{C}(H)$, then $[Z] \in \overline{\mathcal{R}(\delta_{[A][B]})}$. There exists a sequence of operators $(X_n)_n$ in $\mathcal{L}(H)$ and a sequence $(K_n)_n$ of compact operators in $\mathcal{K}(H)$ such that $AX_n - X_nB + K_n \to Z$. But

$$f_0(AX_n - X_nB + K_n) = f_0(AX_n - X_nB) + f_0(K_n) = 0,$$

and thus $f_0(Z) = 0$. It follows that $f_0 \in \mathcal{R}(\delta_{A^*B^*})^o \cap \mathcal{K}(H)^o$, and hence $f \in \mathcal{R}(\delta_{A^*B^*})^o$. Therefore, $\mathcal{R}(\delta_{AB})^o \subseteq \mathcal{R}(\delta_{A^*B^*})^o$. We obtain the reverse inclusion using a similar argument. ∎

Corollary 2.9 *If U and V are two isometries, then (U,V) has the property $(F,P)_{\mathcal{C}_1}$.*

Proof (U,V) is D*-symmetric by Corollary 2.6. It follows from Theorem 2.8 that (U,V) has the property $(F,P)_{\mathcal{C}_1}$. ∎

Theorem 2.10 *Let $A, B \in \mathcal{L}(H)$. If there exist two nonzero elements f and g in H, and $\lambda \in \mathbb{C}$, such that $B(f) = \lambda f$, $B^*(f) \neq \overline{\lambda} f$ and $A^*(g) = \overline{\lambda} g$, then (A,B) is not D*-symmetric.*

Proof Since for all $\lambda \in \mathbb{C}$, $\mathcal{R}(\delta_{AB}) = \mathcal{R}(\delta_{(A-\lambda)(B-\lambda)})$, we may assume without loss of generality that $\lambda = 0$. Note that $B^* f = \omega \neq 0$, where $\omega \perp f$. If $X = \|\omega\|^{-2}(g \otimes \omega)$ and $Y \in \mathcal{L}(H)$, then

$$\begin{aligned}\big\langle (A^*X - XB^*)f, g\big\rangle &= \langle A^*X(f), g\rangle - \langle XB^*f, g\rangle\\ &= \langle 0, g\rangle - \langle X(\omega), g\rangle = -\langle g, g\rangle = -\|g\|^2\end{aligned}$$

and

$$\big\langle (AY - YB)f, g\big\rangle = \langle Yf, A^*g\rangle - \langle 0, g\rangle = 0.$$

Suppose that $A^*X - XB^* \in \overline{\mathcal{R}(\delta_{AB})}^U$. Then there exists a net $(Y_\alpha)_\alpha$ in $\mathcal{L}(H)$ such that for all x and y in H, we have:

$$\big\langle (AY_\alpha - Y_\alpha B)x, y\big\rangle \longrightarrow \big\langle (A^*X - XB^*)x, y\big\rangle,$$

so that

$$0 = \big\langle (AY_\alpha - Y_\alpha B)f, g\big\rangle \longrightarrow \big\langle (A^*X - XB^*)f, g\big\rangle = -\|g\|^2.$$

It follows that $g = 0$. This proves that $A^*X - XB^* \notin \overline{\mathcal{R}(\delta_{AB})}^U$, that is, $\overline{\mathcal{R}(\delta_{AB})}^U \neq \overline{\mathcal{R}(\delta_{A^*B^*})}^U$. Consequently we obtain that (A,B) is not D*-symmetric by Theorem 2.8. ∎

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