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PERIODIC POINTS AND CONTRACTIVE MAPPINGS

BY
TSU-TEH HSIEH[1] AND KOK-KEONG TAN[2]

1. **Introduction.** Let X be a non-empty set and $f:X\to X$. A point $x\in X$ is (i) a fixed point of f iff $f(x)=x$, and (ii) a periodic point of f iff there is a positive integer N such that $f^N(x)=x$. Also a periodic orbit of f is the (finite) set $\{x, f(x), f^2(x), \ldots\}$ where x is a periodic point of f. It has been of interest in investigating those mappings f satisfying the following property

(P) every periodic point of f is a fixed point.

In a metric space, it is known [2] that every contractive mapping (even every iteratively contractive mapping [3]) satisfies (P). It is also proved in [1] that every subcontractive mapping also satisfies (P). In a Hausdorff space (X, τ) whose topology τ is generated by a family $\mathscr{D}$ of pseudometrics on X, the second author [4] found that every nonexpansive and iteratively contractive mapping w.r.t. $\mathscr{D}$ also has the property (P).

In this paper, we generalize all the above results and at the same time weaken the condition of f.

2. **Definitions.** Let (X, τ) be a Hausdorff space whose topology τ is generated by a family $\mathscr{D}$ of pseudometrics on X. The family $\mathscr{D}$ is said to be saturated iff given any finite subfamily, say $\{d_1, \ldots, d_n\}$, of $\mathscr{D}$, the pseudometric d defined by $d(x, y)=\max\{d_i(x, y): i=1, \ldots, n\}$, for all $x, y\in X$, in symbol $d=d_1\vee\cdots\vee d_n$, is also in $\mathscr{D}$. If $f: X\to X$, then f is called

(i) contractive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, for any $x, y\in X$, $d(x, y)>0$ implies $d(x, y)>d(f(x), f(y))$;

(ii) iteratively contractive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, for any $x, y\in X$, $d(x, y)>0$ implies the existence of a positive integer N such that $d(x, y)>d(f^N(x), f^N(y))$;

(iii) subcontractive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, for each $x\in X$, $d(x, f(x))>0$ implies $d(x, f(x))>d(f(x), f^2(x))$;

(iv) iteratively subcontractive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, for each $x\in X$, $d(x, f(x))>0$ implies the existence of a positive integer N such that $d(x, f(x))>d(f^N(x), f^{N+1}(x))$;

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(v) nonexpansive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, $d(x,y)\geq d(f(x),f(y))$, for all $x,y\in X$;

(vi) subnonexpansive w.r.t. $\mathscr{D}$ iff for each $d\in\mathscr{D}$, for each $x\in X$, $d(x,f(x))\geq d(f(x),f^2(x))$.

It is clear from the definitions that (a) each contractive mapping w.r.t. $\mathscr{D}$ is subcontractive and iteratively contractive w.r.t. $\mathscr{D}$ and (b) each subcontractive mapping w.r.t. $\mathscr{D}$ as well as each iteratively contractive mapping w.r.t. $\mathscr{D}$ is iteratively subcontractive w.r.t. $\mathscr{D}$.

3. Main results.

THEOREM. *Let (X,d) be a pseudometric space and $f:X\to X$ be iteratively subcontractive w.r.t. $\{d\}$. Then (I) every periodic orbit of f contains some point x with $d(x,f(x))=0$; (II) if f is either subnonexpansive or contractive w.r.t. $\{d\}$, every periodic orbit of f is of diameter zero (or equivalently $d(x,f(x))=0$ for every periodic point x of f).*

Proof. Without loss of generality, we may assume that X is a periodic orbit of f with N points.

(I) If $d(x,f(x))>0$ for each $x\in X$, then the same proof of Theorem 1 in [3] applies to f and d leads to a contradiction. Hence $d(x,f(x))=0$ for some $x\in X$.

(II) Suppose f is subnonexpansive w.r.t. $\{d\}$. By (I), there exists an $x\in X$ with $d(x,f(x))=0$. But then $d(f^n(x),f^{n+1}(x))=0$, for each $n=0,1,\ldots,N-1$, so that by triangle inequality, the diameter of X is zero.

Next suppose f is contractive w.r.t. $\{d\}$. We shall make use of the substitution principle: If $d(y,z)=0$, then $d(x,y)=d(x,z)$ for all x. Suppose there is an $x_0\in X$ with $d(x_0,f(x_0))>0$, then $\varepsilon=\inf\{d(x,f(x)):x\in X \text{ and } d(x,f(x))>0\}>0$. Choose $x\in X$ with $d(x,f(x))=\varepsilon$. We shall show that

$$(*)\qquad d(f(x),f^k(x))=0, \qquad \text{for all } k=1,2,\ldots.$$

Indeed, (*) is trivial for $k=1$. Suppose $d(f(x),f^k(x))=0$, the substitution principle gives $d(x,f^k(x))=d(x,f(x))=\varepsilon>0$. Since f is contractive w.r.t. $\{d\}$, $d(f(x),f^{k+1}(x))<d(x,f^k(x))=\varepsilon$. By the definition of ε, $d(f(x),f^{k+1}(x))=0$. Hence (*) holds by induction. Thus $0<\varepsilon=d(f(x),x)=d(f(x),f^N(x))=0$, which is a contradiction. Therefore $d(x,f(x))=0$ for all $x\in X$ and the triangle inequality shows that X is of diameter zero.

COROLLARY. *Let (X,τ) be a Hausdorff space whose topology τ is generated by a family $\mathscr{D}$ of pseudometrics on X and $f:X\to X$. If either (i) f is iteratively subcontractive w.r.t. $\mathscr{D}$ and $\mathscr{D}$ is saturated, or (ii) f is iteratively subcontractive and subnonexpansive w.r.t. $\mathscr{D}$, or (iii) f is contractive w.r.t. $\mathscr{D}$, then f satisfies (P).*

Proof. Let $x\in X$ be a periodic point of f and $N=\inf\{n:f^n(x)=x\}$. Suppose f is iteratively subcontractive w.r.t. $\mathscr{D}$ and $\mathscr{D}$ is saturated. If $N>1$, then $f^i(x)\neq f^{i+1}(x)$

for each $i=0, 1, \ldots, N-1$. Since (X, τ) is Hausdorff, for each $i=0, 1, \ldots, N-1$, there exists $d_i \in \mathscr{D}$ with $d_i(f^i(x), f^{i+1}(x))>0$. Define $d=d_1 \vee \cdots \vee d_n$, then $d \in \mathscr{D}$ since $\mathscr{D}$ is saturated. But then $d(f^i(x), f^{i+1}(x))>0$ for each $i=0, 1, \ldots, N-1$, which contradicts Theorem (I). Thus we must have $N=1$ and $f(x)=x$.

Next suppose f is either both iteratively subcontractive and subnonexpansive w.r.t. $\mathscr{D}$ or is contractive w.r.t. $\mathscr{D}$. Then by Theorem (II), $d(x, f(x))=0$ for each $d \in \mathscr{D}$. But then $f(x)=x$ since (X, τ) is Hausdorff. Therefore f satisfies (P).

Corollary (i) and (ii) generalize Propositions 2.2. and 2.1. respectively due to the second author in [4] and also Theorem 1 due to Bryant and Guseman in [1].

REMARK. Even if f is (continuous and) iteratively contractive w.r.t. $\mathscr{D}$ or subcontractive w.r.t. $\mathscr{D}$, it is not hard to construct counterexamples when the conditions "$\mathscr{D}$ is saturated" and "f is subnonexpansive w.r.t. $\mathscr{D}$" are removed from Corollary (i) and (ii) respectively.

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REFERENCES

1. J. Bryant and L. F. Guseman, Jr., *Fixed points of subcontractive mappings*. To appear.
2. M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), pp. 74–79.
3. K. W. Ng, *A remark on contractive mappings*, Canadian Math. Bull. **13** (1970), pp. 111–113.
4. K. K. Tan, *Fixed point theorems for nonexpansive mappings*, Pacific J. Math. **14** (1972), pp. 829–842.

UNIVERSITY OF SASKATCHEWAN, SASKATOON, SASKATCHEWAN, CANADA
DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA