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A Generalization of Integrality

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Abstract. In this paper, we explore a generalization of the notion of integrality. In particular, we study a near-integrality condition that is intermediate between the concepts of integral and almost integral. This property (referred to as the Ω-almost integral property) is a representative independent specialization of the standard notion of almost integrality. Some of the properties of this generalization are explored in this paper, and these properties are compared with the notion of pseudo-integrality introduced by Anderson, Houston, and Zafrullah. Additionally, it is shown that the Ω-almost integral property serves to characterize the survival/lying over pairs of Dobbs and Coykendall.

1 Introduction and Background

The study of various notions of integrality has proven central in the attempt to understand the structure and overring behavior of a commutative ring with identity. In particular, domains with "nice" properties often (but not always) have nice properties with respect to their integral and almost integral closures. A standard result along these lines is the fact that a unique factorization domain (UFD) is (completely) integrally closed. Although the results for other types of "factorization domains" are not quite as strong (for example, half-factorial domains need not be integrally closed), it is the case that many of the stronger results are obtained in tandem with some type of integrality assumption. For example, it was shown in [3] that for $R[x]$ to be a half-factorial domain, it is necessary for the domain R to be integrally closed (as well as having the half-factorial property).

One of the more interesting and useful results in commutative algebra is the fact that Krull dimension is preserved in integral extensions (that is, $\dim(R) = \dim(T)$ if $R \subseteq T$ is an integral extension). Almost integral extensions, however, do not have to possess this property. We will see in this paper that Ω-almost integral extensions do not have to preserve dimension either, but the failure is tamer.

Also central to the focus of this paper is the notion of survival/lying over pairs. We review briefly here to say that an extension $R \subseteq T$ is a lying over extension if given any prime ideal $\mathfrak{P} \subseteq R$, then there is a prime ideal of T lying over $\mathfrak{P}$ (that is, there is a prime ideal $\mathfrak{Q} \subseteq T$ such that $\mathfrak{Q} \cap R = \mathfrak{P}$). The "pair" notion generalizes the more specific "extension" notion in the following sense. We say that $R \subseteq T$ is an X-pair (with X denoting some property of extensions) if given intermediate extensions A and B such that

$$R \subseteq A \subseteq B \subseteq T,$$

then $A \subseteq B$ is an X-extension.

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In this paper, we introduce the notion of Ω-almost integral elements and Ω-almost integral extensions. The inspiration for looking at this property has its roots in the theory of factorization. We recall that an almost integral element over the domain R (with quotient field K) is an element $\omega \in K$ such that there is a nonzero $r \in R$ such that $r\omega^n \in R$, for all $n \geq 1$.

We also recall that the standard proof that an integral element ξ (of the quotient field) is almost integral involves writing ξ as a fraction $\xi = \frac{a}{b}$, with $a, b \in R$. One then uses the equation of integrality

$$\left(\frac{a}{b}\right)^n + r_{n-1}\left(\frac{a}{b}\right)^{n-1} + \cdots + r_1\left(\frac{a}{b}\right) + r_0 = 0,$$

to show that b^{n-1} can function as the "r" in the definition of almost integrality.

With this in mind, we set out to define a "denominator independent" version of the almost integral property. As it turns out, a denominator independent version is precisely the key needed to classify the "near-integral" properties in survival/lying over pairs. What is more, the notion of near-integrality that we will highlight in this paper is precisely what is needed for near-integrality in Prüfer domains.

2 Pseudo-Integrality and Almost Integrality

We begin this section by recalling the pseudo-integral property introduced in [1].

Definition 2.1 Let R be a domain with quotient field K. We say that $\lambda \in K$ is pseudo-integral over R if $\lambda I^{-1} \subseteq I^{-1}$ for some nonzero finitely generated ideal $I \subseteq R$.

An alternative definition of integrality can be stated as follows: if R is an integral domain with quotient field K, an element $\omega \in K$ is integral if there is a nonzero finitely generated ideal $I \subseteq R$ such that $\omega I \subseteq I$. In [1], it was shown that the set of elements pseudo-integral over R forms a subring of K (called the pseudo-integral closure of R).

We now introduce new notions of integrality, which, we shall see, behave quite well in certain overring pair situations.

Definition 2.2 Let R be an integral domain with quotient field K. The element $\omega \in K$ is said to be Ω-almost integral if for all nonzero $b \in R$ such that $b\omega \in R$, then there is a nonnegative integer m_b such that $b^{m_b}\omega^n \in R$ for all $n \geq 1$.

We now refine the definition for the case that m_b is independent of the choice of b.

Definition 2.3 Let R be an integral domain with quotient field K and m a nonnegative integer. We say that $\omega \in K$ is m-almost integral if for all nonzero $b \in R$ such that $b\omega \in R$, then $b^m\omega^n \in R$ for all $n \geq 1$.

Definition 2.4 Let $R \subseteq T$ be an extension of domains having the same quotient field. We say that T is Ω-almost integral over R if every element of T is Ω-almost integral over R. Additionally, we say that R is Ω-almost integrally closed in T if all elements of T that are Ω-almost integral over R are contained in R. In a similar fashion, we say that T is m-almost integral over R if every element of T is m-almost integral over R, and we say that R is m-almost integrally closed in T if all elements of T that are m-almost integral over R are contained in R.

We remark that if R is a domain with quotient field K and $\omega \in K$, then we have the implications: ω is integral $\Rightarrow \omega$ is Ω-almost integral $\Rightarrow \omega$ is almost integral. The analogous chain of implications holds globally (for domains as opposed to elements).

Our first result shows that unit behavior for Ω-almost integrality is analogous to the situation for integral extensions (as opposed to general almost integral extensions).

Proposition 2.5 *Let R be a domain and suppose that $r \in R$ is an element with the property that r^{-1} is Ω-almost integral over some integral extension of R. Then $r \in U(R)$. In particular, if $R \subseteq T$ is an Ω-almost integral extension, then $U(T) \cap R = U(R)$.*

Proof Let A be our integral extension of R and note that $rr^{-1} = 1 \in R \subseteq A$. Since r^{-1} is Ω-almost integral over A, there is a nonnegative integer k such that for all $n \geq 1$, $r^k r^{-n} \in A$. In particular, $r^{-1} \in A$ and so is integral over R. Hence it follows that [8, Theorem 15] $r \in U(R)$. The second statement follows easily. ■

Example 2.6 Consider the extension of domains $\mathbb{Z} + 2x\mathbb{Z}[x] \subseteq \mathbb{Z}[x]$. It is straightforward to check that this extension is, in fact, a 1-almost integral extension that is not an integral extension. This should be contrasted with the case of valuation domains of dimension greater than 1. Indeed, if V_k is a k-dimensional valuation domain with $k \geq 2$ and maximal ideal $\mathfrak{M}$, we select $x \in \mathfrak{M} \setminus \mathfrak{P}$, where $\mathfrak{P}$ is the prime ideal in V_k of coheight one. We also select a nonzero element p that is contained in a nonmaximal prime ideal of V_k. We note that $px^{-n} \in V_k$ for all $n \geq 1$, and hence x^{-1} is almost integral. However, x is a unit in any proper overring of V_k, and hence V_k has no proper Ω-almost integral extension.

We now provide an example that establishes the existence of m-almost integral elements for all m.

Example 2.7 Let $\mathbb{F}$ be a field. We consider the subring of $\mathbb{F}(x, y)$ given by

$$R := \mathbb{F}\Big[y, \frac{y}{x}, \frac{y^m}{x}, \frac{y^m}{x^2}, \frac{y^m}{x^3}, \frac{y^m}{x^4}, \dots\Big].$$

Note that $\frac{1}{x}$ is an element of the quotient field of R, and it is easy to see that, in fact, $\frac{1}{x}$ is m-almost integral (indeed, by construction, if $r \in R$ is such that $r\frac{1}{x}$ is in R, then r is an element of $(y, \frac{y}{x}, \frac{y^m}{x}, \frac{y^m}{x^2}, \dots)$, and hence $r^m(1/x)^n \in R$ for all $n \geq 1$). If we consider the element y, it is easy to see that $y^m(1/x)^n$ for all $n \geq 1$, and no smaller power of y will suffice.

Proposition 2.8 *If $R \subseteq T$ is an Ω-almost (resp. m-almost) integral extension and $S \subseteq R$ a multiplicative set, then $R_S \subseteq T_S$ is Ω-almost (resp. m-almost) integral.*

Proof We prove the result for the "m-almost" statement; the "Ω-almost" statement is identical.

Let $\frac{r_1}{s_1}\frac{t}{s} = \frac{r_2}{s_2} \in R_S$ (with $s, s_1, s_2 \in S$, $r_1, r_2 \in R$, and $t \in T$). Clearing denominators, we obtain $s_2r_1t = r_2s_1s \in R$. Since t is m-almost integral over R, we have that $(s_2r_1)^m t^n = y_n \in R$ for all $n \geq 1$. Dividing the above equation by $s_2^m s_1^m s^n$, we obtain

$$\left(\frac{r_1}{s_1}\right)^m \left(\frac{t}{s}\right)^n = \frac{y_n}{s_2^m s_1^m s^n} \in R_S$$

for all $n \geq 1$. This concludes the proof. ■

Our next example will show that, as opposed to the classical notions of integrality, Ω-almost integrality does not always behave well with respect to the intermediate extensions.

Example 2.9 Consider a 2-dimensional discrete valuation domain, V, with $\mathrm{char}(V) = 2$, residue field the field of 2 elements ($\mathbb{F}_2$), and with the spectrum of V given by $0 \subseteq \mathfrak{P} \subseteq \mathfrak{M}$. We denote the quotient field of V by K, and we consider the $D + M$ construction $R := \mathbb{F}_2 + \mathfrak{P}$. It is straightforward to see that if $x \in \mathfrak{M} \setminus \mathfrak{P}$ is a generator of $\mathfrak{M}$, then both x and x^{-1} are Ω-almost integral over R. Unfortunately, $R[x]$ is a valuation domain (isomorphic to V). The upshot of this is that although x^{-1} is Ω-almost integral over R, it is not Ω-almost integral over the extension $R[x]$.

With this example in mind, we make the following definition.

Definition 2.10 We say that the extension $R \subseteq T$ is strongly Ω-almost integral (resp. strongly m-almost integral) if every element $t \in T$ is Ω-almost integral (resp. m-almost integral) over every intermediate extension A that is integrally closed in T.

Note that from definition it is clear that strong Ω-almost integrality implies Ω-almost integrality. The fact that t is only required to be Ω-almost integral over the intermediate extensions that are integrally closed in T is motivated by the fact due to Gilmer and Heinzer [7] that an element may not be almost integral over a domain even if it is almost integral over the integral closure of the domain.

As it turns out, strongly Ω-almost integral extensions localize. We record the result below.

Proposition 2.11 *If $R \subseteq T$ is strongly Ω-almost integral and $S \subseteq R$ is a multiplicative set, then $R_S \subseteq T_S$ is strongly Ω-almost integral.*

Proof Suppose that D is a domain such that $R_S \subseteq D \subseteq T_S$ and suppose that D is integrally closed in T_S. We must show that T_S is Ω-almost integral over D. We first consider $D \cap T \subseteq T$. Since $D \cap T$ is integrally closed in T, we have that T is Ω-almost integral over $D \cap T$. Since $(D \cap T)_S = D$, we apply Proposition 2.8 to obtain that T_S is Ω-almost integral over D. ■

We now consider the going-up (GU) property. Generally, almost integral extensions need not be going-up (or even lying over), but we will see that strongly Ω-almost extensions have this nice property. We begin with a technical lemma.

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Lemma 2.12 *Let $R \subseteq T$ be a strongly Ω-almost integral extension and let $\alpha \in T$. If $\mathfrak{P} \subseteq R$ is a prime ideal and S is the set complement of $\mathfrak{P}$ in R, then $\mathfrak{P}R[\alpha] \cap S = \varnothing$.*

Proof Suppose that $s \in \mathfrak{P}R[\alpha] \cap S$. We write

$$s = p_0 + p_1\alpha + p_2\alpha^2 + \cdots + p_n\alpha^n$$

with $p_i \in \mathfrak{P}$ for $0 \leq i \leq n$. Multiplying this equation by α^{-n} and transposing, we obtain

$$(s - p_0)(\alpha^{-1})^n - p_1(\alpha^{-1})^{n-1} - \cdots - p_n = 0.$$

As the element $s - p_0 \notin \mathfrak{P}$, we see that α^{-1} is integral over the localization $R_{\mathfrak{P}} = R_S$; equivalently, $\alpha^{-1} \in R_S[\alpha] \subseteq T_S$. We have that $R_S[\alpha^{-1}] \subseteq R_S[\alpha]$.

Since the extension $R \subseteq T$ is strongly Ω-almost integral, we have that $R_S \subseteq T_S$ is strongly Ω-almost integral. Hence α is Ω-almost integral over A, the integral closure of $R_S[\alpha^{-1}]$ in T_S.

Since α is Ω-almost integral over A and $\alpha^{-1} \in A$, we must have that $\alpha \in U(A)$(by Proposition 2.5). Additionally, A is integral over $R_S[\alpha^{-1}]$ and $\alpha^{-1} \in U(A)$. Hence, we see α^{-1} is a unit in $R_S[\alpha^{-1}]$ and so $\alpha \in R_S[\alpha^{-1}]$. This gives the needed containment, and we have $R_S[\alpha] = R_S[\alpha^{-1}]$.

Since $R_S \subseteq R_S[\alpha^{-1}] = R_S[\alpha]$ is an integral extension, it is immediate that

$$S \cap \mathfrak{P}R_S[\alpha] = \varnothing.$$

Hence $S \cap \mathfrak{P}R[\alpha] \subseteq S \cap \mathfrak{P}R_S[\alpha] = \varnothing$. Thus we get a contradiction and the lemma follows. ■

Theorem 2.13 *Let $R \subseteq T$ be a strongly Ω-almost integral extension. The $R \subseteq T$ satisfies going-up.*

Proof We use the characterization from [8, Theorem 41]: if $\mathfrak{P}$ is a prime ideal of R and S is the complement of $\mathfrak{P}$ in R and $\mathfrak{Q}$ is a prime ideal of T maximal with respect to the exclusion of $S := R \setminus \mathfrak{P}$, then $\mathfrak{Q} \cap R = \mathfrak{P}$.

With the notation as above, it is immediate that $\mathfrak{Q} \cap R \subseteq \mathfrak{P}$, so the other containment will suffice. We proceed by assuming that the containment is strict and derive a contradiction.

Suppose that $z \in \mathfrak{P} \setminus (\mathfrak{Q} \cap R)$. Since $\mathfrak{Q}$ (as an ideal of T) is maximal with respect to the exclusion of $S = R \setminus \mathfrak{P}$, the ideal $(\mathfrak{Q}, z)$ intersects S nontrivially. We write $zt + q = s$ for some $t \in T$, $q \in \mathfrak{Q}$, and $s \in S$.

First note that if $q \in R$, then $q \in \mathfrak{P}$ and hence $zt = s - q \in S \cap \mathfrak{P}R[t]$, which contradicts the previous lemma. So we will suppose that $q \notin R$.

In this case, we consider the intermediate extension $A := R[q]$. Our first claim is that the ideal $(z, q)A$ "misses" S (that is, $(z, q) \cap S = \varnothing$). If not, we can write

$$zf(q) + qh(q) = s$$

for some $s \in S$, $f, h \in R[x]$.

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Rewriting the above equation gives that $zr_0 + qF(q) = s$, where $r_0 = f(0)$ and $F(x) = h(x) + z\frac{f(x)-r_0}{x} \in R[x]$. Transposing, we obtain that $qF(q) = s - zr_0$.

Since $zr_0 \in \mathfrak{P}$, we have that $s - zr_0 \in S$. But $qF(q)$ is an element of $\mathfrak{Q} \subseteq T$, and this is a contradiction. We conclude that $(z, q)A \cap S = \varnothing$.

With this in hand, we expand (z, q) to a prime ideal $\mathfrak{P}_0 \subseteq A$ that is maximal with respect to the exclusion of S. Let $S_0 := A \setminus \mathfrak{P}_0$. Note that $S \subseteq S_0$, and hence the equation $zt + q = s$ implies that $zt \in S_0$.

Now consider $\overline{A_T}$, the integral closure of A in T. Since the extension $A \subseteq \overline{A_T}$ is integral, it is also going-up.

But, since $A \subset T$ is again strongly Ω-almost integral, we have from the previous lemma that $\mathfrak{P}_0 A[t] \cap S_0 = \varnothing$. This is our desired contradiction, and the proof is complete. ■

The following corollary records the Krull dimension behavior of strongly Ω-almost integral extensions.

Corollary 2.14 *If $R \subseteq T$ is a strongly Ω-almost integral extension, then* $\dim(R) \leq \dim(T)$.

Proof Since $R \subseteq T$ is going-up, any chain of prime ideals of R

$$\mathfrak{P}_0 \subsetneq \mathfrak{P}_1 \subsetneq \cdots \subsetneq \mathfrak{P}_n \subsetneq R$$

gives rise to a chain of primes in T

$$\mathfrak{Q}_0 \subsetneq \mathfrak{Q}_1 \subsetneq \cdots \subsetneq \mathfrak{Q}_n \subsetneq T$$

with $\mathfrak{Q}_i \cap R = \mathfrak{P}_i$ for all $0 \leq i \leq n$. In particular, $\dim(R) \leq \dim(T)$. ■

It is natural to ask if incomparability holds (which would be a sufficient condition for equality of Krull dimension) in our strongly Ω-almost integral extension. Below, we produce two examples to demonstrate that neither incomparability nor equality of Krull dimension necessarily holds. But first, we define the notion of "survival extension" and "survival pair".

Definition 2.15 Let $R \subseteq T$ be an extension of rings. Let $\mathfrak{I}$ be an ideal of R such that $\mathfrak{I}T \neq T$. Then $\mathfrak{I}$ is said to "survive" in T.

If every ideal of R survives in T, then the extension is known as a "survival extension".

Suppose that for all intermediate rings A, B with $R \subseteq A \subseteq B \subseteq T$, the extension $A \subseteq B$ is a survival extension. Then the pair (R, T) is called a "survival pair".

Example 2.16 We record a couple of examples here that demonstrate the failure of incomparability.

(i) We revisit the ring extension $\mathbb{Z} + 2x\mathbb{Z}[x] \subseteq \mathbb{Z}[x]$. This extension is, in fact, a survival pair, and hence strongly 1-almost integral by the theorem below. Yet the ideals $2\mathbb{Z}[x]$ and $(2, x)\mathbb{Z}[x]$ of $\mathbb{Z}[x]$ are comparable primes both lying over the prime $2x\mathbb{Z}[x] \subseteq \mathbb{Z} + 2x\mathbb{Z}[x]$.

(ii) We revisit another example. Consider our 2-dimensional discrete valuation domain, V, from Example 2.9. The extension $R = \mathbb{F}_2 + \mathfrak{P} \subseteq V$ is a strongly Ω-almost integral extension. To see this, note that if $\alpha \in \mathfrak{M} \setminus \mathfrak{P}$ then it is easy to see that α is 1-almost integral over R. Now observe that if T is any proper intermediate extension $R \subsetneq T \subseteq V$, then $\overline{T_V}$, the integral closure of T in V, coincides with V, and hence the extension is strongly Ω-almost integral. But note that in this case, the ideals $\mathfrak{P}$ and $\mathfrak{M}$ of V both lie over $\mathfrak{P} \subseteq R$. And in this case, $1 = \dim(R) < \dim(V) = 2$.

Theorem 2.17 *Let $R \subseteq T$ be an extension of domains sharing the same quotient field with R integrally closed in T. Then $R \subseteq T$ is a survival pair if and only if $R \subseteq T$ is strongly 1-almost integral.*

We note that the assumption that R is integrally closed in T is necessary. Indeed, even in the case where $R \subseteq T$ is an integral extension, the extension may not even be m-almost integral if there are integral elements that are roots of irreducible polynomials of arbitrarily large degree.

Proof For the first direction, we assume that $R \subseteq T$ is a survival pair. Let $x \in T$ and let $r \in R$ such that $rx \in R$. If $rx^n \in R$ for all $n \geq 1$, then we are done; so we assume that $rx^k \notin R$ for some $k \geq 1$.

We now make a couple of observations. Since $R \subseteq T$ is a survival pair, and hence a lying over pair ([4, Theorem 2.2]), $R_{\mathfrak{M}} \subseteq T_S$ (where $\mathfrak{M} \subseteq R$ is a maximal ideal and $S = R \setminus \mathfrak{M}$) is a survival pair, since lying over pairs localize ([5, Lemma 2.11(c)]). Also, since R is integrally closed in T, $R_{\mathfrak{M}}$ is integrally closed in T_S.

Now since $R \subseteq T$ is a survival pair, the ideal generated by $\alpha := 1 + rx^k$ and r must blow up in $R[\alpha]$ (since this ideal certainly blows up in T). Hence one can find polynomials $p(x)$ and $q(x)$ in $R[x]$ such that

$$\alpha p(\alpha) + rq(\alpha) = 1.$$

With notation as above, we now note that if r is in the maximal ideal $\mathfrak{M}$, then α^{-1} is integral over $R_{\mathfrak{M}}$ and hence must be in $R_{\mathfrak{M}}$. Since the extension $R_{\mathfrak{M}} \subseteq T_S$ is survival and T_S contains the element α, it is also the case that $\alpha \in R_{\mathfrak{M}}$. This implies that $rx^k \in R_{\mathfrak{M}}$.

On the other hand, if $r \notin \mathfrak{M}$, then the fact that $rx \in R \subseteq R_{\mathfrak{M}}$ shows that $x \in R_{\mathfrak{M}}$, and hence $rx^k \in R_{\mathfrak{M}}$.

We conclude that

$$rx^k \in \bigcap_{\mathfrak{M}:\mathrm{m}axi mal} R_{\mathfrak{M}} = R,$$

and this is the desired contradiction.

For the other direction, we assume that $R \subseteq T$ is strongly 1-almost integral. It has been established that if $R \subseteq T$ is any Ω-almost integral extension, then $R \subseteq T$ is going-up. Since $R \subseteq T$ is strongly 1-almost integral, we have that if R_1 is any intermediate extension, then $A \subseteq T$ is going-up, where A is the integral closure of R_1 in T. Hence $A \subseteq T$ is lying over, so $R_1 \subseteq T$ is lying over (hence survival). This completes the proof. ■

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Theorem 2.18 *Let $R \subseteq T$ be domains with quotient fields $K \subseteq F$, where F is algebraic over K. The integral closure of R in T is denoted by $\overline{R_T}$. Then $R \subseteq T$ is a survival pair if and only if the extension $\overline{R_T} \subseteq T$ is strongly 1-almost integral.*

Proof Suppose that $R \subseteq T$ is a survival pair with respective quotient fields $K \subseteq F$.

Since F is algebraic over K, it is well-known that the quotient field of the integral closure of R in T ($\overline{R_T}$) coincides with F. We apply the previous result to obtain that the ring extension $\overline{R_T} \subseteq T$ is strongly 1-almost integral.

On the other hand, assume that $R \subseteq T$ is an extension of domains such that $\overline{R_T} \subseteq T$ is strongly 1-almost integral. To show that $R \subseteq T$ is a survival pair, we merely need to show that if A is any intermediate domain ($R \subseteq A \subseteq T$) and $I \subseteq A$ is an ideal, then I survives in T. Consider the extensions

$$R \subseteq A \subseteq \overline{A_T} \subseteq T,$$

where $\overline{A_T}$ is the integral closure of A in T. Note that since the extension $A \subseteq \overline{A_T}$ is integral, I survives in $\overline{A_T}$. Also, since $\overline{R_T} \subseteq T$ is strongly 1-almost integral and $\overline{A_T}$ is an intermediate extension, we have that $I\overline{A_T}$ survives in T by Theorem 2.17. This concludes the proof.

■

The following corollary is immediate, but worth noting.

Corollary 2.19 *The pair $R \subseteq T$ is a survival (equivalently, lying over) pair if and only if the pair $\overline{R}_T \subseteq T$ is a survival (equivalently, lying over) pair.*

3 Valuation and Prüfer Domains

In this section we outline a major strength of the Ω-almost integral property. Unlike general almost integrality, both valuation and Prüfer domains are Ω-almost integrally closed. By way of contrast, we point out the well-known fact that valuation domains are "rarely" completely integrally closed in the sense that valuation domains are completely integrally closed if and only if they have Krull dimension not exceeding 1.

Proposition 3.1 *Any valuation domain is Ω-almost integrally closed.*

Proof Let V be a valuation domain with quotient field K. Suppose that $\alpha \in K$ is Ω-almost integral over V. If $\alpha \in V$, then we are done, and so we will assume that $\alpha \notin V$. Since V is a valuation domain and $\alpha \notin V$, α^{-1} is necessarily an element of V. Since $\alpha^{-1}\alpha \in V$ and α is Ω-almost integral, it must be the case that $(\alpha^{-1})^m\alpha^n \in V$ for all $n \geq 1$. Hence for sufficiently large N, $\alpha^N \in V$. Since α is integral over V and V is a valuation domain (and hence integrally closed), this implies that $\alpha \in V$, and we have our desired contradiction. ■

The next result is of independent interest and will also allow us to globalize the previous result to the case of Prüfer domains.

Theorem 3.2 *A domain D is Ω-almost integrally closed if $D_{\mathfrak{P}}$ is Ω-almost integrally closed for all prime ideals $\mathfrak{P} \subseteq D$.*

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Proof Assume that $D_{\mathfrak{P}}$ is Ω-almost integrally closed for all $\mathfrak{P} \in \operatorname{Spec}(D)$ and denote the quotient field of D by K. Let $\alpha \in K$ be an element that is Ω-almost integral over D. Since α is Ω-almost integral over D, we have that if $d\alpha \in D$, then there is a nonnegative integer m such that $d^m\alpha^n \in D$ for all $n \geq 1$. We first take care to note that α is Ω-almost integral over $D_{\mathfrak{P}}$. Indeed, if $\frac{r}{s}$ (with $r \in D$ and $s \in D \setminus \mathfrak{P}$) is such that

$$\frac{r}{s}\alpha = \frac{r_1}{s_1} \in D_{\mathfrak{P}},$$

then we can clear the denominator to obtain $rs_1\alpha = r_1 s$. Since α is assumed to be Ω-almost integral over D and $rs_1\alpha \in D$, we have that there is a nonnegative integer m_1 such that $(rs_1)^{m_1}\alpha^n = d \in D$ for all $n \geq 1$. Dividing both sides of the above equation by $(ss_1)^{m_1}$, we obtain

$$\left(\frac{r}{s}\right)^{m_1}(\alpha)^n = \frac{d}{(ss_1)^{m_1}} \in D_{\mathfrak{P}},$$

as desired.

Now, since α is Ω-almost integral over D, it is Ω-almost integral over $D_{\mathfrak{P}}$ for all $\mathfrak{P} \in \operatorname{Spec}(D)$ by the above argument. Since each $D_{\mathfrak{P}}$ is Ω-almost integrally closed by hypothesis, we have that

$$\alpha \in \bigcap_{\mathfrak{P} \in \operatorname{Spec}(D)} D_{\mathfrak{P}} = D,$$

and this concludes the proof. ∎

We remark that at this point the status of the converse to the previous theorem is unknown. It is probably worthwhile to note that localizations do not preserve "ordinary" completely integrally closed domains. A standard example can be found as a localization of the ring of entire functions on the complex plane. The interested reader is referred to [6] or [2, p. 356, Problem 12].

Corollary 3.3 *If D is Prüfer domain, then D is Ω-almost integrally closed.*

Proof Since D is Prüfer, $D_{\mathfrak{P}}$ is a valuation domain for all $\mathfrak{P} \in \operatorname{Spec}(D)$. Apply the previous two results. ∎

4 Polynomial and Power Series Extensions

In this section, we record the behavior of the preservation of Ω-almost integrality in polynomial and power series extensions. We find that the behavior of Ω-almost integrality is akin to the behavior of almost integrality (that is, polynomial behavior is tame, whereas power series behavior is not).

We begin with a useful lemma.

Lemma 4.1 *Let R be a domain, $r \in R$, and ω an element that is Ω-almost integral over R. Then $r + \omega$ and $r\omega$ are Ω-almost integral over R.*

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Proof For the first statement, suppose that $a \in R$ is such that $a(r+\omega) \in R$. Then, clearly $a\omega \in R$, and hence, we can find a nonnegative integer m such that $a^m\omega^n \in R$ for all $n \geq 1$. Therefore, $a^m(r+\omega)^n \in R$ for all $n \geq 1$. The second statement is very straightforward: suppose that $b \in R$ is such that $b(r\omega) \in R$, that is, $(br)\omega \in R$. Then for some $m \geq 1$, $(br)^m\omega^n \in R$ for all $n \geq 1$, and therefore $b^m(r\omega)^n \in R$ for all $n \geq 1$. ■

It is worth noting that the above proof shows that, in fact, if ω is m-almost integral, then so are $r+\omega$ and $r\omega$.

We now show that the property of being Ω-almost integrally closed is preserved in polynomial extensions.

Theorem 4.2 *Let R be a domain. Then $R[x]$ is Ω-almost integrally closed if and only if R is Ω-almost integrally closed.*

Proof ($\Rightarrow$) Suppose that $R[x]$ is Ω-almost integrally closed and that ω is an element of the quotient field of R. If $r\omega \in R \subseteq R[x]$, then since r may be viewed as an element of $R[x]$ and $R[x]$ is Ω-almost integrally closed, there is a positive integer t such that $r^t\omega^n \in R[x]$ for all $n \geq 1$. But since r and ω are elements of the quotient field of R (say K), we have that $r^t\omega^n \in K \cap R[x] = R$ for all $n \geq 1$, and this direction is established.

($\Leftarrow$) Now we suppose that R is Ω-almost integrally closed. For this direction, we will suppose that there is an $f(x)$ that is Ω-almost integral whose coefficients are not Ω-almost integral. We will write

$$f(x) = t_0 + t_1x^{a_1} + \cdots + t_mx^{a_m}$$

with $1 \leq a_1 < a_2 < \cdots < a_m$. We choose this notation because we can then assume that each $t_i \in K \setminus R$ by the previous lemma. We will additionally assume that $f(x)$ is of minimal degree (with respect to the property that it is Ω-almost integral, yet none of its coefficients are in R).

By this minimality assumption on the degree of $f(x)$, we have that if $rt_0 \in R$, then $rt_j \in R$ for all $0 \leq j \leq m$. To see this, observe that if $rt_0 \in R$, then $rf(x) - rt_0 = rt_1x^{a_1} + \cdots + rt_mx^{a_m}$, and hence $rt_1x^{a_1-1} + \cdots + rt_mx^{a_m-1}$ is Ω-almost integral (it is easy to see that if $xg(x)$ is Ω-almost integral, then $g(x)$ is Ω-almost integral). By minimality of the degree of $f(x)$, we must have that all the coefficients, rt_j, are elements of R.

Suppose that $rt_0 \in R$ for some nonzero $r \in R$. By the above observation, it must be the case that $rf(x) \in R[x]$. Since $f(x)$ is Ω-almost integral, we have that there is a nonnegative integer k such that $r^kf(x)^n \in R[x]$ for all $n \geq 1$. In particular, $r^kt_0^n \in R$ for all $n \geq 1$, and hence t_0 is Ω-almost integral, which is a contradiction. Hence all of the coefficients of $f(x)$ are in R, and the theorem is established. ■

To settle the question of the stability of the Ω-almost integrally closed property for power series, we give an example of an Ω-almost integrally closed domain R such that $R[[x]]$ is not Ω-almost integrally closed. The ease and the fundamental nature of the counterexample shows that, in a certain sense, power series extensions do not behave well with respect to Ω-almost integrality (as is the case for integrality).

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Example 4.3 Let V be any valuation domain with $\dim(V) > 1$. It is well known that V is integrally closed but not completely integrally closed. We have already established that V is Ω-almost integrally closed. It is well known that if R is any integrally closed domain that is not completely integrally closed, then $R[[x]]$ is not integrally closed [6, Proposition 13.11]. So we see that $V[[x]]$ is not integrally closed and hence not Ω-almost integrally closed despite the fact that V is Ω-almost integrally closed.

5 Some Ideal Theoretic Properties

In this section, we will look at a couple of ideal-theoretic properties that distinguish the notion of Ω-almost integrality from "ordinary" almost integrality.

Before we begin, we are going to define a couple of notations we have used throughout this section. J_α is used to denote the set of all the denominators of the element $\alpha \in K$, that is, $J_\alpha = \{x \in R \mid x\alpha \in R\}$, where K is the quotient field of R, and $\mathfrak{M}$ is used to denote the R-submodule generated by $\{\alpha^n\}_{n\geq 1}$.

Lemma 5.1 *Let R be a domain with quotient field K. Let $\alpha \in K$ be an element of the quotient field. Then α is Ω-almost integral over R if and only if $J_\alpha \subseteq \sqrt{(J_\alpha : \mathfrak{M})}$.*

Proof Let us assume that α is Ω-almost integral over R, and consider an element x of J_α. Since α is Ω-almost integral over R, there is a positive integer m such that $x^m\alpha^n \in J_\alpha$ for all $n \geq 1$. Therefore, $x^m\mathfrak{M} \subseteq J_\alpha$; in other words, $x^m \in (J_\alpha : \mathfrak{M})$, which implies $x \in \sqrt{(J_\alpha : \mathfrak{M})}$.

For the other direction, let us assume that $J_\alpha \subseteq \sqrt{(J_\alpha : \mathfrak{M})}$ and let $x \in J_\alpha$. Since $x \in \sqrt{(J_\alpha : \mathfrak{M})}$, then $x^m \in (J_\alpha : \mathfrak{M})$ for some positive integer m, and therefore $x^m\mathfrak{M} \subseteq J_\alpha \subseteq R \Rightarrow x^m\alpha^n \in R$ for all $n \geq 1$. Hence α is Ω-almost integral over R. ∎

In fact, in the above situation, it can be further shown that $\sqrt{(J_\alpha : \mathfrak{M})} = \sqrt{J_\alpha}$. We record this result in the next theorem, using the same notation as above.

Theorem 5.2 *Let R be a domain with the quotient field K. Let $\alpha \in K$ be an element of the quotient field. Then α is Ω-almost integral if and only if $\sqrt{(J_\alpha : \mathfrak{M})} = \sqrt{J_\alpha}$.*

Proof We begin by assuming that α is Ω-almost integral.

For the first containment, we suppose that $t \in \sqrt{(J_\alpha : \mathfrak{M})}$, which implies that $t^m \in (J_\alpha : \mathfrak{M})$ for some positive integer m. Hence $t^m\mathfrak{M} \subseteq J_\alpha$, and so $t^m\alpha^n \in J_\alpha \subseteq R$ for all $n \geq 1$. We obtain that $t^m \in J_\alpha$, and so $t \in \sqrt{J_\alpha}$. Thus, $\sqrt{J_\alpha : \mathfrak{M}} \subseteq \sqrt{J_\alpha}$.

For the other containment, assume that $t \in \sqrt{J_\alpha}$. So for some positive integer m, $t^m \in J_\alpha$. Since α is Ω-almost integral by assumption, Lemma 5.1 gives that $J_\alpha \subseteq \sqrt{(J_\alpha : \mathfrak{M})}$. Hence $t \in \sqrt{\sqrt{(J_\alpha : \mathfrak{M})}}$, and so $t \in \sqrt{(J_\alpha : \mathfrak{M})}$. Thus $\sqrt{(J_\alpha : \mathfrak{M})} = \sqrt{J_\alpha}$.

The other implication is straightforward since $J_\alpha \subseteq \sqrt{J_\alpha} = \sqrt{(J_\alpha : \mathfrak{M})}$. Indeed, Lemma 5.1 shows that α is Ω-almost integral. ∎

The following corollary is almost immediate.

Corollary 5.3 *Let α be Ω-almost integral over R. If the ideal $(J_\alpha : \mathfrak{M})$ is a radical ideal, then α is 1-almost integral over R. Additionally, if R is root closed, then $(J_\alpha : \mathfrak{M})$ is a radical ideal.*

Proof Assume first that $(J_\alpha : \mathfrak{M})$ is a radical ideal and suppose that $t\alpha \in R$. Since α is Ω-almost integral over R, there is a nonnegative integer m such that $t^m\alpha^n \in R$ for all $n \geq 1$. Hence $t^m \in (J_\alpha : \mathfrak{M}) = \sqrt{(J_\alpha : \mathfrak{M})}$. We obtain that $t \in (J_\alpha : \mathfrak{M})$, and therefore $t\alpha^n \in J_\alpha \subseteq R$ for all $n \geq 1$. So α is 1-almost integral.

For the second statement, assume that R is root closed, α is 1-almost integral, and $t \in \sqrt{(J_\alpha : \mathfrak{M})}$. It suffices to show that $t \in (J_\alpha : \mathfrak{M})$.

Since $t \in \sqrt{(J_\alpha : \mathfrak{M})}$, there is a nonnegative integer m such that $t^m \in (J_\alpha : \mathfrak{M})$. Hence for all $n \geq 1$, $t^m\alpha^n \in J_\alpha \subseteq R$. Choosing $n = km$, we get $(t\alpha^k)^m \in J_\alpha \subseteq R$ (for all $k \geq 1$). Since R is root closed, we have that $t\alpha^k \in R$ for all $k \geq 1$, and hence, by definition, $t \in (J_\alpha : \mathfrak{M})$. This concludes the proof. ■

We now give an example demonstrating the necessity of the Ω-almost integral condition.

Example 5.4 Consider the domain $R = \mathbb{F}[x, \frac{y}{x}, \frac{y}{x^2}, \dots, \frac{y}{x^n}, \dots]$. We have already mentioned that $\frac{1}{x}$ is an almost integral element over R that is not Ω-almost integral. Note that in this case

$$J_{1/x} = (x), \ \sqrt{(J_{1/x} : \mathfrak{M})} = \left(\{y, \tfrac{y}{x}, \dots, \tfrac{y}{x^n}, \dots\}\right), \text{ and } (x) \supset \left(\{y, \tfrac{y}{x}, \dots, \tfrac{y}{x^n}, \dots\}\right).$$

Also note that (x) is a maximal ideal and thus prime. Hence, $J_{1/x} \nsubseteq \sqrt{(J_{1/x} : \mathfrak{M})}$.

For the sake of completeness, we point out that the R-module $\mathfrak{M}$ is a fractional ideal of R (because $x\mathfrak{M} \subseteq J_\alpha \subset R$) and $J_\alpha \subseteq \mathfrak{M}^{-1}$.

6 Pathologies and a Connection with Pseudo-integrality

In this section, we explore the connection between Ω-almost integrality and the notion of pseudo-integrality developed by Anderson, Houston, and Zafrullah [1]. We also look at some "pathological" behavior of the notion of Ω-almost integrality. Although we have found that Ω-almost integral extensions have some very nice behavior (and give a useful characterization of survival/lying over pairs), the most disturbing pathology of Ω-almost integral elements is that, in general, they do not form a ring. In fact, it is not true in general that products or sums of Ω-almost integral elements are still Ω-almost integral (of course, they *are* almost integral). In general, the set of elements that are Ω-almost integral over R may not even be an R-submodule of the complete integral closure of R.

The first example that we produce shows that an element may be Ω-almost integral over R and fail to be Ω-almost integral over the integral closure of R.

Example 6.1 Consider the ring $R := \mathbb{Z}[\pi] + x\mathbb{R}[x]$. The element $1/\sqrt{\pi}$ is Ω-almost integral over R. To see this, note that if $1/\sqrt{\pi}p(\pi)$, with $p(x) \in \mathbb{Z}[x]$, is in $\mathbb{Z}[\pi]$, the transcendence of π over $\mathbb{Q}$ gives that $p(x)$ is the zero polynomial. Hence if $r1/\sqrt{\pi} \in R$, then $r \in x\mathbb{R}[x]$. It is easy to see that $r(1/\sqrt{\pi})^n \in R$ for all n. But note that $\sqrt{\pi}$ is an element of the integral closure of R, but $1/\sqrt{\pi}$ is not. Hence by Proposition 2.5, we have that although $1/\sqrt{\pi}$ is Ω-almost integral over R, it is not Ω-almost integral over the integral closure of R.

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For the next example, we utilize the same domain to show that sums and products of Ω-almost integral elements need not be Ω-almost integral.

Example 6.2 We once again consider the domain $R := \mathbb{Z}[\pi] + x\mathbb{R}[x]$. Let $\alpha \in \mathbb{R}$ be transcendental over $\mathbb{Q}(\pi)$. We note that α is Ω-almost integral over R. Indeed, if $\alpha p(\pi)$, with $p(x) \in \mathbb{Z}[x]$, is in $\mathbb{Z}[\pi]$ then α is certainly in $\mathbb{Q}(\pi)$, we arrive at a contradiction unless $p(x) = 0$. Hence $p(x) = 0$ and as in the previous example, we see that α is Ω-almost integral. We also record the fact that since $1/\pi - \alpha$ is also transcendental over $\mathbb{Q}(\pi)$, then $1/\pi - \alpha$ is also Ω-almost integral over R.

But as $1/\pi$ cannot be Ω-almost integral over R (since $\pi \in R$ is a nonunit), we have that

$$\alpha + \left(\tfrac{1}{\pi} - \alpha\right) = \tfrac{1}{\pi},$$

in other words, the sum of two Ω-almost integral elements need not be Ω-almost integral.

Repeating the above example replacing addition with multiplication and $\frac{1}{\pi} - \alpha$ with $\frac{1}{\pi\alpha}$ shows that the product of Ω-almost integral elements need not be Ω-almost integral.

We now show that, in general, intersections of Ω-almost integrally closed domains need not be Ω-almost integrally closed.

Example 6.3 Let R be any domain that is integrally closed, but not Ω-almost integrally closed (*e.g.*, $\mathbb{Z} + 2x\mathbb{Z}[x]$). Since R is integrally closed, we can write R as the intersection of its valuation overrings: $R = \cap V$. But since we have shown that any valuation domain is Ω-almost integrally closed, this gives an example of an intersection of Ω-almost integrally closed domains that is not Ω-almost integrally closed.

As it turns out, the concept of m-almost integrality that we have explored can be collapsed to the concept of 1-almost integrality for the integrally closed case. In a certain sense, if one considers an m-almost integral extension, the number $m - 1$ reflects the amount of "noise" generated by the integral elements in the extension.

We first record a result for the case of strongly Ω-almost integral extensions. This result also serves to strengthen Theorem 2.17.

Theorem 6.4 *Let $R \subseteq T$ be an extension of domains with R integrally closed in T. Then $R \subseteq T$ is a strongly Ω-almost integral extension if and only if $R \subseteq T$ is a strongly 1-almost integral extension.*

Proof Since Ω-almost integrality implies 1-almost integrality, there is nothing to prove for the first direction.

For the other direction, we have to show that if A is any intermediate extension of domains with A integrally closed in T, then the extension $A \subseteq T$ is 1-almost integral. Since $A \subseteq T$ is strongly Ω-almost integral, we have that $A \subseteq T$ is going-up by Theorem 2.13.

We now claim that $A \subseteq T$ is a survival pair. To this end, it suffices to show that if $A \subseteq B \subseteq T$ and $I \subsetneq B$ is a proper ideal, then I survives in T. Denote the integral closure of B in T by $\overline{B_T}$ and note that $I\overline{B_T} \subsetneq \overline{B_T}$, since the extension $B \subseteq \overline{B_T}$

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is integral. Additionally, since $R \subseteq T$ is strongly Ω-almost integral, the extension $\overline{B_T} \subseteq T$ is strongly Ω-almost integral and hence, by Theorem 2.13, is going-up (and hence lying over). Hence, $\overline{B_T} \subseteq T$ is a survival extension, and so the proper ideal $I\overline{B_T}$ survives in T. We conclude that I survives in T and the claim is established.

Since $A \subseteq T$ is a survival pair, we have from Theorem 2.17 that the extension $A \subseteq T$ is (strongly) 1-almost integral. So the extension $R \subseteq T$ is strongly 1-almost integral. ■

The next result further quantifies the remarks made above concerning the connections between Ω-almost integrality and 1-almost integrality. Note the similarity to Corollary 5.3; in fact, this result may be considered to be Corollary 5.3 from a different point of view.

Proposition 6.5 *If R is root closed (in particular, if R is integrally closed) and ω is Ω-almost integral over R, then ω is 1-almost integral over R.*

Proof Suppose that $r\omega \in R$. By assumption, there is a nonnegative integer m such that $r^m\omega^n \in R$ for all $n \geq 1$. If k is a nonnegative integer, we choose $n = km$ and note that $(r\omega^k)^m \in R$. Since R is integrally closed, we obtain that $r\omega^k \in R$ for all $k \geq 1$. This concludes the proof. ■

We now give a couple of results to compare and contrast the notions of Ω-almost integrality and pseudo-integrality. The first result shows that any element that is 1-almost integral over R is pseudo-integral, but then it is shown that, in general, pseudo-integral elements are not necessarily Ω-almost integral.

For completeness, we repeat the following result from [1].

Theorem 6.6 *Let V be a valuation domain of the form $F + M$, where F is a field and M is the maximal ideal of V. Let D be a subring of F and $R := D + M$. We have the following.*

(i) *The pseudo-integral closure of R is $D' + M$ (where D' is the pseudo-integral closure of D) if F is the quotient field of D.*
(ii) *The pseudo-integral closure of D is V if F properly contains the quotient field of D.*

This result allows the following example that shows that pseudo-integral elements may fail to be Ω-almost integral.

Example 6.7 Using the above notation, let

$$V = \mathbb{Q}(i)[[x]], \quad M = x\mathbb{Q}(i)[[x]], \quad F = \mathbb{Q}(i), \quad \text{and} \quad D = \mathbb{Z}.$$

As $\mathbb{Q}(i)$ properly contains $\mathbb{Q}$, the above result gives that the pseudo-integral closure of $R = \mathbb{Z} + x\mathbb{Q}(i)[[x]]$ is V. Hence (in particular) the element $\frac{1}{2}$ is pseudo-integral over R, but Proposition 2.5 shows that $\frac{1}{2}$ cannot be Ω-almost integral over R. Thus pseudo-integrality does not imply Ω-almost integrality.

Theorem 6.8 *Let R be a domain and ω an element of the quotient field K that is 1-almost integral over R. Then ω is pseudo-integral over R.*

Proof It suffices to find a nonzero finitely generated ideal $I \subseteq R$ such that $\omega I^{-1} \subseteq I^{-1}$. Assume that $r\omega \in R$ and consider the ideal $I := \big(r, r\omega\big)$. Note that the inverse of I is given by

$$I^{-1} = \left\{ \tfrac{t}{r} \mid t, t\omega \in R \right\}.$$

Observe that $(\omega\frac{t}{r})(r) = t\omega \in R$ and $(\omega\frac{t}{r})(r\omega) = t\omega^2$, which is also in R since ω is 1-almost integral. Hence $\omega I^{-1} \subseteq I^{-1}$, and so ω is pseudo-integral. ∎

The next corollary shows that in the case of integrally closed domains, the notions of pseudo-integrality and Ω-almost integrality are comparable.

Corollary 6.9 *Let R be an integrally closed domain. If ω is Ω-almost integral over R, then ω is pseudo-integral over R.*

Proof By Proposition 6.5, since ω is Ω-almost integral over R, ω is 1-almost integral over R. Hence by Theorem 6.8, ω is pseudo-integral over R. ∎

We conclude by asking if it is the case that every Ω-almost integral element is pseudo-integral. Certainly it is the case that one cannot expect the sum (or product) of Ω-almost integral elements to be Ω-almost integral (see Example 6.2), but the ring generated by all Ω-almost integral elements over R is certainly contained in the complete integral closure of R. But at this time, it is not known whether or not the ring generated by the Ω-almost integral elements is contained in the pseudo-integral closure of R.

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