

Irrational “Coefficients” in Renaissance Algebra

Jeffrey A. Oaks

University of Indianapolis
E-mail: oaks@indy.edu

Argument

From the time of al-Khwārizmī in the ninth century to the beginning of the sixteenth century algebraists did not allow irrational numbers to serve as coefficients. To multiply $\sqrt{18}$ by x , for instance, the result was expressed as the rhetorical equivalent of $\sqrt{18x^2}$. The reason for this practice has to do with the premodern concept of a monomial. The coefficient, or “number,” of a term was thought of as how many of that term are present, and not as the scalar multiple that we work with today. Then, in sixteenth-century Europe, a few algebraists began to allow for irrational coefficients in their notation. Christoff Rudolff (1525) was the first to admit them in special cases, and subsequently they appear more liberally in Cardano (1539), Scheubel (1550), Bombelli (1572), and others, though most algebraists continued to ban them. We survey this development by examining the texts that show irrational coefficients and those that argue against them. We show that the debate took place entirely in the conceptual context of premodern, “cossic” algebra, and persisted in the sixteenth century independent of the development of the new algebra of Viète, Descartes, and Fermat. This was a formal innovation violating prevailing concepts that we propose could only be introduced because of the growing autonomy of notation from rhetorical text.

Premodern algebra, in Greek, Arabic, Latin, and Italian, was a numerical problem-solving technique. To work out a problem by algebra, an unknown number was named in terms of the given names of the powers, the conditions of the problem were applied to set up an equation, and this was then simplified and solved. While this broad approach is comparable to modern algebra, premodern algebraists – and for the time being I mean those who practiced algebra before the sixteenth century – did not solve their problems the same way we do today. Some of the steps they took differ from ours, their phrasing of certain operations makes little sense when interpreted through our symbolic solutions, and even their notations exhibit differences that cannot be attributed to local variation.

One example of a difference in practice is that premodern algebraists consistently worked out their operations before setting up equations, where we do not hesitate to

include operations in equations (see Oaks 2009; Oaks 2010; Christianidis and Oaks 2013). And to simplify an equation like $10 - x = 4x$ Arabic algebraists would “restore” the 10, and then add the x to the $4x$. We have no reason to restore quantities (whatever that would mean) that have something subtracted from them.¹ And in notation a “1” is always given for the coefficient of a term when there is one of them: where we write just x , Italian abacists wrote “ 1ρ ”.²

Taken together, these and other differences in procedure, wording, and notation point to a different understanding of monomials, polynomials, and equations. Premodern polynomials, for example, were treated as collections or aggregations of the names of the powers of the unknown, and not as the linear combinations of the powers that we work with today. Particularly relevant for the present study is that the interpretation of notation was determined by the rhetorical version it represented. This meant that the signs for the powers of the unknown functioned differently than ours do today. The abacist “ ρ ”, for example, does not stand for the value of the unknown the way our x does, but indicates a *kind* of number.

In this article I focus on one particular difference in practice: that premodern algebraists in Arabic, Latin, and Italian did not allow the “number” (coefficient) of a term to be irrational. They restricted their coefficients to (positive) rational numbers, even if irrational numbers were commonplace and unquestioned in medieval logic. As an example, we multiply $\sqrt{18}$ by x to get $\sqrt{18}x$, but Maestro Biaggio, a fifteenth century abacist, writes instead the rhetorical equivalent of $\sqrt{18x^2}$.³ Here the “number” of x^2 ’s is rational, while the square root of the term is perfectly valid as an irrational number (the value of x is found in this problem to be $\sqrt{200} - 10$).

This conceptual framework of premodern algebra was inherited by European algebraists in the sixteenth century. The symptomatic differences in practice, wording, and notation that distinguish it from modern algebra remained in force, too, but with one exception: some algebraists now allowed their coefficients to be irrational. This first occurs in a limited way in Christoff Rudolff (1525), then more abundantly in Girolamo Cardano (1539), and later in Johann Scheubel (1550), Pedro Nuñez (1567), Rafael Bombelli (1572), and Simon Stevin (1585). But most algebraists, including Michael Stifel, Jacques Peletier, and Niccolò Tartaglia, continued with the old way by putting the whole term under the root.

This development was not a step toward the modern algebra of Viète, Fermat, and Descartes. The appearance of irrational coefficients and the controversies that it sparked took place entirely within the conceptual context of premodern algebra, where polynomials remained aggregations and “coefficients” were at least nominally regarded

¹This kind of “restoration” is explained in Oaks and Alkhateeb 2007. Very briefly, the $10 - x$ was regarded as a diminished 10, so it must be restored to a whole 10. The simplification of this particular equation is from Abū Kāmil’s problem (T3) (Abū Kāmil 1986, 44.21; Abū Kāmil 2012, 325.10).

²For Arabic notation, see Abdeljaouad 2002; Oaks 2012. For Italian notation, see Oaks 2010.

³“la radice di 18 censj” (Biagio 1983, 53).

as numbers that count the term. It is only in this context that the issue makes sense, so this is the conceptual setting in which the various texts must be assessed. When Viète later devised an algebra founded in geometry he created a radically new notation to go with it, and it is out of this that Descartes’s notation and its subsequent incarnations emerged.

I begin by reviewing the ways medieval algebraists expressed monomials affected by irrational numbers. The medieval practice is presented in the context of the aggregations interpretation with particular attention to notation, and this is put into broader perspective by a comparison with how fractions and multitudes of roots were expressed in arithmetic. Then I examine sixteenth century books to show how the restriction on “coefficients” was relaxed in some algebraists. Contemporary attitudes are then brought to light by examining a debate over this issue in mid-seventeenth-century Italy in which the participants cite the earlier algebraists. I conclude by addressing two related questions: What change had occurred in algebra that allowed some sixteenth-century algebraists to work with irrational coefficients? And what effect did this change have on the medieval concepts?

Some Preliminary Remarks

Premodern algebra is almost always presented in the manuscripts in rhetorical form. Natural language by its nature admits ambiguities in meaning, but when expressing algebra this meaning is controlled in important ways by the mathematics. For example, we know that Abū Kāmil’s “a root of five hundred *māls*” corresponds to our $\sqrt{500x^2}$ and not $\sqrt{500}x^2$ because interpreting it as the latter is mathematically wrong in the context of the problem being solved. The examples I have chosen to quote in this article are mathematically unambiguous. Abū Kāmil’s phrase, for instance, is the result of multiplying “ten things” by “a root of five,” which corresponds to our $10x$ by $\sqrt{5}$. Even taking into consideration the conceptual differences between the Arabic words and our modern notation, the underlying computations still have to make sense.

The sources for this study consist of every medieval Arabic, Latin, Italian, and early modern European book treating algebra down to 1630 that I could locate, plus a few from later in the seventeenth century. I consulted 33 books in medieval Arabic, seven in medieval Latin (not including translations of al-Khwārizmī and Abū Kāmil), 27 in Italian (fourteenth and fifteenth centuries), and more than fifty written in Latin and European vernacular languages between 1500 and 1630. The reference list shows only works from which I cite passages or give references.

A word on the word “coefficient”: Algebraists before Viète called the coefficient the “number” or “quantity” of a term. For example, Ibn al-Hā’im (1387) begins his rule for finding x^2 in the equation $x^2 = 4x + 5$ with: “The square of the number (*idda*) of roots is sixteen” (Ibn al-Hā’im 2003, 95:2). Maestro Dardi (1344), after arriving at

the equation $\sqrt{8x^2} = \frac{2}{3}x^3$, writes “Multiply the quantity (*quantità*) of cubes by itself, which is $2/3$.”⁴ Most often, though, algebraists would simply write “the roots” or “the cubes” to mean the *number* of roots or cubes.

It is important to keep in mind that what is at stake in this article is not the concept of an irrational number, but rather the concept of an algebraic monomial. To use modern terminology, I investigate how algebraists understood the relationship between the power of the unknown and the coefficient. Irrationals come into play because it is with them that we see a difference in practice. The concepts of irrational numbers presented in some sixteenth-century books like those of Regiomontanus (1533), Stifel (1544), and Stevin (1585)⁵ appear to have no connection with the introduction of irrational coefficients that we find in Rudolff 1525, Cardano 1539, Stevin 1585, etc. This is not surprising. The rules for operating with irrational roots were already well-established in the ninth century, and irrationals had remained a staple of calculation in practical arithmetic and algebra ever since. Whatever musings on the nature of irrationals we find in sixteenth-century authors would have had no effect on actual calculations, and it is with calculations that I am concerned.

Last, modern notation can still be useful to clarify the medieval operations as long as one keeps the medieval concepts in mind. Thus I use it throughout this article.

Monomials in Medieval Algebra and the Aggregations Interpretation

To show the ways that Arabic algebraists constructed and worked with monomials I need to first outline the main features of medieval algebraic problem solving, including the way they conceived of the polynomials that make up the two sides of an equation. The name given to the first-degree unknown in Arabic algebra is *shay'* (“thing”), though it is sometimes called *jidhr* (“root”). This corresponds to our “*x*.” Its square, our x^2 , is called a *māl*, literally “sum of money,” “treasure,” “property,” “wealth.”⁶ The cube of a “thing” is called a *ka'b* (“cube”), and higher powers are written as some combination of *māl* and *ka'b*, like *māl ka'b* for x^5 and *māl māl māl māl* for x^8 . Units are often counted in dirhams, a silver coin, but also with words meaning “units” or “in number,” and often the term is dropped altogether. So “ten dirhams,” “ten units,” “ten in number,” and simply “ten” all mean the number 10.

In books that describe algebra in detail the explanation of the various rules is followed by a collection of solved problems. There are some notable differences between how medieval algebraists solved problems and how we solve them today. One difference is that although these algebraists worked freely with irrational roots, they would not allow

⁴“Multiplica la quantità de cubi in sé, cioè $2/3$. . .” (Dardi 2001, 107.15).

⁵See for example, Bos 2001, 135ff; Malet 2006; Rommevaux-Tani 2014.

⁶Because there is no good English translation of *māl*, I leave it untranslated. I write its plural with the English suffix: *māls*.

them to serve as the "number" of a term. As an example of how medieval algebraists worked with monomials involving roots I present the enunciation and the first solution to problem (41) from Abū Kāmil's late ninth century *Book of Algebra*.⁷ The enunciation reads:

Ten: you divided it into two parts. You divided each one of the parts by the other, and you added them, so it yielded a root of five dirhams.

To reformulate this in modern algebraic notation we can name the parts a and b , so $a + b = 10$ and the condition states that $\frac{a}{b} + \frac{b}{a} = \sqrt{5}$. In the solution Abū Kāmil names one of the parts "a thing," and then he manipulates the operations in terms of the names of the powers to form an equation:

Its rule is that you make one of the parts a thing $[x]$, and the other ten less a thing $[10 - x]$. So multiply each one of them by itself and add them, so it yields a hundred dirhams and two *māls* less twenty things $[100 + 2x^2 - 20x]$. Keep this in mind. Then multiply one of the parts by the other, which is a thing by ten less a thing, so it yields ten things less a *māl* $[10x - x^2]$. Multiply this by a root of five, so it yields: a root of five hundred *māls* less a root of five *māls māl* equal a hundred dirhams and two *māls* less twenty roots $[\sqrt{500x^2} - \sqrt{5x^4} = 100 + 2x^2 - 20x]$.

The equation is then simplified and solved:

So restore a root of five hundred *māls* by a root of five *māls māl* and add it to a hundred dirhams and two *māls* less twenty things. And restore the hundred dirhams and two *māls* less twenty things by the twenty things and add it to a root of five hundred *māls*.⁸ It yields: twenty things and a root of five hundred *māls* equal a hundred dirhams and two *māls* and a root of five *māls māl* $[20x + \sqrt{500x^2} = 100 + 2x^2 + \sqrt{5x^4}]$.

Return everything you have to one *māl*, which is that you multiply the two *māls* and a root of five *māls māl* by a root of five dirhams less two dirhams $[\sqrt{5} - 2]$. You multiply it by a root of five dirhams less two dirhams because if you divided a dirham by two and a root of five, it resulted in one root of five less two dirhams, as is clear to you.⁹ So you multiply everything you have by a root of five less two dirhams to get: a *māl* and a root of fifty thousand dirhams less two hundred dirhams equal ten things $[x^2 + \sqrt{50000} - 200 = 10x]$.

⁷This is problem <47> in Rashed's edition (Abū Kāmil 2012, 427). Abū Kāmil gives two more solutions after the one given here.

⁸For an explanation of "restoration" (*al-jabr*), see Oaks & Alkhateeb 2007. Ordinarily for the second restoration an algebraist would restore the "hundred dirhams and two *māls*" and not the whole "hundred dirhams and two *māls* less twenty things." The Latin translation includes the "less twenty things," but judging by Levey's garbled translation, the Hebrew translation omits it (Sesiano 1993, l. 2428; Levey 1966, 152.5).

⁹i.e., $\frac{1}{2+\sqrt{5}} = \sqrt{5} - 2$.

Then halve the things,¹⁰ so it yields five. Multiply it by itself, so it yields twenty-five dirhams. Subtract from it a root of fifty¹¹ thousand dirhams less two hundred dirhams. This leaves two hundred twenty-five dirhams less a root of fifty thousand. A root of that subtracted from five¹² is one of the two parts, and added to five is the other part [$5 - \sqrt{225 - \sqrt{50000}}$ and $5 + \sqrt{225 - \sqrt{50000}}$].

In the first part of the solution Abū Kāmil multiplies the $\sqrt{5}$ by “ten things less a *māl*” ($10x - x^2$) to get “a root of five hundred *māls* less a root of five *māls māl*” ($\sqrt{500x^2 - \sqrt{5}x^4}$). We would put the powers outside the square roots like this: $\sqrt{500}x - \sqrt{5}x^2$. Abū Kāmil, instead, insists on taking the square roots of the whole terms.¹³ Once the operations have been performed he sets up and simplifies his equation, which in modern notation is $20x + \sqrt{500x^2} = 100 + 2x^2 + \sqrt{5}x^4$. Instead, we would write it as $(20 + \sqrt{500})x = 100 + (2 + \sqrt{5})x^2$. In our version it is much easier to identify the linear and quadratic terms. Even with his formulation Abū Kāmil understands that to set the “number” of *māls* to 1 he must multiply everything by the reciprocal of $2 + \sqrt{5}$.

Abū Kāmil does not work with fractions in this problem, but many problems in Arabic algebra are solved with fractions. One example is this equation from al-Khwārizmī’s problem (25): “twenty-five parts of a hundred forty-four parts of a *māl* and sixteen dirhams less three roots and a third of a root equal . . . a thing” ($\frac{25}{144}x^2 + 16 - 3\frac{1}{3}x = x$) (al-Khwārizmī 2009, 187.6–8; my translation).¹⁴

Clearly the Arabic “number” of a term does not possess the same meaning as our modern coefficient. I explain this in some detail in my “Polynomials and Equations in Arabic Algebra” (Oaks 2009, §5), and more succinctly in “Medieval Arabic Algebra as an Artificial Language.” There I explain medieval polynomials by decoding al-Karajī’s equation “. . . ten things less a *māl*, and that equals four things and five dirhams” (Saidan 1986, 202.8) ($10x - x^2 = 4x + 5$):

Medieval algebraists conceived of polynomials differently than we do today. For us, a polynomial is constructed from the powers of x with the operations of scalar multiplication and addition/subtraction. In other words, it is a linear combination of the powers. By contrast, Arabic polynomials contain no operations at all. In the expression “four things,”

¹⁰i.e., take half of the number of “things.”

¹¹The Arabic MS mistakenly has “five” here (p. 89.7), but the Latin and Hebrew translations write it correctly (Sesiano 1993, l. 2451; Levey 1966, 152.14).

¹²The MS mistakenly has “a root of five” instead of just “five.” The Latin and Hebrew translations write it correctly (Sesiano 1993, l. 2453; Levey 1966, 152).

¹³Abū Kāmil does not explain how to multiply a root by number in his book, but he does explain how to convert multiple roots into a single root in the introduction to his book, like: “if we wanted to double a root of sixteen, we multiplied two by two to get four, then we multiplied four by sixteen to get sixty-four. A root of sixty-four is eight, which is double a root of sixteen” (Abū Kāmil 1986, 33.19; Abū Kāmil 2012, 301.9).

¹⁴I express “three roots and a third of a root” as $3\frac{1}{3}x$ because it reflects how it was written in medieval Arabic notation.

the “four” is not multiplied by “things.” Instead, it merely indicates *how many* things are present. Think of “four things” like “four bottles.”

Further, the phrase “four things and five dirhams” entails no addition, but is a collection of nine items of two different kinds. Think of it like “four bottles and five cans.” The *wa* (“and”) connecting the things and the dirhams is the common conjunction. It does not take the meaning of the modern word “plus.”

The “ten things less a *māl*” on the other side of the equation describes an amount which is a *māl* short of “ten things.” Think of it as a collection of ten items which have been diminished by a *māl*. Similarly, if I take a bite out of an apple, I can describe the result as “an apple less a bite.” “Ten things less a *māl*,” like the bitten apple, is a static object: it is only our *description* of it which seems to imply a subtraction. *Illā* (“less”) is the negative counterpart to “and.” It does not mean “minus.” (Oaks 2007, 548)¹⁵

Even grammatically the number of a term is “how many” there are. Premodern algebraists, in Arabic, Latin, and Italian, speak of “a thing,” making the noun singular when there is one, and “two things,” “three things,” etc., making “thing” plural when there is more than one. It would make no sense to multiply “a root of five” by “a thing” and get “a root of five things” ($\sqrt{5}x$). One *can* talk about the aggregation “five *māls*,” though, and because this represents a number, its root can be taken. So the result of the multiplication is “a root of five *māls*” ($\sqrt{5x^2}$). To borrow a phrase from Jacob Klein, the number of a term is “a determinate number of determinate things” (Klein 1968, 131), where these “things” are the species, or powers of the unknown: dirhams, things, *māls*, cubes, etc.

Abū Kāmil may present the solution to his problem rhetorically, but he would have worked out the calculations in some kind of notation on a dust-board or another erasable surface. Books were regarded as transcriptions of lectures, and since notation serves no purpose when reading aloud, it is not written in the manuscripts. We know of the Arabic algebraic notation that developed in the western part of the Islamic world around the twelfth century only because some textbook authors show it to instruct students in its use. Even there the problems and solutions are still written out in words. In Italian algebra notation starts to creep into the rhetorical presentations, but the declinations of the signs and the words that appear within it suggest that it was intended to be pronounced. Only in the sixteenth century do we begin to see notation shed its dependence on the rhetorical parts of the text (Oaks 2010; Oaks 2012).

Premodern concepts are naturally reflected in the algebraic notations found in Greek and medieval manuscripts, and they remained in force in books treating algebra

¹⁵It is true that today coefficients are introduced to students first learning algebra as “how many.” My point is that we cannot read premodern polynomials with the modern, theoretical idea of polynomials in mind, where the coefficient is regarded as being multiplied by the term.

in sixteenth-century Europe.¹⁶ What is important for this study is the relationship between the coefficient and the power. As an example, consider the equation “ $1\mathfrak{z} + 8\mathfrak{z}\mathfrak{e} - 24 \text{ æquatur } 72$ ” from Michael Stifel’s 1544 *Arithmetica Integra* (Stifel 1544, fol. 241b). The first-degree unknown is shown by the cossic symbol “ \mathfrak{z} ”, and the second-degree unknown with the symbol “ $\mathfrak{z}\mathfrak{e}$ ”. Translated into modern notation the equation is $x^2 + 8x - 24 = 72$. Stifel’s version may look tantalizingly modern, but the “ $8\mathfrak{z}\mathfrak{e}$ ”, for instance, was not read like our “ $8x$.” There the “ 8 ” and the “ $\mathfrak{z}\mathfrak{e}$ ” play different roles than their modern counterparts. Where our x represents the value of the first-degree unknown, Stifel’s $\mathfrak{z}\mathfrak{e}$ denotes a kind or denomination (*denominatio*) of number (Stifel 1544, fol. 228ff; see also fol. 7b). In fact, as Stifel himself notes, the cossic signs are much like denominations of coins (Stifel 1544, fol. 79bff; fol. 231a).¹⁷ Writing “ $1\mathfrak{z} + 8\mathfrak{z}\mathfrak{e}$ ” is like writing “1 euro and 8 dollars,” a particular amount of money. We do not include a coefficient of “1” for the x^2 in our $x^2 + 8x$ because x^2 denotes the value of a particular unknown number. But Stifel could not write “ $\mathfrak{z} + 8\mathfrak{z}\mathfrak{e}$ ” because it would be like saying “euros and 8 dollars.” This leaves unanswered the question of how many euros there are. Premodern polynomials are inventories in which each denomination or kind requires a number (coefficient) to go with it. By itself $\mathfrak{z}\mathfrak{e}$ is a *kind* of number, and only with a coefficient, like “ $1\mathfrak{z}\mathfrak{e}$ ” or “ $8\mathfrak{z}\mathfrak{e}$ ”, does it stand for a *value*. This is why Stifel and others consistently place a “1” before a sign when there is only one of them. An important consequence of this is that the cossic signs $\mathfrak{z}\mathfrak{e}$ and \mathfrak{z} are not subject to operations the way our x and x^2 are. Only with a coefficient can they be multiplied, added, etc. to form algebraic expressions. For this reason in particular the concatenation of the number with the sign cannot mean multiplication. Simple numbers like the 24 and 72 in Stifel’s equation require no particular sign. The entire polynomial “ $1\mathfrak{z} + 8\mathfrak{z}\mathfrak{e} - 24$ ” is a collection of nine objects of two different kinds, diminished by 24 objects of a third kind.

From Arabic to Latin and Italian

The practice of applying roots to entire terms in algebra was consistently applied throughout the medieval period. ‘Alī al-Sulamī (ca. 10th century), for example, multiplies “a thing” by “a root of twenty” to get “a root of twenty *māls*” (‘Alī al-Sulamī, ff. 61b–62a; see also fol. 97a). At one point al-Karajī (early 11th century) multiplies “a root of ten by ten less two things” to get “a root of a thousand dirhams less a root of forty *māls*” (Problem (II.49) (Saidan 1986, 204.2). In the western part of

¹⁶I have described the Arabic and Italian notations in Oaks 2012 and Oaks 2010, §4. There are differences in the ways the notations functioned in these three languages, but the fundamental idea of aggregations is common to all of them.

¹⁷Both al-Bīrūnī (eleventh century) and Luca Pacioli (1494) also compare the names of the powers of the unknowns in algebra with denominations of coins in their presentations of algebra (al-Bīrūnī 1934, 37–38; Pacioli 1494, fol. 112a.42).

the Islamic world Ibn al-Bannā’ (d. 1321) multiplies $\sqrt{3}$ by $5x$ to get $\sqrt{75x^2}$ (Saidan 1986, 537.10). Similar examples can also be found in the works of al-Māhānī (mid 9th century), al-Samaw’al (12th century), and Ibn al-Hā’im (1387).¹⁸ Other algebraists do not show the right kinds of computations to illustrate this, including among others al-Khwārizmī, Ibn Turk, al-Khayyām, Sharaf al-Dīn al-Ṭūsī, Ibn Badr, and al-‘Āmilī.

Most medieval Latin texts are translations or adaptations from Arabic, so it should be no surprise that the “number” of a term is always rational there, too. In problem <113> of the *Liber Mensurationum*, a twelfth-century translation from Arabic, the side of an equilateral triangle is named a *rem* (“thing”), making the altitude “radix trium quartarum census” ($\sqrt{\frac{3}{4}x^2}$) (Busard 1968, 111). In his *Liber Abaci* (1228) Fibonacci took over many problems involving roots from Abū Kāmil, but he also included some that are not currently known from any Arabic source. In one of these, problem (66), he multiplies “res, et denari 10; que multiplicata in radicem de 5, faciunt radicem 5 censuum, et radicem 500 denariorum” ($(x + 10) \cdot \sqrt{5} \rightarrow \sqrt{5x^2} + \sqrt{500}$) (Fibonacci 1857, 430.27; Fibonacci 2002, 582.33). Jean de Murs’s 1343 *Liber Quadrupartitum* also reproduces many problems that ultimately come from Abū Kāmil, and he shows the same treatment of roots (Jean de Murs 1990, 408, 414, 442, 450, 452). Other Latin books, like the *Liber Augmenti et Diminutionis* and the *Liber Mahameleth*, show no examples of an irrational root multiplied by an algebraic term.

When multiplying a monomial by a root, Italian abacists work the product as it was done in Arabic. Maestro Benedetto of Florence (1463) gives a full explanation of the product of $\sqrt{12}$ by $8x$ (Note: *censo* is the Italian translation of the Arabic *māl*):

Multiply the root of 12 by 8 things. First you render the root of 12 rational, whereby you multiply the root of 12 by itself, making 12. And similarly you multiply 8 things by itself, making 64 *censi*. And you multiply 12 by 64 *censi* to make 768 *censi*, and you say that this multiplication is the root of 768 *censi*, and the same goes for each of the other names.¹⁹

The same treatment is found in every abacus book I have seen that deals with such computations, down through Luca Pacioli’s 1494 *Summa de Arithmetica*. Among many examples, Pacioli posits the side of a square as “una cosa” (i.e. “a thing,” or x), and he calculates its diagonal as “radice di 2 *censi*” (“root of two *censi*,” or $\sqrt{2x^2}$). In this problem he sets up the equation “the *censo* is equal to 4 things and root of 8 *censi*”

¹⁸Examples: (Ben Miled 1999, 146.8; al-Samaw’al 1972, 216.5; Ibn al-Hā’im 2003, 254.3).

¹⁹“Multiplicha la radice di 12 vie 8 chose. Prima arrecheraj la radice di 12 a rationale, dove moltiplicheraj la radice di 12 in se medesima, fanno 12 e similmente moltiplicheraj 8 chose in sé medesime, fanno 64 *censi* e moltiplicheraj 12 vie 64 *censi*, fanno 768 *censi* e diraj che quella moltiplicatione sia la radice di 768 *censi* e chosì di ciaschuno degli altrj nomj” (Benedetto da Firenze 1982, 26.25). By “names” he means the names of the powers of the unknown.

$(x^2 = 4x + \sqrt{8x^2})$, where the right side of the equation is obtained from doubling $2x + \sqrt{2x^2}$.²⁰

An interesting slip-up occurs in an anonymous fourteenth-century *Trattato di Geometria Pratica*. At one point in the solution to problem [246] the author wants to calculate $(45 + x^2) - (\sqrt{27} - x)^2$, after which the result is to be divided by 12. In the excerpt translated below, the $\sqrt{27} - x$ is squared, resulting in $27 - \sqrt{108}x + x^2$. Note the irrational number of *cose*! Then the remainder after subtracting is found to be $18 + \sqrt{108}x$, which is then divided by 12. As the author is about to divide the $\sqrt{108}x$ he notices that the term is not proper, so he converts it to $\sqrt{108x^2}$ and then proceeds with the division, getting $\sqrt{\frac{3}{4}x^2}$ (in this book “1/c” means “1 *cosa*”, and “1/z” means “1 *zenso*,” a variation on *censo*):

multiplying the root of 27 less 1/c by itself makes 27 less the root of 108 *co[se]* and plus²¹ 1/z. Subtracting it from 45 [and] 1/z leaves 18 plus the root of 108/c. You must divide by double the base, which is Ab, which is by 2 by 6, making 12. Divide 18 by 12 to get 1 1/2; divide the root of 108/c, which is to say 108/z and not *co[se]*. Divide [the root of] 108/z by 12: render 12 a root, which is 144, to get 3/4, and the root of 3/4 of z comes out.²²

In all other instances in this book, including the calculations later in the same problem on the next two pages, the whole term is put under the root.²³

Recall that to multiply a root by a quantity, Maestro Benedetto first made the quantity a root by squaring it. In the medieval Italian books this was done even when the quantity had more than one term. In problem (27) of an anonymous *Trattato d'Algebra* from the 1390's the author needs to multiply $25 - x^2$ by $\sqrt{14}$. Rather than multiply the $\sqrt{14}$ by each term as Abū Kāmil would have done, this algebraist first converts $25 - x^2$ into $\sqrt{625 - 50x^2 + x^4}$, and then multiplies it by $\sqrt{14}$ to get $\sqrt{8750 - 700x^2 + 14x^4}$:

for such a multiplication it is necessary to render the 25 less a *censo* into a root, which will be the root of 625 less 50 *censi* plus a *censo di censo*, and this quantity multiplied against the root of 14 makes the root of 8750 less 700 *censi* plus 14 *censi di censi*.²⁴

²⁰“lo censo è iguali a 4 cose e R di 8 censi” (Pacioli 1494 II, fol. 54a, line 7 of par. 4).

²¹Here I write “plus” to translate the Italian *più*. This word has a meaning close to “more,” “further,” or the colloquial “in addition to.” It does not designate the arithmetical operation of addition (see Oaks 2010, §3.7).

²²“multiplica radice di 27 men 1/c in sè, fa 27 men radice di 108 co. e più 1/z, trallo di 45 1/z, rimane 18 più radice di 108/c, dielo partire per lo doppio della basa, cioè .A.b., coì in 2 via 6, fa 12; parti 18 in 12, vien 1 1/2, parti radice di 108/c, che ssi vuole dire 108/z, e non co., parti [radice di] 108/z in 12, recha 12 a radice, che è 144, vien 3/4 e radice di 3/4 di z viene” (Anonimo Fiorentino 1993, 172.23).

²³Problems [67] p. 83, [68] p. 85, [96] p. 97, [153] p. 127, and [172] p. 135.

²⁴“per la quale multiplicazione fare sarà di bisogno d'arechare lo 25 meno uno censo a radice, che sarà radice di 625 meno 50 censi più uno censo di censo, e questa quantitate multiplicada chontro alla radice di 14 fa radice

I have found only two other examples of an Italian abacist multiplying an irrational root by a polynomial, and in both cases the operation is performed the same way. Piero della Francesca, in his *Trattato d'Abaco* (ca. 1470–80), multiplies $\sqrt{18}$ by $10 - 2x$ to get $\sqrt{1800 - 720x + 72x^2}$, and Raffaello Canacci, in his *Ragionamenti d'Algebra i problemi* (ca. 1495), multiplies $15 - x$ by $\sqrt{18}$ to get $\sqrt{18x^2 - 540x + 4050}$ (Piero della Francesca 1970, 164; Canacci 1983, 25.31).

Nicholas Chuquet, too, follows the traditional way in his *Triparty en la Science des Nombres*, composed in French in 1484. For example, at one point he writes “Ores multiplie .1.¹ par R.² 72. m̄. 1. si auras R.² 72.² m̄. 1.¹”. In modern notation this is $x \cdot (\sqrt{72} - 1) \rightarrow \sqrt{72x^2} - x$ (Marre 1880, 813.15).

The problems in Abū Kāmil's *Book of Algebra* continued to be translated and copied in medieval Europe. His problem (41), translated in the previous section, reappears several times. It is of course in the Latin translation preserved in a fourteenth-century manuscript and in the fifteenth-century Hebrew translation of his book, and it is also found in Fibonacci's *Liber Abaci*, Jean de Murs's *Liber Quadripartitum*, and in a fifteenth-century medieval Italian translation of Fibonacci's chapter on algebra (Sesiano 1993, l. 2426; Levey 1966, 150; Fibonacci 1857, 434.26; Fibonacci 2002, 587.4; Jean de Murs 1990, 441; Salomone 1984, 58, 63).²⁵ The same treatment of roots is found in all these incarnations.

So far I am aware of no counterexamples to my claim that medieval algebraists forbid the number of a term to be irrational. The anonymous Italian abacist nearly provided one, but he caught and corrected his error. Because someone else may have made the same kind of error, there may be a counterexample or two lurking in some book I have not consulted. Such counterexamples, if they are found, would not alter the fact that algebraists in Arabic, Latin, and Italian deliberately and routinely insisted on rational coefficients. That practice, which might seem odd to us, is a natural outcome of their concept of a monomial.

Multitudes of Roots and Fractions in Arithmetic

This idea of “a number of” a particular object was not restricted to algebra. It also applies to the ways medieval mathematicians expressed roots and fractions. It will be worthwhile to take a digression into arithmetic now to explain this, not just for the broader arithmetical context it provides, but also because it will give us the background to understand a transition in the way roots of numbers were expressed by some European mathematicians beginning with Descartes.

di 8750 meno 700 censi più 14 censi di censi” (Franci & Pancanti 1988, 127). This is from the same problem as Abū Kāmil's (41), except with $\sqrt{14}$ in place of $\sqrt{5}$.

²⁵In Jean de Murs a different solution of Abū Kāmil is presented, but it shows the same way of working with roots.

Let us back up and look at Abū Kāmil's answer to problem (41): $5 - \sqrt{225 - \sqrt{50000}}$ and $5 + \sqrt{225 - \sqrt{50000}}$. His "root of fifty thousand" is the result of multiplying 100 by $\sqrt{5}$. He says "a root of fifty thousand" ($\sqrt{50000}$) and not "a hundred roots of five" ($100\sqrt{5}$), which we would prefer, because "a hundred roots of five" is a *hundred* numbers. To express it as a *single* number he squares the 100 and multiplies it by the five to get "a root of fifty thousand." While $100\sqrt{5}$ is a perfectly acceptable number to us, medieval algebraists preferred one root to a multitude of roots.

Multitudes of roots crop up sometimes in the course of solving problems, and they are always converted to a single root in the end. 'Alī al-Sulamī, for example, poses the problem of adding $\sqrt{10}$, $\sqrt{40}$, $\sqrt{90}$ and $\sqrt{250}$. He converts the second, third, and fourth numbers to "two roots of ten," "three roots of ten," and "five roots of ten" respectively, and then he adds them all to get "eleven roots of ten." He then gives the instruction to "make this the root of one number," and his answer is $\sqrt{1210}$ ('Alī al-Sulamī, fol. 29b.8).

Al-Hawārī expresses the idea of multitudes in his commentary on Ibn al-Bannā's *Talkhīṣ* (1305): "Add half the root of twenty to two roots of five. Half the root of twenty is less than one root, so we convert it to one root, as seen in the chapter on division. After working the multiplication it becomes a root of five. And two roots of five are more than one root, so we convert it to one root, to get a root of twenty" ([al-Hawārī 2013, 179.20).²⁶

Al-Khwārizmī explains how to convert a multitude of roots and fractions of a root into a single root in his algebra book. To double the root of a quantity "you multiply two by two, then by the quantity. The root of the result becomes twice the root of that quantity" (al-Khwārizmī 2009, 131; my translation). He gives similar instructions for triple the root, half of the root, etc. Later, in the chapter on mensuration, he multiplies 5 by $\sqrt{75}$ by squaring the five, and then multiplying the 25 by 75 to get $\sqrt{1875}$ (al-Khwārizmī 2009, 219). Here is one more example, from al-Karājī (1011/12 CE): "Two roots of ten is a root of what quantity? Multiply two by two to get four. Multiply it by ten to get forty. So a root of forty is two roots of ten" (Saidan 1986, 121.19). Think of "two roots of ten" as the pair ($\sqrt{10}$ $\sqrt{10}$), and not as the single number $2\sqrt{10}$.

This rule is likewise applied in all medieval Latin and Italian books I have seen. This next problem comes from an anonymous fourteenth-century *Trattato dell'Alcibra Amuchabile*:

I want to multiply 3 by the root of 34. Do it like this: you must render the 3, which is the number, to a root, and say 3 by 3 makes nine. Then you must multiply the root of

²⁶Of course he could have stopped after the first step and added the $\sqrt{5}$ to $2\sqrt{5}$ to get $3\sqrt{5}$, which is $\sqrt{45}$, but al-Hawārī chose to reduce this to a previously solved problem type, that of adding two single roots.

nine by the root of 34. Nine by 34 makes 306, and you say that multiplying 3 by the root of 34 makes the root of 306, and it is done.²⁷

Mathematicians continued to write single roots throughout the sixteenth century. Cardano multiplies “R cu 5” by 10 to get “R cu 5000” ($\sqrt[3]{5} \cdot 10 \rightarrow \sqrt[3]{5000}$) (Cardano 1539, 188); Bombelli writes “2 via R.q. 5 fa R.q. 20” ($2 \cdot \sqrt{5} \rightarrow \sqrt{20}$) (Bombelli 1966, 159); and Stevin doubles $\sqrt{735}$ to get $\sqrt{2940}$ (Stevin 1585, 207). Two examples of many in Viète show that he prefers $\sqrt{59319}$ to our $39\sqrt{39}$ and $\sqrt{32}$ to $4\sqrt{2}$ (Viète 1615, 7, 118; Viète 1646, 87, 153). Mathematicians in the early seventeenth century continued this practice, including Christoph Clavius (1608), Albert Girard (1629), Marino Ghetaldi (1630), and Thomas Harriot (1631). Harriot, for instance, writes $\sqrt{18252}$ instead of $78\sqrt{3}$ (Harriot 1631, 101; Girard 1629, 23; Ghetaldi 1630, 221, 224). The earliest modern treatment of roots I have found is in René Descartes’s *La Geometrie* (French, 1637). On page 379 he writes the equation “ $x^3 - \sqrt{3}xx + \frac{26}{27}x - \frac{8}{27\sqrt{3}} \propto 0$ ”; and on page 380 he writes the numbers “ $\frac{2}{9}\sqrt{3}$ ”, “ $\frac{1}{3}\sqrt{3}$ ”, and “ $\frac{4}{9}\sqrt{3}$ ”. I will come back to this development below.

Multitudes of numbers extend even to fractions. In Arabic there are two basic ways of expressing fractions. If the denominator is ten or less, they say it just as we do, as “seven tenths” or “three fourths.” In this latter fraction the “three” indicates *how many* “fourths” there are. Even our own words “numerator” and “denominator” reflect this conception. The “denominator” is the *name* or kind of object, and the “numerator” is their number. In Arabic, when the denominator is larger than ten and the fraction cannot be reduced to some combination of smaller fractions, the language of “parts” is used. For $\frac{4}{13}$ one says “four parts of thirteen parts of a dirham.” Here it is the “parts” that are counted. A “dirham” (i.e. the unit) is partitioned into 13 equal parts, and the fraction is four of those parts. In medieval Italian the texts generally show fractions the way we do, using Arabic numerals and the division bar.

So in medieval arithmetic and algebra, numbers can be counted just like bricks, chickens, and shoes. “Five chickens,” “five sevenths,” “five roots of three,” and “five things” ($5x$) are all collections of five objects. Only in the case of roots is it possible to turn the multitude into a single number. It is clear, then, why medieval algebraists never allowed the “number” of a term to be irrational even if it complicates the expressions. It simply makes no sense to speak of a first degree term as “a root of three things.” Our $\sqrt{3}x$ is perfectly fine because the $\sqrt{3}$ is regarded as a number *multiplied* by the x , or as a quantity that *scales* the x . It is not how many x ’s there are.

²⁷“Jo voglio multiprichare 3 via radicie di 34. Fa così: convienti rechare il 3, ch’è numero, a radice e di 3 via 3 fa nove, adunque ti conviene multiprichare radicie di nove via radicie di 34, che nove via 34, che fa 306 e dirai che multiprichando 3 via radicie di 34 fa radicie di 306 ed è fatto” (Simi 1994, 17).

Irrational Coefficients in Sixteenth-Century European Algebra

Just as the case in Arabic, Italian algebra was largely the domain of practitioners. It was not taught in the early universities, and only in the sixteenth century did it draw the serious attention of more theoretically minded mathematicians. Many significant advances were made then, most notably the solutions to general cubic and quartic equations, which are linked to the introduction of negative and complex numbers. One development that has gone unnoticed is that some sixteenth-century algebraists began to allow the “number” of a term to be irrational.

The earliest text I know to deviate from the medieval rule is Christoff Rudolff's 1525 *Coss*.²⁸ Rudolff makes the adjustment in four instances in 240 problems. In each case the “number” of the term is a binomial. In one problem he multiplies “ $1\text{ }2\text{e} + 2 + \sqrt{2}\text{ }9$ ” by “ $1\text{ }2\text{e}$ ” to get “ $1\text{ }3 + 2 + \sqrt{2}\text{ }2\text{e}$ ”, or in modern notation, $(x + 2 + \sqrt{2}) \cdot x \rightarrow x^2 + (2 + \sqrt{2})x$. He adds this comment in parentheses immediately after: “Note that the binomial would be taken as a quantity, or would be spoken as $2 + \sqrt{2}$, 2e ” (Kaunzner & Röttel 2006, 113, 243a).²⁹ Rudolff is aware that it is not customary for the “quantity” of a term to be “deaf” (i.e. irrational), so he has taken the step of treating it *as if it were* “spoken.” Had he followed the old way he would have expressed the product as “ $1\text{ }3 + 2\text{ }2\text{e} + \sqrt{2}\text{ }2\text{e}$ ”, where the last term corresponds to our $\sqrt{2x^2}$, like we saw in Abū Kāmil.

In the other instances Rudolff multiplies the binomials $8 + \sqrt{20}$, $18 + \sqrt{648}$, and $10 + \sqrt{18}$ by “ $1\text{ }2\text{e}$ ” to get “ $8 + \sqrt{20}\text{ }2\text{e}$ ”, “ $18 + \sqrt{648}\text{ }2\text{e}$ ”, and “ $10 + \sqrt{18}\text{ }2\text{e}$ ” respectively (Kaunzner & Röttel 2006, 62, 63, 118, 196d, 197c, 249b). Rudolff's notation is not ambiguous because he uses the symbol “ 9 ” for units. Thus “ $1\text{ }3 + 2 + \sqrt{2}\text{ }2\text{e}$ ” means $x^2 + (2 + \sqrt{2})x$, while “ $1\text{ }3 + 2\text{ }9 + \sqrt{2}\text{ }2\text{e}$ ” would be $x^2 + 2 + \sqrt{2}x$. In all other cases Rudolff follows tradition by putting everything under the root. In one problem, for example, he takes the square root of “ $\frac{3}{2}\text{ }3$ ” ($\frac{3}{2}x^2$) to get “ $\sqrt{\frac{3}{2}\text{ }3}$ ” (Kaunzner & Röttel 2006, 107, 235d).

Marco Aurel translates many of Rudolff's problems in his *Libro Primero de Arithmetica Algebraica* (Spanish, 1552), including his first, third, and fourth problems with irrational binomials (Aurel 1552, ff. 106b, 122a, 133a). He translates Rudolff's explanation as “nota el binomio $2 + \sqrt{2}$, se toma por sola 1 cantidad, y al presente por 2e ” (Aurel 1552, fol. 122a). Also, one of Rudolff's examples is found among the many problems translated by George Henischus in his *Arithmetica Perfecta et Demonstrata* (Latin, 1609)

²⁸The full title is *Behend vmd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden*. A facsimile of the book is included in Kaunzner & Röttel 2006. I thank Albrecht Heeffer for first pointing this out to me.

²⁹“merck das binomium würt für ein quantitet genomen / als $2 + \sqrt{2}$ würt gesprochen 2e ”.

(Henischus 1609, 374).³⁰ Like Rudolff, both Aurel and Henischus put the whole term under the root in other problems. Both authors appear to have merely copied Rudolff, and may have had no opinion about his binomial coefficients. (I discuss Stifel’s edition of Rudolff’s book below.)

Girolamo Cardano is the next algebraist after Rudolff to admit irrational numbers of terms. In his *Practica Arithmetice, & Mensurandi Singularis* (Latin, 1539), Cardano makes allowance not just for binomials, but for any irrational number. He explains that when taking the root of “ce” (the abbreviation for *censo*, the second degree) or “ce ce” (the abbreviation for the fourth degree), one puts the root *after* the “co” (the abbreviation for the first degree *cosa*) or the “ce” (“R” is an abbreviation for *Radix*, “Root”):

R of a number is a number, such as R 10 is R 10. R *censuum* is co, such as R 7 ce is co R 7. R ce ce ce is ce, as R 10 ce ce ce is ce R 10.³¹ (Cardano 1539, ff. 90b last line, 91a.14, 98a.28)

This way he can write the old “R 7 ce” ($\sqrt{7x^2}$) as “co R 7” ($\sqrt{7}x$). He puts the irrational number after the power in some of his subsequent problems, though he retains the old way in others. Three examples are “10 m R 24 æquantur R 6 ce” ($10 - \sqrt{24} = \sqrt{6x^2}$), “co R cu 8” ($\sqrt[3]{8x}$), and “R 4 ce que sit co R 4” ($\sqrt{4x^2}$, which is $\sqrt{4}x$). He applied the same scheme in his *Ars Magna* of 1545. There, for example, we find “rebus R 12” for $\sqrt{12}x$ (Cardano 1545, fol. 11a.11). And one later example from his 1570 *De Regula Aliza* is the equation “cubus æquantur rebus, R cu.100 p.: 10” ($x^3 = \sqrt[3]{100}x + 10$) (Cardano 1570, 19.10).

Pedro Nuñez follows Cardano by putting the root after the name of the term in his *Libro de Algebra en Arithmetica y Geometria* (Spanish, 1567). He explains:

Since things are the roots of *censo*s, as was shown, to say R 5 ce [$\sqrt{5x^2}$] is like co R 5 [$\sqrt{5}x$]. The proof is very clear, because certainly multiplying co R 5 by itself gives us 5 ce [$(\sqrt{5}x)^2 \rightarrow 5x^2$] and we get the same by multiplying R 5 ce [$\sqrt{5x^2}$] by itself. And for the same reason R 7 ce ce will be ce R 7 [$\sqrt{7x^4} = \sqrt{7}x^2$] (Nuñez 1567, fol. 141a.9).³²

Like Cardano, Nuñez writes roots both ways. Later on the page he has “1 ce es ygual a 2 ð R 8 ce . . . Porque tanto vale como dezir, que 1 ce es ygual a 2 ð co R 8 . . .” ($x^2 = 2 + \sqrt{8x^2}$. . . This is the same as saying that $x^2 = 2 + \sqrt{8}x$).³³

³⁰This calculation is repeated on page 394, and there is a correction for page 374 on the second page of the errata.

³¹“R numeri numerus est, ut R 10 est R 10. R *censuum* est co ut R 7 ce est co R 7. R ce ce est ce veluti R 10 ce ce est ce R 10.”

³²“Que por quanto las cosas son raizes delos censo, tanto montara, dezir R 5 ce como co R 5. La prueua es muy clara, porque cierto es, que multiplicando co R 5 en si, haremos 5 ce y lo mismo haremos multiplicando R 5 ce en si. Y por la misma razon R 7 ce ce sera ce R 7.”

³³Nuñez gives the same explanation on fol. 139b. See also ff. 134a, 206b, 231b.

Johann Scheubel uses irrational coefficients in his *Brevis Regularum Algebrae Descriptio* (Latin, 1550).³⁴ Because he never puts an entire term under a root, he can keep the number on the left. He uses the abbreviation “ra.” for *radix*, the name of the first-degree unknown, and “pri.” for *prima*, the name of the second-degree unknown. In one problem he multiplies “ $\sqrt{448}$ N – 1 ra.” by “1 radix” to get “ $\sqrt{448}$ radicum – 1 pri” ($(\sqrt{448} - x) \cdot x \rightarrow \sqrt{448}x - x^2$) (Scheubel 1550, 66). Like Rudolff, his symbol for units (N) prevents ambiguity. His “ $32 - \sqrt{512}$ ra” (Scheubel 1550, 56) corresponds to our $(32 - \sqrt{512})x$, while “ $\sqrt{448}$ N – 1 radice” is $\sqrt{448} - x$ (ibid., 66).

Rafael Bombelli consistently puts the root on the left in his *L'Algebra* (Italian, 1572). He explains how to multiply an irrational root (“R.q.”) by a monomial (a *dignità*, or “rank”) in terms of the example $\sqrt{5} \cdot 2x \rightarrow \sqrt{20}x$, which he prefaces with a warning against those who also square the rank:

Sometimes it happens that a R.q. is multiplied by a rank, and it is the position of some authors that one must square them both, which even if they manage to do many times, nevertheless it carries the ranks so far that there is no type [*Capitolo*] that matches it.³⁵ So in order not to encounter this setback, keep the following in mind. Multiply R.q. 5 by 2 \downarrow [$\sqrt{5}$ by $2x$]. This proposal is like multiplying *tanti* [i.e. a first degree term] by number, since these R.q. are themselves number (though they can only be named in power, for they do not have a side), so that multiplying 2 by R.q. 5 makes R.q. 20 [$\sqrt{20}$], whose sign you put next to \downarrow , and will make R.q. 20 \downarrow [$\sqrt{20}x$]. (Bombelli 1966, 158–159; Bombelli 1579, 206)³⁶

Here is how he takes the square root of an even-power term:

and if you have to take the side [i.e. square root] of 20 \updownarrow [$20x^2$], take half of the \updownarrow which is 1 and put it in the semicircle to get \updownarrow . Then you take the side of 20, which will be R.q. 20, and this you put next to \downarrow , making R.q. 20 \downarrow [$\sqrt{20}x$]. (Bombelli 1966, 159.24; Bombelli 1579, 207)³⁷

³⁴This work is inserted at the beginning of his *Euclidis Megarensis, Philosophi & Mathematici Excellentissimi, Sex Libri Priores, De Geometricis principijs*.

³⁵I.e. it matches no type of simplified equation. He might mean that the apparent degree of the equation surpasses those that are given solutions.

³⁶“E perchè alcuna volta accade moltiplicare R.q. via una dignità, e da alcuno Autore è stato posto che si debbia quadrare l'uno e l'altro, il che se riesce assai volte, nondimeno porta tanto avanti le dignità che non vi è poi Capitolo per agguagliarlo, però per non incorrere in questo inconveniente tenghisi l'infrascritto ordine. Moltiplichisi R.q. 5 via 2 \downarrow ; questa proposta è come a moltiplicare tanti via numero, perchè queste R.q. anch'elle sono numero, ma non si possono nominare se non in potentia, per non havere lato, che moltiplicato 2 via R.q. 5 fa R.q. 20, al quale pongasi il segno al pari del \downarrow , e farà R.q. 20 \downarrow .”

³⁷“e se si havesse à pigliare il lato di 20 \updownarrow , piglisi il mezo delle \updownarrow ch'è 1, e pongasi nel semicirculo fa \downarrow , poi si pigli il lato di 20, che sarà R.q. 20, e questo si ponga al pari à \downarrow , farà R.q. 20 \downarrow .”

He then adds that when the power is odd, one uses his version of parentheses. These consist of an "L" for "(", and an "J" for ")". So the square root of "6 3" ($6x^3$) is "R.q. L 6 3 J" (Bombelli 1966, 159 last line).

Simon Stevin, on the other hand, employs the symbol "X" to separate an irrational root from the symbol of the power in his *L'Arithmetique* (French, 1585). His $\sqrt{9}X(2)$ corresponds to our $\sqrt{9}x^2$, while $\sqrt{9}X(2)$ is $\sqrt{9x^2}$.

For $\sqrt{9}X(2)$, which is to say root of 9 seconds, the $\sqrt{}$ refers only to 9, & points to (2), where the dividing mark is denoted by X: thus $\sqrt{9}X(2)$ is valued as much (seeing that $\sqrt{9}$ makes 3) as 3 (2). But when the radical number is an incommensurable arithmetical number, like $\sqrt{5}X(2)$, it must remain as it is. But without this dividing mark X, like $\sqrt{9}X(2)$, it will be like saying root of nine seconds. Considering (because there is no division mark) that the $\sqrt{}$ refers to 9, and to (2), $\sqrt{9}X(2)$ is valued as 3 (1). For example, $\sqrt{3}8(3)$ [$\sqrt[3]{8x^3}$] is valued as 2 (1). (Stevin 1585, 40)³⁸

I checked all printed books dealing with algebra I could locate down to 1630, which is about the time Viète's *algebra nova* became widely known. In the four and a half decades after Stevin's *L'Arithmetique* I found only two other books admitting irrational numbers of terms, both from 1620. Clément Cyriaque de Mangin appended algebraic treatments to practical geometry problems in his edition of Jean Errard's *La Geometrie et Pratique Generale d'Icelle*. In one problem, for example, he squares " $\sqrt{21} + 1R - 2$ " ($\sqrt{21} + x - 2$) to get " $1q + \sqrt{84R} + 25 - 4R - \sqrt{336}$ " ($x^2 + \sqrt{84}x + 25 - 4x - \sqrt{336}$). Pietro Antonio Cataldi's *Elementi delle Quantità Algebratiche* shows the new way in many examples. In one of these he squares "5 cose piu rad 7" ($5x + \sqrt{7}$) to get "25 censi piu rad 700 cose piu 7" ($25x^2 + \sqrt{700}x + 7$). (Cyriaque de Mangin 1620, 52; Cataldi 1620, 15). Cataldi's use of the letter "L" for parentheses shows that he was influenced by Bombelli.

Algebraists Who Continued to Follow the Traditional Way

Many prominent sixteenth-century algebraists continued with the old way. Michael Stifel works only with roots of entire terms in his *Arithmetica Integra* (Latin, 1544). For example, in one problem he multiplies "12e" by " $\sqrt{3}12500$ " to get " $\sqrt{3}125003$ "

³⁸"Comme $\sqrt{9}X(2)$, c'est à dire racine de 9 secondes, mais considéré que la $\sqrt{}$ se refere seulement au 9, & point à (2), se que denote la marque de separation X: de forte que $\sqrt{9}X(2)$, vaut autant (veu que $\sqrt{9}$, facit 3) comme 3 (2); Mais quand le nombre radical sera a nombre Arithmetique incommensurable, comme $\sqrt{5}X(2)$, il faut qu'il demeure ainsi; Mais sans icelle separation de la marque X, comme $\sqrt{9}X(2)$, Ce sera aussi à dire racine de 9 secondes, mais considéré (par ce qu'il n'y a point de marque de separation) que la $\sqrt{}$ se refere & a 9, & a (2), de forte que $\sqrt{9}X(2)$, vaut autant comme 3 (1). Item $\sqrt{3}8(3)$ autant comme 2 (1)."

$(x \cdot \sqrt{12500} \rightarrow \sqrt{12500x^2}$. Note that “ $\sqrt{\text{R}}$ ” is the sign for square root) (Stifel 1544, fol. 287a).³⁹ In this book he criticizes Rudolff’s notational innovation:

Note here that in the clause $\sqrt{\text{R}648\text{R}}$, the sign R that you see placed on the right side is considered to be a sign for R because of the sign $\sqrt{\text{R}}$ which is on the left side. If, for example, 6 is multiplied by R , the result is R . But if $\sqrt{\text{R}} 6$ is multiplied by R , the result is $\sqrt{\text{R}} \text{R}$ and not $\sqrt{\text{R}} \text{R}$, as Christoph wanted. When $\sqrt{\text{R}} 6$ is multiplied by R then $\sqrt{\text{R}} \text{R}$ takes the place of R . I give this warning because of the numbers my reader might see in the way Christoph works out his problem 46, so that we may know that my numbers are correct as placed. So $\sqrt{\text{R}} 648\text{R} + \text{R}$ should be taken as a number of roots not because of the addition sign, as Christoph seems to have thought, but because each part refers to a number of roots. (Stifel 1544, ff. 283a-b)⁴⁰

Later, in his 1553 edition of Rudolff’s *Coss*, Stifel converted the four binomials to the old way. For example, he writes “ $\text{R} + \sqrt{20\text{R}}$ ” in place of Rudolff’s “ $8 + \sqrt{20\text{R}}$ ” (Rudolff 1553, ff. 201a, 206a, 386b, 409a).

Niccolò Tartaglia explains that one must take the roots of entire terms his *La Sesta Parte del General Trattato di Numeri, et Misure* (Italian, 1560):

For example, the R of 3 ce cannot be taken, but is represented in this way: R 3 ce, and similarly the R of 5 ce is R 5 ce, and so forth for the other non-square numbers. (Tartaglia 1560, fol. 3b)⁴¹

Here “R 3 ce” corresponds to our $\sqrt{3x^2}$. He puts this into practice in his problems. For example, at one point he multiplies “R $\frac{3}{4}$ cen.” ($\sqrt{\frac{3}{4}x^2}$) by “ $\frac{1}{2}$ co” ($\frac{1}{2}x$) to get “R $\frac{3}{16}$ ce ce” ($\sqrt{\frac{3}{16}x^4}$) (Tartaglia 1560, fol. 32b).⁴²

Christoph Clavius, too, consistently works with the old way in his *Algebra* (Latin, 1608). In one problem he multiplies $150 - \sqrt{4500}$ by x to get $150x - \sqrt{4500x^2}$ (Clavius 1608, 367–8).

³⁹For other examples see ff. 232a, 246a–248a, 251a, 282a–283a, 285a–285b, etc.

⁴⁰“Observabis autem hic, quod in ista particula $\sqrt{\text{R}648\text{R}}$ hoc signum R quod positum vides a parte dextra, reputatur pro signo isto R , propter signum hoc $\sqrt{\text{R}}$, quod stat a parte sinistra. Nam si (exempli gratia) 6 sint multiplicanda per R , tunc fiunt R . Si aut $\sqrt{\text{R}} 6$ sit multiplicandum per R , tunc sit $\sqrt{\text{R}} \text{R}$: & non sit $\sqrt{\text{R}} \text{R}$, ut Christophorus voluit. Quando enim $\sqrt{\text{R}} 6$ multiplicatur per R , tunc recipitur $\sqrt{\text{R}} \text{R}$ pro R . Ista moneo propter numeros, quos Lector meus in Christophoro fortassis videbit, circa operationem huius exempli eius 46 ut sciat meos numeros correctius esse positos. Itaque $\sqrt{\text{R}} 648\text{R} + \text{R}$ haberi debet pro uno numero radicum: non propter signum additiorum, id quod Christophorus videtur existimasse, sed qui ultra que pars sit numerus radicum.”

⁴¹“Essempi gratia la R de 3 ce. non si puo cauare, ma se representarà in questa forma R 3 ce.& cosi la R de 5 ce. è R 5 ce. & cosi discorrendo negli altri de numero non quadrato.”

⁴²For another example see the last page of Book 2 of his *Nova Scientia* (1558).

François Viète works with two different kinds of algebra. One of them, his *logistique numerosa*, corresponds to traditional numerical algebra, and the other, *logistique speciosa*, is his new algebra founded in geometry. Viète has only one opportunity to express an irrational number of a term among the samples of *logistique numerosa* in his works, but he neither follows the new way, nor does he put the whole term under the root. In chapter 5 of *De Emendatione Aequationum* (Latin, 1615) he writes the equation “1C. – lc. 18. in 1N. æquatur 6” (Viète 1615, 91; Viète 1646, 140),⁴³ or in our notation, $1x^3 - \sqrt[3]{18} \cdot 1x = 6$. Viète writes the preposition *in* (“by”) to indicate that the $\sqrt[3]{18}$ is multiplied by the “1N”. This $\sqrt[3]{18}$ is not the “number” of the term, since Viète still retains the “1” before the “N.” This “1C. – lc. 18. in 1N” is Viète’s numerical version of “A cubus – Lc. B solido-solidi in A” ($A^3 - \sqrt[3]{B^{SS}} \cdot A$),⁴⁴ so he has simply transferred the operation from *logistique speciosa* to the version in *logistique numerosa*.

Other books exhibiting the traditional way include: Étienne de la Roche’s *L’Arismethique Nouuellement Composee* (French, 1520), Francesco Ghaligai’s *Summa de Arithmetica* (Italian, 1548), Jacques Peletier’s *L’Algebre* (French, 1554) and *De Occulta Parte Numerorum Libri Duo* (Latin, 1560), M. Valentin Mennher de Kempten’s *Practique pour Brievement Apprendre à Ciffrer, & Tenir Livre de Comptes, avec la Regle de Coss, & Geometrie* (French, 1556), Juan Pérez De Moya’s *Tratado Mathematicasen que Secontienen Cosas de Arithmetica, Geometria, Cosmographia*, etc. (Spanish, 1573), Anton Schultze’s *Arithmetica oder Rechenbuch* (German, 1600), Nicolaus Petri’s *Practique om te Leeren Rekenen* (Dutch, 1605), Christophorus Dibadius’s *In Arithmetica Irrationalium* (Latin, 1605), Anthoni Smyters’s *Arithmetica* (German, 1612), Ludolf van Ceulen’s *Fundamenta Arithmetica et Geometrica* (Latin, 1615), Joannes Lantz’s *Institutionum Arithmeticarum* (Latin, 1619), Claude Gaspar Bachet’s *Diophanti Alexandrini Arithmeticonum Libri sex et De Numeris Multangulis Liber Unus* (Latin, 1621), Hermann Follinus’s *Algebra sive Liber de Rebus Occultis* (Latin, 1622), and D. Henrion’s *Sommaire de l’Algebre* (French, 1623).⁴⁵ In his *Coss* (German, 1525 MS) Adam Ries puts the whole terms under roots, but he does not operate with radicals enough to know if he might have admitted irrational numbers of terms (Ries 1992, 417–420).

The 1984 edition of Dionigi Gori’s 1544 *Libro e Trattato della Praticha d’Alcibra* appears to show the new way, but the two instances are due to transcription errors and problems with the manuscript (Gori 1984, 23, 25).⁴⁶ Likewise the *Arithmetica* by Giels Van den Hoecke (Dutch, 1545, originally published 1537) appears to apply the new way on fol. 91a, but a comparison with fol. 85b shows that it is a misprint. Cajori also

⁴³In the *Opera Omnia* the “lc” is replaced with “ \sqrt{C} ”.

⁴⁴I follow Witmer’s scheme in Viète 1983 for translating Viète’s notation.

⁴⁵Examples are found in La Roche 1520, ff. 68a, 68b, 70a; Ghaligai 1548, ff. 107b–108a; Peletier 1554, 196ff; Peletier 1560, 57ff; Mennher 1556, fol. 156b; Pérez de Moya 1573, 488; Schultze 1600, fol. 238b; Petri 1605, fol. 207b; Dibadius 1605, 6th page of the Prolegomena; Smyters 1612, 55; Ceulen 1615, 232; Lantz 1619, 169, 171; Bachet 1621, 444; Follinus 1622, 155; Henrion 1623, 78.

⁴⁶I thank Raffaella Franci for checking the Siena manuscript for me.

notes misprints in the one passage he translates from this book (Cajori 1928–29 vol. 1, 138; Hoecke 1545).

Some books do not show the right kinds of calculations to know whether their authors subscribed to the new way or not. These include Heinrich Schreiber's *Ayn new kunstlich Buech* (German, 1521) and *Ein new künstlichbehend und gewiss Rechenbüchlin* (German, 1544), Francesco Feliciano's *Libro Arithmetica & Geometria* (Italian, 1545), Bento Fernandes's *Tratado da Arte de Arismetica* (Spanish, 1555), Casparo Peucer's *Logistice Astronomica* (Latin, 1556), Robert Recorde's *The Whetstone of Witte* (English, 1557), Jean Borrel's *Logistica* (Latin, 1559), Petrus Ramus's *Algebra* (Latin, 1560) and *Arithmetices Libri Duo, et Algebræ Totidem* (Latin, 1592), Wilhelm Klebiz's *Insulæ Melitensis* (Latin, 1565), Henricus Brucaeus's *Mathematicarum Exercitationum Libri Duo* (Latin, 1575), Guillaume Gosselin's *De Arte Magna* (Latin, 1577),⁴⁷ Bernard Salignac's *Degalensis Arithmeticae* (Latin, 1580), Zacharias Lochner's *Tractätlein* (German, 1583), Ioseppo Unicornò's *De l'Arithmetica Universale* (Italian, 1598), Nicolaus Raimarus Ursus's *Arithmetica Analytica* (German, 1601), Heinrich Roselen's *Wolgegrünt Kunst und artig Rechenbuch von allerhand Regulen des Kauffmanschafft* (German, 1629), and Albert Girard's *Invention Nouvelle en l'Algebre* (French, 1629).

Coefficients in Viète's *algebra nova* and Its Successors

Our word “coefficient” originated with François Viète's 1591 *Isagoge in Artem Analyticem*. In his *logistice speciosa* the word *coëfficiens* refers to a known yet undetermined magnitude multiplied by a power of an unknown magnitude. One purpose of coefficients is to make the terms in an equation homogeneous. Take for example the equation “A cubus plus B plano ter in A, æquetur D solido” ($A^3 + 3B^P \cdot A = D^S$) (Viète 1615, 10). The “A cubus” is a third-degree term, so to make the second term homogeneous with it the first degree unknown A is multiplied by (3 copies of) the two-dimensional coefficient “B plano.” The “ter” (“thrice”) is not a coefficient, but functions like the medieval “number” of the term to say how many B plano's are multiplied by the A. Note that the “A cubus” is not preceded by a “1”. This is because it represents a value, and not a kind or denomination. Also, the coefficients are explicitly multiplied by the terms, as indicated by the preposition “in.” Viète's notation is modern in these fundamental respects.

Viète's *logistice speciosa* is an algebra founded in magnitudes (*magnitudines*). In order to show an equation numerically he switches to *logistice numerosa*. Here the coefficients are given specific numerical values and the equations are expressed with the premodern

⁴⁷Gosselin puts a whole term under the root when he writes in one place “Sit æquale L7, L3L, multiplicabimus utramque quantitatem in se, fient 7 æqualia 3L.” (“If $\sqrt{7}$ and $\sqrt{3x}$ are equal, we multiply each quantity by itself, to get 7 equals 3x.”), but we do not know how he would have expressed the result of multiplying an irrational root by 1L (Gosselin 1577, fol. 68b).

notation Viète borrowed from Xylander’s translation of Diophantus (1575). The equation above is rewritten as “1C + 6N æquatur 7” ($x^3 + 6x = 7$). The “1” placed before the “C” is one indication that the “C” and “N” represent kinds, while the numbers 1 and 6 tell how many of each of them is present. In Viète’s works the coefficients in *logistica speciosa* are always letters, while those in *logistica numerosa* are always numbers.

Starting with Descartes some algebraists assumed a unit line, which allowed numbers to serve as coefficients in their versions of *logistica speciosa*. Like in Viète, their symbols for the unknowns or variables no longer stand for kinds, but for the numerical values themselves. We have already seen above Descartes’s equation “ $x^3 - \sqrt{3}xx + \frac{26}{27}x - \frac{8}{27\sqrt{3}} \infty 0$ ”, where the x^3 requires no “1” before it, and the $\sqrt{3}$ is a coefficient multiplied by the xx . Two other examples are Abraham de Graaf’s equation $xx \infty \frac{1}{2}y + y\sqrt{\frac{1}{4}}$ from his *De Beginselen van de Algebra of Stelkonst* (Dutch, 1672) and John Wallis’s equation $78\sqrt{3} = 3ddd\sqrt{3} + 3dee\sqrt{3}$ from *A Treatise of Algebra both Historical and Practical* (1685) (de Graaf 1672, 160; Wallis 1685, 178). In Wallis’s “ $3dee\sqrt{3}$ ” the 3, the d , the two e ’s and the $\sqrt{3}$ all signify the values of numbers that are understood to be multiplied by one another by virtue of concatenation. The roots appear on the right in these two equations only to prevent the topless $\sqrt{\quad}$ from being applied to the whole term.

In any numerical algebra that ultimately derives from Viète’s *logistica speciosa* there is no difficulty with irrational coefficients because all elements in a term are understood to be multiplied together. Irrational numbers of terms only pose a problem in cossic, or pre-Viètan algebra, where the coefficient is understood to *count* the term.

With the adaptation of *logistica speciosa* to numbers the notion of a single root vs. a multitude of roots vanishes. Descartes writes $\frac{8}{27\sqrt{3}}$ instead of $\frac{64}{243}$, de Graaf shows $2\sqrt{17}$ rather than $\sqrt{68}$, and Wallis has $14\sqrt{2}$ instead of $\sqrt{392}$ (de Graaf 1672, 163; Wallis 1685, 177). For these mathematicians “ $2\sqrt{17}$ ” is a number composed through multiplication of 2 by $\sqrt{17}$. It is not *two* roots of 17. Those algebraists who restricted their *logistica speciosa* to magnitudes continued writing single roots. Viète wrote $\sqrt{59319}$ and Harriot $\sqrt{18252}$ because for them the old *logistica numerosa*, founded on a concept of polynomials as aggregations, still reigned over numbers. And of course practitioners of cossic algebra in the seventeenth century likewise also continued to work with single roots.

Cossic Algebra post-Viète: Benedetto Maghetti’s *quesiti*

The old and new algebras coexisted for many decades after Viète’s ideas began to take hold around 1630. To name just four examples, Carlo Renaldini’s *Opus Mathematicum* (Latin, 1655), J. R. Brassier’s *Regula Cos, of Algebra* (Dutch, 1663), Andrés Puig’s *Arithmetica Especulativa, y Practica, y Arte de Algebra* (Spanish, 1672), and Giuseppe Maria Figatelli’s *Trattato Aritmetico* (Italian, 1678) all teach algebra the traditional way. In these authors we find the same mix of the old and new ways of representing irrational

roots of terms. Renaldini follows the new way, Brassier and Figatelli follow the old way, and Puig does not perform the right operations for us to know (Renaldini 1655, 109; Brassier 1663, 35; Figatelli 1678, 218).

The same mathematician could even be found working with both cossic algebra and Viète's *logistica speciosa*, as Giovanni Camillo Gloriosi did in his *Exercitationum Mathematicarum Decas Tertia*, henceforth *terza Deca*, as it was called in Italian (Latin, 1639). In the first *exercitatio*, Gloriosi cites Viète and Ghetaldi by name, and he presents theorems using Viète's notation. In the third *exercitatio*, on the other hand, all work is done with pre-Viètan concepts and notation.

I will not give lists of every post-1630 book with cossic algebra to show who followed the old way and who followed the new way. Rather, I will give an account of a controversy over this issue that erupted in Italy toward the middle of the century that gives us an idea of how contemporary mathematicians understood the issue. In 1638 Benedetto Maghetti, a physician in Ancona, published a collection of nine questions (*quesiti*) in a single folio titled "To all those who practice mathematics."⁴⁸ The *quesiti* were posted all over Italy and in a couple of neighboring countries. The first eight *quesiti* ask for the square, cube, fourth, or fifth root of a given polynomial in order to solve an equation, and the ninth asks for a number satisfying a certain condition. The purpose of these questions, stated on the same sheet, was "to teach the way to extract roots of such composite numbers, both with and without algebraic ranks," in other words, to find the roots of polynomials.⁴⁹

The second and seventh *quesiti* show that Maghetti followed the new way of writing terms. Here is the second *quesito*:

Question II. The square root of $9cc - 24qc + 46qq - 40c - rq180c + 25q + rq320q - rq500N + 5 = 94368 - rq5$. One wants the square root without parentheses, as is always intended, and the value of $1N$.⁵⁰

Maghetti is asking for the square root of the polynomial on the left side of the equation, which in modern notation is $9x^6 - 24x^5 + 46x^4 - 40x^3 - \sqrt{180}x^3 + 25x^2 + \sqrt{320}x^2 - \sqrt{500}x + 5$. This root is then equated to the number $94368 - \sqrt{5}$ and the equation is then solved.

⁴⁸"A tutti quelli che professano matematiche." Maghetti's original posting is reproduced in Gloriosi 1639, 105-108.

⁴⁹"per significarli le presenti Questioni esser da me mandate alle stampe per porgere occasione al lettore mi renda avisato se da altri sia stato insegnato il modo d'estrarre le radici di simili numeri composti con Dignità Algebratiche, e senza" (Gloriosi 1639, 105). In Maghetti 1639, 6, he writes that he was inspired for this project by the works of Viète.

⁵⁰"Quesito II. La radice quadra di $9cc - 24qc + 46qq - 40c - rq180c + 25q + rq320q - rq500N + 5 = 94368 - rq5$. si desidera la Radice quadra senza parentesi, e sempre s'intende così in tutte, & il valore di $1N$ " (Gloriosi 1639, 107). Gloriosi writes the ranks with capital letters. Except for "N" I converted them to lower case to match Maghetti's way of writing them.

Maghetti published his solutions and explained his techniques the following year in his 184-page book *Analisi o Risoluzione de Quesiti*. He finds the square root of the polynomial in the second *quesito* to be " $3c - 4q + 5N - rq5$ " ($3x^3 - 4x^2 + 5x - \sqrt{5}$) (Maghetti 1639, 123). Maghetti was aware that writing terms with roots as he does might cause some controversy, so he defends his position early in the book:

Multiplying $rq3$ by $2N$ makes $rq12N$ [$\sqrt{3} \cdot 2x \rightarrow \sqrt{12}x$], which gives rise to a doubt: whether having to square the 2, the number of N , means that one must also square the rank [i.e. the power], and say $rq4q$ [$\sqrt{4x^2}$].

If I must say what I feel, I believe that one can do either one, because either way one can take the square root, or cube root, etc.⁵¹

Maghetti then mentions that Bombelli allows for irrational numbers of terms while Clavius performs the operation in the traditional way. He concludes "I am persuaded to follow Bombelli, and others can follow who they want."⁵²

One of those who answered Maghetti's *quesiti* was Gloriosi, whose solutions occupy the fifth *exercitatio* in the *terza Deca*. Gloriosi was committed to the old way of expressing terms with irrational roots, so he asserts that the polynomial in the second *quesito* should have been written " $9CC - 24QC + 46QQ - 40C - Rq.180CC + 25Q + Rq.320QQ - Rq.500Q + 5$ " (Gloriosi 1639, 111–2). To account for Maghetti's "error," Gloriosi suggested "Perhaps it was from neglect (*per incuriam*) that our author thought that $3C - 4Q + 5N$, multiplied by $Rq.5$, makes $Rq.45C - Rq.80Q + Rq.125N$, which is not true. It is in fact $Rq.45CC - Rq.80QQ + Rq.125Q$."⁵³ Gloriosi justified his correction by citing the abacus rule that if two terms to be multiplied are of "different natures" (*diversæ naturæ*), they must be reduced to one nature before multiplying. In this case a term expressed with a root is of a different nature than one without a root. As Maestro Benedetto explained above, both terms should be squared, and then one takes the root of their product. But of course the issue at hand is not about the process of multiplication, but about how to express the result.

The *terza Deca* was published just a short time before Maghetti's *Analisi*, so Maghetti had not yet seen it when he wrote his book (Maghetti 1639, 180–1; Gloriosi 1641,

⁵¹"multiplichisi $rq3$ via $2N$ fa $rq12N$ qui nasce un dubbio, se havendosi a quadrare il 2 numero di N si debba anco quadrare la dignità, e dir $rq4q$. Se hò da dire il mio senso credo si possa fare l'uno, e l'altro modo, perche nell'uno, e nell'altro modo si può cavar la radice quadra, ò cuba, &c." (Maghetti 1639, 20).

⁵²"m'induco à seguitare il Bombello, & altri segue chi gli pare" (Maghetti 1639, 22).

⁵³"fortassis per incuriam putavit autor quod $3C - 4Q + 5N$ dum multiplicantur cum $Rq.5$ efficiant $Rq.45C + Rq.80Q - Rq.125N$ quod verum non est, efficiunt enim $Rq.45CC + Rq.80QQ - Rq.125Q$." (I corrected the mistakes in sign in my translation) (Gloriosi 1639, 112).

47, 56).⁵⁴ When Maghetti finally did read Gloriosi's remarks he responded with his *Apologia*, a short treatise of 39 pages published in 1640. Now, instead of writing that people can choose whatever notation they want, he maintains that his is the correct way. He explains his position already on the title page: "Where it is proven with very sure demonstrations that one must not square the rank while multiplying a number with rank by a square root, nor cube with a cube root, nor square the square with a fourth root, etc."⁵⁵ Maghetti writes that he "must respond to that fraternal warning made by Giovanni Camillo Gloriosi at the end of the fifth *exercitatione* of his *terza Deca*."⁵⁶ He felt particularly insulted by Gloriosi's accusation that it was *from neglect* that he wrote his polynomials the way he did. In the first 17 pages of the book, where he sets out his defense, Maghetti cites the offending phrase five times.

Maghetti was undoubtedly pleased to find that Gloriosi himself had followed the new way in the sixth *exercitatione* of the *terza Deca* (Maghetti 1640, 6). There, on page 127, Gloriosi gives instructions that are remarkably close to Bombelli's: "When the number does not have a root, and the power has an even index, then the sign of the root is prefixed to the number, and the index is halved, and this gives rise to the sought-after root."⁵⁷ Gloriosi then gives the example that the square root of "28Q" is to be written as "R28N." Maghetti does not cite it, but Gloriosi writes after this that if the index of the power is odd, one uses parentheses. The examples are "33C" and "25C," whose roots are written as "R(33C)" and "R(25C)" respectively. In other places Gloriosi follows the old way. Just before this, on page 125, he multiplies "R24" by "1Q" to get "R24QQ." After explaining his reasoning again, Maghetti concludes "From this one sees that that which I have written and thought & is true, is not otherwise *from neglect*."⁵⁸ In his subsequent arguments Maghetti simply presumes that his reading of the notation is correct.

The debate did not stop there. Gloriosi responded to Maghetti's *Apologia* with another treatise, his 56-page *Responsio Ioannis Camilli Gloriosi ad Apologiam Benedicti Maghetti*, published in 1641. Gloriosi was in a bind for having applied the new way in his *terza Deca*. So he reformulated his argument by appealing to the need to avoid ambiguity, first criticizing those authorities who follow the new way. Gloriosi calls

⁵⁴Gloriosi had sent Maghetti a letter explaining that he had solved all of the *quesiti*, which Maghetti notes in his book.

⁵⁵"Dove si prova con Dimostrazioni certissime non doversi quadrar la dignità, mentre nell'operare occorrà moltiplicare numero con dignità via Radice Quadra, ne cubare via rc, ne quadro quadrare via rqq, &c."

⁵⁶"si deve rispondere à quella fraterna amonizione fatta da Giovan Camillo Gloriosi nella fine della quinta esercitazione della sua terza Deca" (Maghetti 1640, 4).

⁵⁷"Quando numerus non habet radicem, & potestas parem habet indicem, tunc numero præfigatur signum Radicale, & ab indice fumatur dimidium, & procreabitur radix quæsita" (Gloriosi 1639, 127).

⁵⁸"Da tutto questo si veda, che quello, che ho scritto, e pensato, & è vero, e non altrimenti *per incuriam*" (Maghetti 1640, 19). He quotes Gloriosi on page 4, and the other instances where he mentions the phrase *per incuriam* are on pages 7, 11, 12, and 16.

Bombelli’s notation uncertain and ambiguous (“incerti & ambigui”), and the operations of Nuñez and Stevin are said to be illegitimate and to breed confusion (“operatio illegitima est, & confusionem parit”) (Gloriosi 1641, 10). Of course the notations of these mathematicians were not ambiguous at all, but this claim allowed Gloriosi to address his own ambiguity in the *terza Deca*. He writes that “uncertainty and ambiguity are introduced to algebraic operations if the sign of the radical does not refer to the whole term,”⁵⁹ and this leads him to propose a convention: that the “radical sign refer to the whole term, to the number as well as the rank.”⁶⁰

Like Maghetti, in all his subsequent arguments Gloriosi simply presumes that his reading of the notation is correct to show that his opponent is wrong. This way he is able to accuse Maghetti of “mixing the cards” and to make remarks like “I say that my procedure is good and his is false.”⁶¹

Salvator Grisio, working in Rome, was another person who attacked Maghetti’s way of writing terms with irrational roots. Grisio’s 1641 book *Antanalisi a Quesiti Stampati nell’Analisi di Benedetto Maghetti* devotes 144 pages to the issue. Grisio takes Maghetti’s claim that “rq3 by 2N makes rq12N” as implying that 2N multiplied by 2N gives 4N (Grisio 1641, 5). This is because he assumes that Maghetti follows the abacus rule that one squares the 2N before multiplying by the rq3, when in fact he only squares the 2. Throughout the book Grisio presumes that the R *must* apply to the whole term. He does not seem to understand that it can be a matter of convention.

Grisio appeals to many authorities to support his view. He admits that Bombelli explains the multiplications in a way that conforms with Maghetti, but he asserts that Bombelli does not follow this rule in practice. The example he quotes is Bombelli’s multiplication of “R.Q(4N.–6.)” by “3N” ($\sqrt{4x-6}$ by $3x$), resulting in “R.Q.(36C–54Q.)” ($\sqrt{36x^3-54x^2}$).⁶² But this is an example with parentheses, which Bombelli explains along with his other notations. Bombelli does in fact apply his rule for monomials in his worked-out problems, but mainly near the end.⁶³ Grisio later comments on Cardano’s way of putting the irrational root after the name, he mentions that Stifel condemns Rudolff’s binomial coefficients, and he writes that Stevins’s separation mark “ χ ” “is not a necessary thing, and serves nothing.”⁶⁴ Despite Grisio’s familiarity with the algebraic literature from Pacioli down to his own time, he misses the point entirely.

Even farther from any understanding of the issue is the Sicilian priest Pietro Emmanuele, who published the short book *Risposta alli Quesiti di Benedetto Maghetti* in

⁵⁹“Sed quia in operationibus Algebraicis inducitur incertitudo quaedam & ambiguitas, si signum radicale non referatur ad totum compositum” (Gloriosi 1641, 11).

⁶⁰“signum radicale referatur ad totum compositum, ad numerum scilicet & ad dignitatem” (Gloriosi 1641, 11).

⁶¹“scambia le carte,” “io dico che’l proceder mio è buono & il suo è falso” (Gloriosi 1641, 24, 14).

⁶²Grisio has changed Bombelli’s notation to conform with his own, and he writes on page 2 that he used the 1579 edition (Grisio 1641, 7; Bombelli 1579, 218).

⁶³In problems 87, 229, 246, 249, 252, 255, 259, 271, and 272.

⁶⁴“non è cosa necessaria, nè serve à nulla” (Grisio 1641, 12, 21, 38).

Palermo, also in 1641. I have not located any copy of this book, but many quotations from it are preserved in Daniele Spinola's *La Bietolata* (Italian, 1647), a dialogue ridiculing Emmanuele's work.⁶⁵

Emmanuele does not address the issue of irrational numbers of terms. Rather, he sets his sights on solving Maghetti's questions *philosophically* (*filosoficamente*), that is, through alchemy. It is true that Emmanuele does not touch on the main topic of this paper, but I describe his project anyway in order to complete what we know about the influence of Maghetti's *quesiti*.

Emmanuele laments "that algebra, abounding in mathematical precepts, has no place in philosophy," so he wants to show "that the way to solve those *quesiti* is veiled (like Arcanum) in the enigma of Apollo when he taught the art of foretelling to Cassandra."⁶⁶ The allegory is spelled out a few pages later. Apollo (the incarnation of Philosophy, i.e. alchemy), being in love with Cassandra (Arithmetic), teaches her the art of foretelling (solving equations). But once she has learned the art she refuses Apollo's love (Arithmetic shuns Philosophy). Thus Emmanuele sets out to restore Philosophy to Arithmetic.

Spinola gives many quotations of Emmanuele's analysis of the first *quesito*, which is the last one Emmanuele solves. In this *quesito* Maghetti asks for the square root of " $4qc + 12qqc + 25cc + 44qc + 46qq + 40c + 25q$ " and for the value of $1N$ when its root is equal to 969514 (Gloriosi 1639, 107).⁶⁷ Emmanuele has no intention of solving the problem mathematically. Instead, he interprets the finding of the "root" (*radice*) as being the search for the "principle & origin of the thing."⁶⁸ He writes that

the art of making the philosopher's stone is hidden in the figurate numbers according to their mystical signification . . .

The first name of the said septinomial is 4QCC, of which the 4 is the number of Mercury, and the figure QCC (according to Diophantus) has exponent eight, the appropriate number of the element of fire.⁶⁹

⁶⁵The author writes under the pseudonym Landino Alpesei, an anagram of Daniele Spinola.

⁶⁶"che l'algebra, per la quale le matematiche abbondano di precetti, non habbia luogo nella filosofia", "che il modo di solvere detti quesiti sia velato (come Arcano) nell'Enigma d'Apolline quando insegnò l'arte dell'indovinare à Cassandra" (Spinola 1647, 132).

⁶⁷Maghetti uses Bombeli's term *tanto* in place of " $1N$ ".

⁶⁸"principio, & origine della cosa" (Spinola 1647, 152).

⁶⁹"l'arte di fare la pietra filosofale essere ascosa nelli numeri figurati seconda la loro mistica significatione . . . Il primo nome del detto settinomio è 4QCC del quale il 4 è numero proprio del Mercurio, e le figure QCC (secondo Diofanto) hanno per esponente otto, numero appropriato all'elemento del fuoco, onde detto primo nome 4QCC si espone con dire" (Spinola 1647, 160).

The analysis continues in this way for each term of the polynomial, and its root is found to be “the production of Mercury.”⁷⁰

Not only does Emmanuele extract the “root” of the polynomial “philosophically,” but he also proposes to find the value of 1N by “leaving Diophantus and following Pythagoras.”⁷¹ So he treats the number 969514 similarly. He notes that the sum of its digits is 34, which leads him to a four by four magic square and to the “temperament of the philosophical Mercury.” He concludes that the value of 1N “will be the priceless stone.”⁷² Spinola writes “Whoever believes this is crazy.”⁷³

Synthesis

At this point it is natural to ask one of those inherently deep and complex questions about scientific development: What caused some sixteenth-century algebraists to allow for irrational numbers of terms? It is not simply that these mathematicians decided to introduce a new convention in notation. The very way algebraists conceived of monomials forbids coefficients from being irrational. So while Rudolff’s notational innovation may have simplified the solutions of some problems, it was a violation of how monomials had been understood for centuries.

I suspect that in part the development was made possible by the fact that in the early sixteenth century algebraic notation was in the process of becoming autonomous from speech, allowing syntax to take precedence over semantics. With notation replacing verbal expositions for calculations some algebraists could view their notation more formally and take algebra in directions that would have been absurd if expressed rhetorically. This formal treatment of algebraic notation is an example of what Frits Staal has called the “generosity” of an artificial language.⁷⁴ It was in a similar vein that some of the same mathematicians began working with complex and negative numbers around the same time.

If the introduction of irrational coefficients were accompanied by a corresponding shift in algebraic concepts, in other words, if it were something more than just a formal extension of the notational syntax, then the foundation of the medieval aggregations interpretation of polynomials would have been compromised. But I do not see any evidence that this happened. In terms like Cardano’s “co R 8,” the irrational number (here “R 8”, or $\sqrt{8}$) was still treated as if it were the number that *counts* the term. Two features of sixteenth-century algebra suggest this. First, the medieval relationship between the

⁷⁰“la radice del settinomio metallo, quale estratta filosoficamente si ritrova essere la productione del Mercurio” (Spinola 1647, 166).

⁷¹“lasciando Diofanto, e seguendo Pitagora” (Spinola 1647, 166).

⁷²“temperamento del Mercurio filosofico,” “sarà la inapprezzabil pietra” (Spinola 1647, 166).

⁷³“E pazzo ch’il crede” (Spinola 1647, 167).

⁷⁴Staal was fond of quoting d’Alembert: “algebra is generous: she often gives more than is asked from her” (Staal 2007, 405).

number and the sign for the power was retained (see Oaks 2010, 47; Oaks 2012). A “ \mathfrak{r} ”, “co”, “ra,” “ $\mathfrak{1}$,” or “ $\textcircled{1}$ ” does not stand alone if there is only one of them, but the “1” is always written: “ $1\mathfrak{r}$ ”, “1 co,” “1 ra.,” “1 $\mathfrak{1}$,” and “1 $\textcircled{1}$.”⁷⁵ If, by contrast, irrational coefficients were thought of as scalars, or as numbers multiplied by the power, then the signs would have ceased to be kinds of number and would have become values, allowing for them to operate alone like Viète’s A or Descartes’s x . In fact, the only two algebraists who offer brief explanations of their irrational coefficients still situate them in premodern terms. Rudolff urges his readers to regard irrational coefficients as if they were “spoken,” and Bombelli asserts that irrational roots, “though they can only be named in power,” are still numbers, so they too can serve as coefficients.

Second, algebraists continued to solve problems by working the operations before setting up equations.⁷⁶ Irrational numbers of terms did not open the door for operations in equations, which is further evidence that a term like “co R 8” does not denote the multiplication of “co” by “R 8.” Although these irrational numbers look and behave more like modern coefficients than their medieval counterparts, the conceptual shift that that we might associate with them appears not to have occurred. That would come only with the new algebra of Viète, Descartes, and Fermat, in which operations are an integral part of equations.

The debate between Maghetti, Gloriosi, and Grisio over irrational numbers of terms took place after the publication of Descartes’s *La Geometrie* in 1637. In hindsight it was a debate framed in terms of an algebra that was already out of date. Yet Gloriosi for one was not a mathematician who lingered behind the times. He was not only aware of recent developments in algebra, but he took an active part in them. What made the debate relevant was that traditional algebra was still widely practiced in the middle years of the seventeenth century, and at that time no one could have foreseen its demise in the coming decades. With its demise the debate over irrational numbers of terms passed away unresolved.

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⁷⁵Cardano will sometimes have a lone “co” in his verbal explanations, but this is a rhetorical abbreviation for “[a] cosa.”

⁷⁶As in medieval algebra, exceptions were made for divisions and roots (see Oaks 2009, §6).

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