ON RATIONALITY OF ALGEBRAIC FUNCTION FIELDS

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Let A be an algebraic function field with a constant field k which is an algebraic number field. For each prime p of k, we consider a local completion k_p and set $A_p = A \otimes_k k_p$. Then we have the question: Is it true that A/k is a rational function field (i.e., A is a purely transcendental extension of k) if A_p/k_p is so for every p? In this note we shall discuss the question in a slightly different and hence easier case. In place of an algebraic number field k in the above we shall take an algebraic function field as the constant field of A and shall show that the above question has a negative answer.

The proof for it essentially depends on that the so-called Hasse's norm theorem for cyclic extensions of algebraic number fields does not hold in our algebraic function fields case, as is shown in the following lemma.

LEMMA. Let $K = k(x_1, \ldots, x_n)$ be a rational function field over k. Let σ be an automorphism of K/k defined in such a way that $x_i^\sigma = x_{i+1}$ $(x_{n+1} = x_1)$. Let G be a cyclic automorphism group of K generated by σ and F the G-invariant subfield of K in the sense of Galois theory. Then there exists an element a in K such that a is a norm at K_p/F_p for every p but is not a norm at K/F for k given in the proof.

Proof. According to the class field theory (cf. for example [1]), there exist an algebraic number field k and an integer n such that $k^{*n} \subsetneq \bigcap_p k_p^{*n}$, where k^* denotes the multiplicative group of non zero elements of k, and k^{*n} the subgroup of k^* consisting of n-th powers of elements of k^*. Let $a \in \bigcap_p k_p^{*n} - k^{*n}$. Then a is a norm at K_p/F_p, but it cannot be a norm at K/F since $a \notin k^{*n}$.

Let $y_1, \ldots, y_{n-1}$ be algebraically independent elements over K_p (hence over K), and consider $A = K(y_1, \ldots, y_{n-1})$. Let σ' be an automorphism of A/K such that $y_1^{\sigma'} = y_2, \ldots, y_{n-2}^{\sigma'} = y_{n-1}$ and $y_{n-1}^{\sigma'} = a(y_1 \cdots y_{n-1})^{-1}$ with a in the above lemma. σ' generates a cyclic group G' of order n. In this case, $\mathrm{norm}_{G'}(y_1) = a$. On the

https://doi.org/10.4153/CMB-1969-044-0 Published online by Cambridge University Press

other hand, the automorphism σ in the above lemma is extended to an automorphism of A by setting $y_i^{\sigma} = y_i$. Now set $\bar{\sigma} = \sigma\sigma'$. $\bar{\sigma}$ generates a cyclic automorphism group $\bar{G}$ of A. Let B be the $\bar{G}$-invariant subfield of A. This is an analogue of the Brauer field defined in [2]. B and K are linearly disjoint in a sense of A. Weil and BK = A.

THEOREM 1. B_p/F_p is a rational function field for every p.

Proof. Let b be an element in k_p such that $a = b^n$, and set $z_i = b^{-1}y_i$ $(i = 1, \ldots, n-1)$. Naturally, $K_p(z_1, \ldots, z_{n-1}) = K_p(y_1, \ldots, y_{n-1})$. More important is that $\mathrm{norm}_{\bar{G}}(z_1) = 1$, which is easily verified. In order to show Theorem 1, we shall show that σ and $\bar{\sigma}$ are conjugate in the group of all automorphisms of $K_p(z_1, \ldots, z_{n-1})$. For that, we extend $K_p(z_1, \ldots, z_{n-1})$ to a rational function field $K_p(t_1, \ldots, t_n)$ with n variables $t_1, \ldots, t_n$ in such a way that $z_i = t_i t_{i+1}^{-1}$ $(i = 1, \ldots, n-1)$. We also extend the automorphism $\bar{\sigma}$ to an automorphism ρ of $K_p(t_1, \ldots, t_n)$ by setting $t_i^{\rho} = t_{i+1}$ $(t_{n+1} = t_1)$. It is easy to verify that the extension ρ of $\bar{\sigma}$ is well defined. Now we shall find an automorphism τ of $K_p(t_1, \ldots, t_n)$ such that $\tau\sigma = \rho\tau$. As a matter of fact, consider an automorphism τ such that $t_i^{\tau} = x_i t_1 + x_{i-1} t_2 + \ldots + x_{i-n+1} t_n$ (here $x_k = x_j$ if $k \equiv j \bmod n$). Then $(t_i^{\tau})^{\sigma} = t_{i+1}^{\tau} = (t_i^{\rho})^{\tau}$ (we set $t_i^{\sigma} = t_i$) and $(x_i^{\tau})^{\sigma} = x_{i+1} = (x_i^{\rho})^{\tau}$. Hence $\tau\sigma = \rho\tau$ as was asserted. From the definition of τ, we see that τ induces an automorphism of $K_p(z_1, \ldots, z_{n-1})$ which we shall denote by the same τ. Then $\tau\sigma\tau^{-1} = \bar{\sigma}$, which implies σ and $\bar{\sigma}$ are conjugate. Now set $y_i' = y_i^{\tau^{-1}}$. Then $[F_p(y_1, \ldots, y_{n-1})]^{\tau^{-1}} = F_p(y_1', \ldots, y_{n-1}')$ and the latter rational function field should coincide with the $\bar{G}$-invariant subfield B_p, which completes the proof of Theorem 1.

THEOREM 2. B/F is not a rational function field.

Proof. Assume that B/F were a rational function field. Then there would exist an F-homomorphism ϕ of B to F and ∞ which induces a discrete valuation of B. While a basis $u_1, u_2, \ldots, u_n$ of K/F is also a basis of A/B (because B and K are linearly disjoint and BK = A as was mentioned before Theorem 1), every element of A is expressed as $u_1b_1 + \ldots + u_nb_n$ with some elements b_i in B. What is more important, we have $(u_1b_1 + \ldots + u_nb_n)^{\bar{\sigma}} = u_1^{\sigma}b_1 + \ldots + u_n^{\sigma}b_n$. Now consider a set

$I = \{u_1 b_1 + \ldots + u_n b_n \mid b_i^{\phi} \neq \infty, \ i = 1, \ldots, n\}$ and a set $P = \{u_1 b_1 + \ldots + u_n b_n \mid b_i^{\phi} = 0, \ i = 1, \ldots, n\}$. I is a ring and P is a prime ideal of I, and $I/P \cong K$ in an obvious manner. We shall identify I/P with K in the above isomorphism. Then the automorphism $\bar{\sigma}$ of I induces σ on K as was remarked first. It will be shown that $y_1 \in I - P$. In fact, express $y_1 = u_1 b_1 + \ldots + u_n b_n$, and find an element b in B such that $y_1 b \in I - P$. The element b exists because ϕ induces a discrete valuation on B. Set $y_1 b \bmod P = c \in K$ in the above identification of I/P and K. Then $\mathrm{norm}_G(c) = \mathrm{norm}_{\bar{G}}(y_1 b) \bmod P = ab^n \bmod P = a(b^{\phi})^n$, from which we can conclude that b^{ϕ} is neither 0 nor ∞. Therefore we could take $b = 1$, in other words $y_1 \in I - P$. Now let u be an element in K such that $y_1 \bmod P = u$ in the above identification. Then $\mathrm{norm}_G(u) = \mathrm{norm}_{\bar{G}}(y_1) \bmod P = a$. This is a contradiction, because a is not a norm at K/F. Thus B/F is not a rational function field.

Theorems 1 and 2 show that the algebraic function field B with the constant field F which is also an algebraic function field is not rational even if it is so everywhere locally. As a final remark, we have seen that the proofs in the above are more or less connected with the theory of the generic splitting field of simple algebras over algebraic number fields. In this regard the author owes much to the beautiful work of P. Roquette [2].

REFERENCES

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