Extension of a Formula by Cayley to Symmetric Determinants

By Lars Gårding

(*Received* 25*th November*, 1946. *Read* 17*th January*, 1947.)

It has been proved by Cayley that if $x_{11}, x_{12}, x_{21} \ldots$ are independent variables, $x = \det(x_{ik})$, $\xi = \det(\xi^{ik})$, $(i, k = 1, \ldots n)$, where $\xi^{ik} = \partial/\partial x_{ik}$, then by formal derivation $\xi x^a = a(a+1)\ldots(a+n-1)\, x^{a-1}$. This is a special case of the formula [1]

$$(1) \qquad \xi^{i_1 \ldots i_m k_1 \ldots k_m} x^a = a(a+1)\ldots(a+m-1)\, x^{a-1} x^{i_1 \ldots i_m k_1 \ldots k_m}$$

where $m = 1, \ldots, n$ and $\xi^{i_1 \ldots} = \det(\xi^{ik})$ with $i = i_1, \ldots i_m$; $k = k_1, \ldots k_m$ and $x^{i_1 \ldots}$ is the algebraical complement of $x_{i_1 \ldots i_m k_1 \ldots k_m} = \det(x_{ik})$, $(i = i_1, \ldots i_m;\ k = k_1, \ldots k_m)$, in $x = x_{1 \ldots n 1 \ldots n}$.

In this note it will be shown that (1) holds also for symmetric determinants where $x_{ik} = x_{ki}$, provided that $\xi^{ik} = \frac{1}{2}\partial/\partial x_{ik}$, $\xi^{ii} = \partial/\partial x_{ii}$, $(i \neq k)$, and the factor on the righthand side is replaced by $a(a+\frac{1}{2})\ldots (a + \frac{1}{2}(m-1))$.

Let the sequences $i_1 \ldots i_m i_{1'} \ldots i_{m'}$ and $k_1 \ldots k_m k_{1'} \ldots k_{m'}$ be obtained from $1 \ldots n$ by even permutations. That is, $i_1 \ldots i_m$ is a set of any m of the first n integers, while $i_{1'} \ldots i_{m'}$ is also such a set but not necessarily the same set. Expanding $x_{i_{1'} \ldots i_{m'} k_{1'} \ldots k_{m'}}$ we have

$$\Sigma_{p'=1}^{m'} (-1)^{p'} x_{i_{1'} k_{p'}} x_{i_{2'} \ldots k_{p'-1} k_{p'+1} \ldots}$$
$$= \Sigma_{p'=1}^{m} x_{i_{1'} k_{p'}} x^{i_1 \ldots i_m i_{1'} k_1 \ldots k_m k_{p'}}.$$

Now write $\boldsymbol{i} = i_1 \ldots i_m$ and $\boldsymbol{k} = k_1 \ldots k_m$, put $x^{\boldsymbol{i}i\boldsymbol{k}k} = 0$ if $i \epsilon \boldsymbol{i}$ or $k \epsilon \boldsymbol{k}$ and use the sum convention of tensor calculus, all sums running from 1 to n. The result is

$$(2) \qquad (n-m)\, x^{\boldsymbol{ik}} = x_{ik}\, x^{\boldsymbol{i}i\boldsymbol{k}k} = x_{ik}\, x^{\boldsymbol{i}i\boldsymbol{k}k}$$

With $y_{ik} = x^{ik}$, Jacobi's formula gives $y_{\boldsymbol{ik}} = x^{m-1} x^{\boldsymbol{ik}}$ and $y^{ik} = x^{n-2} x_{ik}$ so that (2) gives the identity

$$(3) \qquad (n-m)\, y y_{\boldsymbol{ik}} = y^{ik}\, y_{\boldsymbol{i}i\boldsymbol{k}k}.$$

The symmetry $x_{ik} = x_{ki}$ is used in

---

[1] H. W. Turnbull, "The Theory of Determinants, Matrices, and Invariants," London, (1928), p. 116.

https://doi.org/10.1017/S0013091500024810 Published online by Cambridge University Press

LEMMA 1. *Let* $u \doteq i_1 \ldots i_m$ *and* $u_r = i_1 \ldots i_{r-1} i_{r+1} \ldots i_m$ *and analogously for* $\boldsymbol{k}$ *and* $\boldsymbol{k}_s$. *Then*

$$m\, x_{\boldsymbol{ik}} = \Sigma^m_{r,\, s=1} (-1)^{r+s} x_{k_s \boldsymbol{i}_r i_r \boldsymbol{k}_s}.$$

Expanding the righthand side, with $\boldsymbol{k}_{ss'} = (\boldsymbol{k}_s)_{s'}$ written to denote the effect of suppressing both k_s and $k_{s'}$ from the set k, one gets

$$\Sigma^m_{r,\, s=1}\Big((-1)^{r+s} x_{k_s i_r} x_{\boldsymbol{i}_r \boldsymbol{k}_s} + \Sigma_{s'<s} (-1)^{r+s+s'} x_{k_s k_s'} x_{\boldsymbol{i}_r i_r \boldsymbol{k}_{ss}'}$$
$$+ \Sigma_{s'>s} (-1)^{r+s+s'+1} x_{k_s k_s'} x_{\boldsymbol{i}_r i_r \boldsymbol{k}_{ss'}} \Big) = m x_{\boldsymbol{ik}}$$
$$+ \Sigma_r \Sigma_{s'>s} (-1)^{r+s+s'} (x_{k_s k_s'} - x_{k_{s'} k_s}) x_{\boldsymbol{i}_r i_r \boldsymbol{k}_{ss}} = m x_{\boldsymbol{ik}}.$$

LEMMA 2.

$$\xi^{ik} x_{ii\boldsymbol{k}k} = \tfrac{1}{2}(n-m)(n-m+1)\, x_{\boldsymbol{ik}}.$$

Let $\boldsymbol{i}_r$ and $\boldsymbol{k}_s$ be defined as before. Differentiating every x_{rs} in $x_{i\boldsymbol{i}k\boldsymbol{k}}$ and picking out its co-factor one gets

$$2\xi_{ik} x_{i\boldsymbol{i}k\boldsymbol{k}} = 2(\xi_{ik} x_{ik})\, x_{\boldsymbol{ik}} + 2\Sigma^m_{s=1} (\xi^{ik} x_{ik_s}) (-1)^s x_{\boldsymbol{i}k\boldsymbol{k}_s}$$
$$+ 2\Sigma^m_{r=1} (\xi^{ik} x_{i_r k}) (-1)^r x_{i\boldsymbol{i}_r \boldsymbol{k}} + 2\Sigma^m_{r,\, s=1} (\xi^{ik} x_{i_r k_s}) (-1)^{r+s} x_{i\boldsymbol{i}_r k \boldsymbol{k}_s}.$$

Now $2\xi^{ik} x_{rs} = \delta^i_r \delta^k_s + \delta^i_s \delta^k_r$, where $\delta^i_k = 0,\ 1$ according as $i \neq k,\ i = k$. Hence

$$n(n+1)\, x_{\boldsymbol{ik}} + (n+1)\Sigma^m_{s=1} (-1)^s x_{\boldsymbol{i}k_s\boldsymbol{k}_s} + (n+1)\Sigma^m_{r=1} (-1)^r x_{i_r \boldsymbol{i}_r \boldsymbol{k}}$$
$$+ \Sigma^m_{r,\, s=1} (-1)^{r+s} x_{i_r \boldsymbol{i}_r k_s \boldsymbol{k}_s} + \Sigma^m_{r,\, s=1} (-1)^{r+s} x_{k_s \boldsymbol{i}_r i_r \boldsymbol{k}_s}.$$

The last term is given by Lemma 1, the others are plainly multiples of $x_{\boldsymbol{ik}}$. Summing one gets $(n(n+1) - 2m(n+1) + m^2 + m)\, x_{\boldsymbol{ik}}$ $= (n-m)(n-m+1)\, x_{\boldsymbol{ik}}$, which is the desired result.

THEOREM. *If* $\boldsymbol{i} = i_1 \ldots i_m$ *and* $\boldsymbol{k} = k_1 \ldots k_m$ *then*

(4) $$\xi^{\boldsymbol{ik}} x^a = h(a, m)\, x^{a-1} x^{\boldsymbol{ik}}$$

where $h(a, m) = \Pi^m_{k=1} (a + \tfrac{1}{2}(k-1))$, *and* $m = 1, \ldots n$.

If $\xi_{\boldsymbol{ik}}$ is the algebraical complement of $\xi^{\boldsymbol{ik}}$, an equivalent form of (4) is (5) $\quad \xi_{\boldsymbol{ik}} x^a = h(a, n-m)\, x^{a-1} x_{\boldsymbol{ik}}$, $(m = 0, \ldots n-1)$.
When $m = n-1$ one has $\xi_{\boldsymbol{ik}} = \xi^{ik}$ and $\xi^{ik} x^a = a x^{a-1} (\xi^{ik} x_{rs})\, x^{rs} = a x^{a-1} x^{ik}$, so that the theorem is true in this case. Now by virtue of (2), $(n-m)\, \xi_{ik} = \xi^{ik} \xi_{i\boldsymbol{i}k\boldsymbol{k}}$, so that proceeding by induction and using Lemma 2

https://doi.org/10.1017/S0013091500024810 Published online by Cambridge University Press

and (3) we have

$$(n-m)\,\xi_{ik}\,x^{a} = \xi^{ik}h(a,\ n-m-1)\,x_{iikk}\,x^{a-1}$$
$$= h(a,\ n-m-1)\,x^{a-2}((a-1)\,x^{ik}\,x_{iikk} + x\xi^{ik}\,x^{iikk})$$
$$= (n-m)\,h(a,\ n-m-1)\,(a-1+\tfrac{1}{2}(n-m+1))\,x^{a-1}x_{ik}$$
$$= (n-m)\,h(a,\ n-m)\,x^{a-1}\,x_{ik}\,.$$

Hence (5) follows for antisymmetrical determinants x and ξ: (4) is valid with a suitable $h(a,\ m)$ if $m=1$ and n is even and also if $m=n=2$ or 4, but probably in no other cases and certainly not in general.

THE INSTITUTE OF MATHEMATICS,
LUND UNIVERSITY,
LUND, SWEDEN.