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ASYMPTOTIC DIMENSION AND GEOMETRIC DECOMPOSITIONS IN DIMENSIONS 3 AND 4

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Abstract

We show that the fundamental groups of smooth 4-manifolds that admit geometric decompositions in the sense of Thurston have asymptotic dimension at most four, and equal to four when aspherical. We also show that closed 3-manifold groups have asymptotic dimension at most three. Our proof method yields that the asymptotic dimension of closed 3-dimensional Alexandrov spaces is at most three. Thus, we obtain that the Novikov conjecture holds for closed 4-manifolds with such a geometric decomposition and for closed 3-dimensional Alexandrov spaces. Consequences of these results include a vanishing result for the Yamabe invariant of certain 0-surgered geometric 4-manifolds and the existence of zero in the spectrum of aspherical smooth 4-manifolds with a geometric decomposition.

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1. Introduction

The uniformization theorem for topological surfaces, proved by Koebe [35] and Poincaré [47], showed that geometric structures can effectively classify distinct families of manifolds. Their geometric classification scheme involves the three constant sectional curvature two-dimensional geometries.

In dimension three, constant sectional curvature manifolds are insufficient to include all 3-manifolds. W.T. Thurston defined a model geometry as a complete, simply connected Riemannian manifold $\mathbb{X}$ such that the group of isometries acts transitively on $\mathbb{X}$ and contains a discrete subgroup with a finite-volume quotient. A manifold X is said to be *geometrizable*, in the sense of Thurston, if X is diffeomorphic to a connected sum of manifolds that admit a decomposition into

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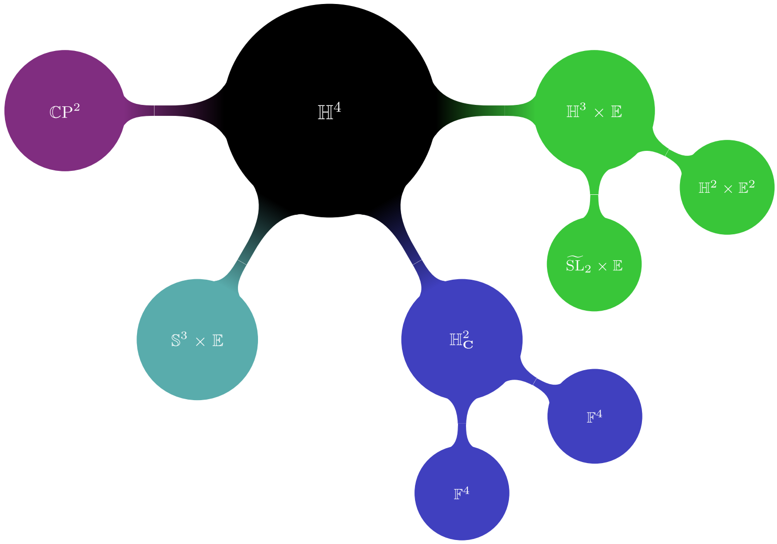

FIGURE 1. A sketch of a 4-manifold X whose connected summands admit a decomposition into Thurston geometries. Each circular region represents a geometric manifold, tagged with its model geometry. Different regions indicate parts that are either geometric or have a proper geometric decomposition. The lower-most region represents a submanifold X_1, which decomposes into pieces modeled on the geometries $\mathbb{H}^2_{\mathbb{C}}$ and $\mathbb{F}^4$, glued along nilpotent boundaries N_1, N_2. In the region on the right, we see a submanifold X_2 that decomposes into $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$, and $\widetilde{\mathbb{SL}}_2 \times \mathbb{E}$ pieces, glued along flat boundaries F_1, F_2. The strips joining different regions represent connected sums. Let the central, real hyperbolic piece be X_0. Then $X = X_0 \# X_1 \# X_2 \# S^3 \times S^1 \# \mathbb{C}P^2$.

pieces, each modeled on a Thurston geometry. In dimension three, there are eight *model geometries*, and manifolds modeled in these serve as building blocks that when assembled produce a global description of a 3-manifold. Comprehensive descriptions of the model geometries in dimension three may be found in Thurston's book [58], and in a survey by Scott [52]. The success of the geometrization program in dimension three by Thurston and Hamilton–Perelman [44] leads us to wonder about the nature of geometrizable manifolds in higher dimensions. Filipkiewicz [22] classified all maximal four-dimensional model geometries. The following list includes all of the four-dimensional Thurston geometries that admit finite-volume quotients.

$$\begin{array}{ccccccc} \mathbb{S}^4, & \mathbb{CP}^2, & \mathbb{S}^3 \times \mathbb{E}, & \mathbb{H}^3 \times \mathbb{E}, & \widetilde{\mathbb{SL}}_2 \times \mathbb{E}, & \mathbb{N}il^3 \times \mathbb{E}, & \\ \mathbb{N}il^4, & \mathbb{S}^2 \times \mathbb{E}^2, & \mathbb{H}^2 \times \mathbb{E}^2, & \mathbb{S}ol^4_{m,n}, & \mathbb{S}ol^4_1, & \mathbb{S}ol^4_0, & \\ \mathbb{S}^2 \times \mathbb{S}^2, & \mathbb{S}^2 \times \mathbb{H}^2, & \mathbb{E}^4, & \mathbb{F}^4, & \mathbb{H}^4, & \mathbb{H}^2 \times \mathbb{H}^2, & \mathbb{H}^2_{\mathbb{C}} \end{array}$$

Detailed explanations and examples for all of these geometries are available in the work of Hillman [31, page 133] and Wall [61]. To keep our exposition short, we recommend that interested readers should consult those sources. In Figure 1, we show a schematic example of one of these manifolds.

We have previously studied the minimal volume entropy problem [55], and the existence of Einstein metrics [15] on manifolds in this family.

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Gromov defined the asymptotic dimension, asdim Γ, of a metric space Γ as a coarse analog of the Lebesgue covering dimension [28] (for details see Section 2.1 below).

Here, we show the following theorem, which is our first main result.

THEOREM 1.1. *Let X be a closed orientable 4-manifold that is geometrizable in the sense of Thurston. Then, the asymptotic dimension of $\pi_1(X)$ is at most 4. Moreover, when X is aspherical,* asdim $\pi_1(X)$ *is equal to* 4.

In the case of 3-manifolds, by the same methods, we show our second, main result.

THEOREM 1.2. *Let Y be a closed 3-manifold. Then* asdim $\pi_1(Y) \leq 3$. *Moreover, when Y is aspherical,* asdim $\pi_1(Y)$ *is equal to* 3.

Our proof uses the geometrization of 3-manifolds, specifically, the description of $\pi_1(Y)$ as a graph of groups. Although it may be known to some experts, Theorem 1.2 improves upon the available published bounds for asdim $\pi_1(Y)$ [20, 39]. In related work, Ren showed that such a $\pi_1(Y)$ has finite decomposition complexity [48]. However, a bound was not made explicit. In the special case of Hadamard 3-manifolds, Theorem 1.2 was shown previously by Lang and Schlichenmaier [36].

Alexandrov spaces are a generalization of smooth manifolds with bounded curvature, in the sense that they include all limits of sequences of smooth manifolds with sectional curvatures bounded below. Briefly, they are locally complete, locally compact, connected length spaces that satisfy a lower curvature bound in the triangle-comparison sense.

Following the same terminology as for 3-manifolds, an Alexandrov three-dimensional space Y is called geometric, with a given model Thurston geometry, if Y can be written as a quotient of that geometry by some cocompact lattice. A closed Alexandrov three-dimensional space is said to admit a geometric decomposition if there exists a collection of spheres, projective planes, tori and Klein bottles that decompose Y into geometric pieces. A geometrization theorem for Alexandrov 3-spaces was shown by F. Galaz-García and Guijarro. They showed that a closed three-dimensional Alexandrov space admits a geometric decomposition into geometric three-dimensional Alexandrov spaces [25, Theorem 1.6]. Moreover, they proved that an Alexandrov 3-space Y may be presented as the quotient of a smooth 3-manifold Y^* by the action of an isometric involution [25, Lemma 1.8].

The universal cover of an Alexandrov space is, by definition, the simply connected cover with the induced metric structure that makes the covering map into a local isometry. Here, the fundamental group of a compact Alexandrov space will be seen as a discrete group of isometries of its universal cover. More details and related rigidity results for Alexandrov 3-spaces may be found in the recent survey by Núñez-Zimbrón [40].

We obtain the following third main result as a consequence of Theorem 1.2.

THEOREM 1.3. *Let Y be a closed 3-dimensional Alexandrov space. Then* asdim $\pi_1(Y) \leq 3$.

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Our proof crucially uses the quasi-isometric invariance property of asdim, applied to a specific action of a group on the universal covering space of the 3-manifold Y^* that produces Y after quotienting out by the isometric involution.

Recall that a space is aspherical if its universal cover is contractible. Our main theorems yield information about the Baum–Connes, Novikov, and zero-in-the-spectrum conjectures, contained in the following corollaries. Although we give more details about these topics in Section 2, we point interested readers to available reviews of these conjectures, for example, by Yu [65], Davis [17], Ferry *et al.* [21], and Weinberger [62].

COROLLARY 1.4. *Let X be a manifold from Theorem 1.1. Then:*

(i) *the coarse Baum–Connes conjecture holds for X;*
(ii) *the Novikov conjecture holds for X; and*
(iii) *if X is aspherical, then its universal cover $\widetilde{X}$ has a zero in the spectrum.*

The second item above includes manifolds constructed using complex hyperbolic pieces, which are not included in previous related work on higher graph manifolds [2, 14, 24]. Moreover, these earlier results do not apply to dimension four, because they depend on arguments from surgery theory that are not yet known to hold for groups of exponential growth [23].

Lott showed that, for a closed geometric 4-manifold X, zero is in the Laplace–Beltrami spectrum of $\widetilde{X}$ [38, Proposition 18]. By comparison, Corollary 1.4 is shown by different methods. It subsumes previous work on aspherical geometric manifolds, and further includes all the aspherical manifolds in Theorem 1.1.

Similarly, Theorem 1.3 has the following consequences.

COROLLARY 1.5. *Let Y be a be a closed 3-dimensional Alexandrov space. Then:*

(i) *the coarse Baum–Connes conjecture holds for Y;*
(ii) *the Novikov conjecture holds for Y; and*
(iii) *if X is aspherical, then its universal cover $\widetilde{X}$ has a zero in the spectrum.*

Previous related work showing the Novikov conjecture holds for singular spaces includes that of Ji on buildings [34], and on torsion-free arithmetic subgroups of connected, rational, linear algebraic groups [33].

A conjecture attributed to Gromov–Lawson–Rosengberg states that there do not exist Riemannian metrics with positive scalar curvature on compact aspherical manifolds. As a consequence of Theorem 1.1 and work of Yu [63] (and, alternatively, of Dranishnikov [18]), the aspherical manifolds in Theorem 1.1 do not admit Riemannian metrics of positive scalar curvature. This result was recently shown for all aspherical smooth 4-manifolds by Chodosh–Li [12] and by Gromov [29]. Nevertheless, our methods provide an independent proof for the manifolds in Theorem 1.1.

A natural approach to understanding how topology and geometry are coupled is by minimizing the curvature that a Riemannian manifold may have. One way to achieve this is to minimize a norm of a curvature tensor. Let (M, g) be a compact Riemannian

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manifold with a smooth metric g. Consider a conformal class of Riemannian metrics, $\gamma := [g] = \{u \cdot g \mid M \xrightarrow{u} \mathbf{R}^+\}$.

The *Yamabe constant* of (M, g) is defined as

$$\mathcal{Y}(M,\gamma) := \inf_{g\in\gamma} \frac{\int_M \mathrm{Scal}_g d\mathrm{vol}_g}{(\mathrm{Vol}(M,g))^{2/n}}.$$

Here, Scal_g denotes the scalar curvature and $d\mathrm{vol}_g$ denotes the volume form of g. The Yamabe invariant of a manifold M is then defined to be $\mathcal{Y}(M) := \sup_\gamma \mathcal{Y}(M,\gamma)$.

Next, we present an application of Theorem 1.1 to the study of the Yamabe invariant.

COROLLARY 1.6. *Let X be a manifold from Theorem 1.1. If the geometric pieces of X are modeled on the geometries*

$$\begin{array}{ccccc} \mathbb{E}^4, & \mathbb{H}^3\times\mathbb{E}, & \mathbb{H}^2\times\mathbb{E}^2, & \mathbb{N}il^4, & \mathbb{S}ol_1^4, \\ \widetilde{\mathrm{SL}}_2\times\mathbb{E}, & \mathbb{N}il^3\times\mathbb{E}, & \mathbb{S}ol_{m,n}^4, & \mathbb{S}ol_0^4, & \textit{or } \mathbb{F}^4, \end{array}$$

then the Yamabe invariant of $X\#k(S^3\times S^1)$, with $k\in\{0,1,2,\dots\}$, vanishes.

This improves upon results by the second-named author [54, Lemme 1.4, Proposition 2.6(i)], covering only closed $\mathbb{E}^4, \mathbb{H}^3\times\mathbb{E}$, or $\mathbb{H}^2\times\mathbb{E}^2$ manifolds. In those cases, the existence of a nonpositive sectional curvature metric obstructs the existence of positive scalar curvature metrics.

When restricted to the special case of symplectic manifolds, Corollary 1.6 includes previously known results, shown by the second-named author with an additional hypothesis [54, Lemme 2.4], and by the second-named author and Torres [56].

Wall [61] showed that there is a close relationship between geometric structures and complex surfaces. So, in Corollary 1.6, there is some overlap with the work of LeBrun, who, as a part of a *tour-de-force* of results on the Yamabe invariant, showed that compact complex surfaces of Kodaira dimension zero or one have null Yamabe invariants [37]. For compact complex surfaces that admit a geometric structure listed in Corollary 1.6, we now have an independent proof that their Yamabe invariant is zero. For example, the compact complex surfaces known as Inoue surfaces are exactly those admitting one of the geometries $\mathbb{S}ol_0^4$ or $\mathbb{S}ol_1^4$ [61]. Albanese recently showed that Inoue surfaces have zero Yamabe invariants [1], and Corollary 1.6 now gives an alternative proof.

Finally, we take this opportunity to include the following result that recovers part of LeBrun's aforementioned theorem [37], and for which we can now produce a simple proof (given what is needed for the previous Corollary).

LEMMA 1.7. *Let X be an aspherical compact complex surface of Kodaira dimension at most one and which is not of class VII. Then $\mathcal{Y}(X) = 0$.*

Here, as usual, the Kodaira dimension $\kappa = \limsup_{m\to\infty}(\log(P_m(X))/\log m)$, where $P_m(X)$ is the dimension of the space of holomorphic sections of the mth tensor power of the canonical line bundle of X, and $\kappa := -\infty$ if $P_m(X) = 0$ for all m. Surfaces of

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Kodaira dimension $-\infty$ that are not Kähler are called surfaces of class VII. These include Inoue surfaces with vanishing second Betti numbers (featured in Corollary 1.6), Hopf surfaces (which are known to be geometric, but are not aspherical), certain compact elliptic surfaces, and surfaces with a *global spherical shell*, which have positive second Betti numbers and are not aspherical. These are conjecturally all the minimal surfaces of class VII. Similar results were previously shown for symplectic 4-manifolds by Torres and the second-named author [56, Theorem 2].

The relevant definitions for the concepts appearing in Corollaries 1.4, 1.5, and 1.6 are found in Sections 2.8, 2.9, 2.10, and 2.11. The proofs of each of the items in Corollaries 1.4 and 1.5 appear as Corollaries 2.25, 2.29, and 2.31.

The proofs of Theorems 1.1 and 1.2 rely on close examinations of the fundamental groups involved; both are in Section 3. We use various properties and operations on groups to bound the asymptotic dimension from both sides. We rely on the fact that the asymptotic dimension of lower-dimensional manifolds is finite. Extending our techniques to higher-dimensional manifolds would first require knowing that the asymptotic dimension of all lower dimensions (appearing on the boundaries) is finite, which is currently an open question. Related results are available for higher graph manifolds, due to the second-named author in collaborations with Connell [14], and with Bárcenas and Juan Pineda [2]. All of these strategies are reminiscent of the original work of Wall [60, Section 12] on codimension-one splittings along a hypersurface, and of Cappell [8, 9] on amalgamated products. However, those arguments from classical surgery theory need to assume that the dimension of the manifold is at least five.

2. Preliminaries and proofs of Corollaries 1.4, 1.5, and 1.6

2.1. Definition of asymptotic dimension. Gromov introduced the concept of the asymptotic dimension of a metric space (X, d) [28]. There are several equivalent definitions. The following definition is the one that we use.

Definition 2.1. We say that the asymptotic dimension of (X, d) does not exceed n, written asdim $X \leq n$, if, for each $D > 0$, there exist $B \geq 0$ and families $\mathcal{U}_0, \ldots, \mathcal{U}_n$ of subsets that form a cover of X such that:

(i) for all $i \leq n$ and all U in $\mathcal{U}_i$, their diameter satisfies diam $(U) \leq B$; and
(ii) for all $i \leq n$ and all U and V in $\mathcal{U}_i$, if $U \neq V$, then $d(U, V) > D$.

Although internalizing this definition may take some time, we recommend consulting the friendly and accessible exposition by Bell [4]. In Figure 2, we see a specific cover, by bricks on a plane, illustrating the two points of Definition 2.1. First, all bricks are isometric, so their diameter is the same. Second, different bricks need to be translated at least a distance D to match, and this quantity depends on the size of the brick (itself determined by its diameter B).

Let Γ be a finitely generated group and let S be a finite generating set. The word length with respect to S, denoted by l_S, of an element $\gamma \in \Gamma$ is the smallest

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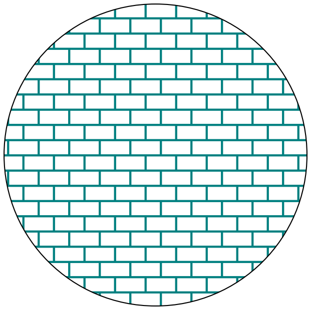

FIGURE 2. This covering by bricks is helpful in understanding why the asdim of $\mathbf{R}^2$ is at most two. Observe that a point in the plane either lies in the interior of a brick, or it lies on the boundary of a brick. In the former case, there is a neighborhood contained in the brick. In the latter, there are two options: it either lies exactly at the point where three bricks meet or it does not. In either of these cases, the neighborhood of the point will intersect at most $(\dim(\mathbf{R}^2) + 1)$ bricks.

integer $n \geq 0$ for which there exist $s_1, \ldots, s_n \in S \cup S^{-1}$ such that $\gamma = s_1 \cdots s_n$. The word metric, denoted by d_S, is defined as $d_S(\gamma_1, \gamma_2) = l_S(\gamma_1^{-1}\gamma_2)$. A finitely presented group, equipped with the word metric, is a metric space. We refer the reader to the work of Bell and Dranishnikov [6] for multiple examples of groups and spaces with finite asymptotic dimension. For a finitely generated group Γ, the asymptotic dimension is a group property, that is, it is independent of the choice of generators [6, Corollary 51].

LEMMA 2.2.

(i) [49, Example 9.6] *The Euclidean n-dimensional space* $\mathbb{E}^n$ *has asymptotic dimension equal to n.*

(ii) [50] *The real hyperbolic n-dimensional space* $\mathbb{H}^n$ *has asymptotic dimension equal to n.*

(iii) [6, Proposition 60] *Let* Γ *be a finitely generated group. Then* $\operatorname{asdim} \Gamma = 0$ *if and only if* Γ *is finite.*

A pair of metric spaces $(X_1, d_1), (X_2, d_2)$ is called *quasi-isometric* if there exists a map $f : X_1 \to X_2$ and constants $B > 0$ and $C \geq 1$ such that:

(1) for every pair of points x, y in X_1,

$$\frac{1}{B} \cdot d_1(x, y) - C \leq d_2(f(x), f(y)) \leq B \cdot d_1(x, y) + C; \text{ and}$$

(2) every point of X_2 lies within a C-neighborhood of the image $f(X_1)$.

A well-known property of the asymptotic dimension is that it is an invariant of the quasi-isometry type of a finitely generated group Γ [28]. As a consequence of the Milnor–Švarc lemma, if M is a compact Riemannian manifold with universal cover $\widetilde{M}$

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and finitely generated group $\pi_1(M)$, then $\widetilde{M}$ is quasi-isometric to $\pi_1(M)$ (with the word metric). Therefore [6, Corollary 56],

$$\operatorname{asdim}(\widetilde{M}) = \operatorname{asdim}(\pi_1(M)). \tag{2-1}$$

As an illustrative example, and because we need it later on, we focus now on the case of surface groups and show the well-known fact that they have asymptotic dimensions bounded above by 2.

LEMMA 2.3. *Let* Γ *be the fundamental group of a closed* 2*-manifold. Then* $\operatorname{asdim}\Gamma \leq 2$.

PROOF. By the uniformization theorem for surfaces, Γ may be represented as either the trivial group, a flat 2-manifold group Γ_F, or the fundamental group of a genus $g \geq 2$ surface with a hyperbolic metric Γ_H. Observe that Γ_F is quasi-isometric to $\mathbb{E}^2$ and Γ_H is quasi-isometric to $\mathbb{H}^2$. Therefore, by Lemma 2.2 and Equation (2-1), in all these cases, $\operatorname{asdim}\Gamma \leq 2$. □

The following result was shown by Carlsson and Goldfarb.

LEMMA 2.4 [11, Corollary 3.6]. *Let* Γ *be a compact lattice in a connected Lie group* G *and let* K *be its maximal compact subgroup. Then* $\operatorname{asdim}(\Gamma) = \dim(G/K)$.

Next, we recall the definition of a coarse space, following Roe's book [49].

DEFINITION 2.5. Let X be a set. A collection of subsets $\mathcal{E}$ of $X \times X$ is called a coarse structure, and the elements of $\mathcal{E}$ are called entourages if the following axioms are satisfied.

(i) A subset of an entourage is an entourage.
(ii) A finite union of entourages is an entourage.
(iii) The diagonal $\Delta_X := \{(x, x) \mid x \in X\}$ is an entourage.
(iv) The inverse E^{-1} of an entourage E is an entourage: that is,

$$E^{-1} := \{(y, x) \in X \times X \mid (x, y) \in E\}.$$

(v) The composition E_1E_2 of entourages E_1 and E_2 is an entourage: that is,

$$E_1E_2 := \{(x, z) \in X \times X \mid \text{there exists } y \in X, (x, y) \in E_1, \text{ and } (y, z) \in E_2\}.$$

The pair $(X, \mathcal{E})$ is called a coarse space.

For example, topological manifolds M are coarse spaces, where entourages may be defined as neighborhoods of points in $M \times M$ [49, Ch. 2].

A pair of coarse spaces X, Y is *coarse equivalent* if there exist coarse maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the compositions $f \circ g$ and $g \circ f$ are close to the identity maps on Y and X, respectively. An action of a discrete group Γ on a metric space X is proper if, for every compact subset $B \subset X$ and for all but finitely many γ in Γ, the intersection $\gamma(B) \cap B = \emptyset$. Let Γ be a discrete group acting properly

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on a proper metric space X. Then the asymptotic dimensions of Γ and X satisfy the following relationship.

THEOREM 2.6 [33, Proposition 2.3]. *Let (M, d) be a proper metric space. If a finitely generated group Γ acts properly and isometrically on M, then, for any point $x \in M$, the map $(\Gamma, d_S) \longrightarrow (\Gamma x, d)$, $\gamma \longrightarrow \gamma \cdot x$ is a coarse equivalence, and hence*

$$\text{asdim}\, \Gamma \leq \text{asdim}\, M.$$

The following extension theorem of Bell and Dranishnikov [6] covers the case of an exact sequence.

THEOREM 2.7 [6, Theorem 63]. *Let $1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1$ be an exact sequence with G finitely generated. Then*

$$\text{asdim}\, G \ \leq \text{asdim}\, H \ + \ \text{asdim}\, K. \tag{2-2}$$

The previous theorem is crucial for geometric decompositions that are injective at the level of the fundamental group, as these determine a splitting into a graph of groups. There are also some cases, involving decompositions into $\mathbb{H}^2 \times \mathbb{H}^2$ pieces, that fail to be π_1-injective. We will treat that situation with the following result.

THEOREM 2.8 [5, Finite Union Theorem]. *Suppose that a metric space is presented as a union of subspaces $A \cup B$. Then*

$$\text{asdim}\, A \cup B \leq \max\{\text{asdim}\, A,\ \text{asdim}\, B\}.$$

Let $(X_1, \mathcal{E}_{X_1})$ and $(X_2, \mathcal{E}_{X_2})$ be coarse spaces. Denote by $p_i : X_1 \times X_2 \longrightarrow X_i$ the projection to the ith factor. The *product coarse structure* is defined as

$$\mathcal{E}_{X_1} * \mathcal{E}_{X_2} := \{E \subseteq (X_1 \times X_2)^2 \mid (p_1 \times p_2)(E) \in \mathcal{E}_{X_i} \text{ for } i \in \{1, 2\}\}.$$

The following proposition was shown by Grave [26, Proposition 20].

PROPOSITION 2.9. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be coarse spaces. Then*

$$\text{asdim}(X \times Y, \mathcal{E}_X * \mathcal{E}_Y) \leq \text{asdim}(X, \mathcal{E}_X) + \text{asdim}(Y, \mathcal{E}_Y). \tag{2-3}$$

However, in general, the equality in Equation (2-3) does not hold (see [6, 26]).

2.2. Fundamental groups of geometrizable manifolds. Let M be an orientable smooth four manifold that admits a proper geometric decomposition. A standard argument using the Seifert–van Kampen theorem shows that $\pi_1(M)$ is isomorphic to an amalgamated product $A *_C B$ or to an HNN-extension $A*_C$.

Here, A is the fundamental group of one of the geometric pieces.

Let Γ be a graph with vertex set V and directed edge set E. A graph of groups over Γ is an object $\mathcal{G}$ that assigns to each vertex v a group G_v, and to each edge e a group G_e, together with two injective homomorphism $\phi_e : G_e \longrightarrow G_{i(e)}$ and $\phi_{\bar{e}} : G_e \longrightarrow G_{t(e)}$. Here, $\bar{e}$ is the edge with reverse orientation, the vertex $i(e)$ is the initial vertex of e, and the vertex $t(e)$ is the final vertex of e.

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An orientable smooth 4-manifold that admits a proper, π_1-injective, geometric decomposition has a fundamental group that is isomorphic to a graph of groups constructed as an iterated amalgamated product [31, 55]. Observe that the only smooth 4-manifolds that admit geometric decompositions that are not π_1-injective are those with irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ pieces [31].

Bell and Dranishnikov proved the following results about the asymptotic dimensions of amalgams [6, Theorem 82].

THEOREM 2.10. *Let A and B be finitely generated groups and let C be a subgroup of both. Then*

$$\operatorname{asdim}(A *_C B) \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}.$$

2.3. Aspherical geometrizable 4-manifolds. Hillman obtained the following classification of closed aspherical 4-manifolds with a geometric decomposition.

THEOREM 2.11 [31, Theorem 7.2]. *If a closed 4-manifold M admits a geometric decomposition, then either:*

(1) *M is geometric; or*
(2) *M is the total space of an orbifold with general fiber S^2 over a hyperbolic 2-orbifold; or*
(3) *the components of $M \backslash \cup S$ all have geometry $\mathbb{H}^2 \times \mathbb{H}^2$; or*
(4) *the components of $M \backslash \cup S$ have geometry $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}}_2 \times \mathbb{E}^2$; or*
(5) *the components of $M \backslash \cup S$ have geometry $\mathbb{H}^2_{\mathbb{C}}$ or $\mathbb{F}^4$.*

In cases (3), (4), and (5) $\chi(M) \geq 0$, and in cases (4) and (5) M is aspherical.

In the geometric case (1), M is aspherical only when its model geometry is aspherical, and thus it must be modeled on $\mathbb{E}^4$, $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2_{\mathbb{C}}$, $\widetilde{\mathbb{SL}}_2 \times \mathbb{E}$, $\mathbb{N}il^3 \times \mathbb{E}$, $\mathbb{N}il^4$, $\mathbb{S}ol^4_1$, $\mathbb{S}ol^4_{m,n}$, or $\mathbb{S}ol^4_0$. In case (2), M is never an aspherical manifold. Hillman expresses precise conditions under which such an orbifold bundle with a geometric decomposition is not geometric [31, Theorem 10.2].

In case (3), we may or may not obtain aspherical manifolds, although some easy constructions are well known to produce aspherical examples [31].

2.4. Hyperbolicity and relative hyperbolicity. A geodesic metric space (X, d) is called δ-hyperbolic for $\delta \geq 0$ if $d(x', y') \leq \delta$ whenever $x, y, z \in X$, x' and y' lie on the geodesics from z to x and y, respectively, and if

$$d(x', z) = d(y', z) \leq (1/2)(d(x, z) + d(y, z) - d(x, y)).$$

Let Γ be a finite group with finite generating set S. The Cayley graph of Γ, with respect to S, is the graph $C(\Gamma, S)$ whose vertices are the elements of Γ and whose edge set is $\{(\gamma, \gamma \cdot s) \mid \gamma \in \Gamma, s \in S \backslash \{e\}\}$. We say that Γ is hyperbolic if the Cayley graph $C(\Gamma, S)$ associated to Γ is a δ-hyperbolic metric space, for some $\delta > 0$. Gromov observed that hyperbolic groups have finite asymptotic dimension [28], and a short proof was made available by Roe [50].

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THEOREM 2.12. *Finitely generated hyperbolic groups have finite asymptotic dimension.*

Let Γ be a group and consider a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of Γ, indexed by Λ. Let X be a subset of Γ. We say that X is a relative generating set of Γ with respect to the collection $\{H_\lambda\}_{\lambda\in\Lambda}$ if Γ is generated by $X \cup (\bigcup_\lambda H_\lambda)$. Let $F(X)$ be the free group with basis X. Then the group Γ can be expressed as the quotient group of the free group $F = (*_{\lambda\in\Lambda} H_\lambda) * F(X)$. We say that the group Γ has a relative presentation

$$\langle X, H_\lambda, \lambda \in \Lambda \mid R = 1, R \in \mathcal{R} \rangle$$

if the kernel N of the natural homomorphism $\epsilon : F \longrightarrow \Gamma$ is a normal closure of a subset $\mathcal{R} \in N$ in the group F.

If X and $\mathcal{R}$ are finite, then we say that the group Γ is finitely presented relative to the collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$.

Let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda}(H_\lambda \backslash \{1\})$. A word W in the alphabet $X \cup \mathcal{H}$ that represents 1 in the group Γ admits an expression in terms of the elements of $\mathcal{R}$ and F given by

$$W =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i. \tag{2-4}$$

Here, $=_F$ denotes equality in the group F, $R_i \in \mathcal{R}$, and $f_i \in F$ for $i = 1, \ldots, k$. The relative area of W, denoted by $\mathrm{Area}^{rel}(W)$, is the smallest number k in a representation of the form in Equation (2-4).

A group Γ is hyperbolic relative to a collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ if Γ is finitely presented relative to the collection and there is a constant $L > 0$ such that, for any word $W \in X \bigcup (\bigsqcup_\lambda (H_\lambda \backslash \{1\}))$ that represents the identity in Γ, we have $\mathrm{Area}^{rel}(W) \leq L\|W\|$.

The next result that we need was shown by Dahmani–Yaman [16, Corollary 0.2] for groups that are hyperbolic relative to a family of virtually nilpotent subgroups, and by Osin [41, Theorem 1.2] in a more general form.

THEOREM 2.13. *Let Γ be a finitely generated group that is hyperbolic relative to a finite collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. If each of the groups H_λ has finite asymptotic dimension, then* $\mathrm{asdim}\,\Gamma < \infty$.

2.5. Nagata dimension. Let X be a metric space and consider a family $\mathcal{B} = (B_i)_{i\in I}$ of subsets of X, with index set I. For some constant $D \geq 0$, the family $\mathcal{B}$ will be called D-bounded if, for all $i \in I$, $\mathrm{diam}\, B_i := \sup\{d(x, x') \mid x, x' \in B_i\} \leq D$.

The multiplicity of the family is defined as the infimum over all integers $n \geq 0$ such that every point in the metric space X is in at most n elements of $\mathcal{B}$. Let $s > 0$ be a constant. The s-multiplicity of the family $\mathcal{B}$ is the infimum over all n such that every subset of X of diameter $\leq s$ intersects at most n elements of $\mathcal{B}$.

DEFINITION 2.14. Let X be a metric space. The Nagata dimension of X, denoted by $\dim_N X$, is the infimum of all integers n such that there exists a constant c such that, for all $s > 0$, X has a $(c \cdot s)$-bounded covering with s-multiplicity at most $n + 1$.

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In Figure 2, we have a family $\mathcal{B} = (B_i)_{i\in I}$ of bricks of sides $l_1 \leq l_2$. These bricks constitute a $(c \cdot s)$-bounded collection of subsets of $\mathbf{R}^2$, where c equals $\sqrt{(l_1^2 + l_2^2)}/s$. Now, to see that the multiplicity of that family is at most three, we need to observe the following cases. Let x be a point in $\mathbf{R}^2$ inside a brick and consider a ball $B(x, r)$ centered on x of radius $r \leq l_1/2$. Then $B(x, r)$ intersects just one of the bricks.

Now, let x be a point in the boundary of two or three bricks and consider again a ball $B'(x, r)$ of radius $r \leq l_1/2$ with center in x. Then $B'(x, r)$ intersects two bricks if x lies on the boundary of exactly two bricks, and it intersects three bricks if x lies on the corner of a brick. We have exhibited a family of subsets of X with multiplicity at most three, and therefore the Nagata dimension of $\mathbf{R}^2$ is at most two.

REMARK 2.15. By definition, the Nagata dimension is an upper bound for the asymptotic dimension, that is,

$$\dim_N X \geq \operatorname{asdim} X.$$

The following result by Lang–Schlichenmaier [36] will be useful later.

THEOREM 2.16 [36, Theorem 3.7]. *Let X be an n-dimensional Hadamard manifold whose sectional curvature K satisfies $-b^2 \leq K \leq -a^2$ for some positive constants $b \geq a$. Then $\dim_N X = n$.*

For example, we obtain the following corollary.

COROLLARY 2.17. *The Nagata dimension of $\mathbb{H}^2_{\mathbf{C}}$ is equal to four.*

PROOF. Recall that the sectional curvature of the Bergman metric on $\mathbb{H}^2_{\mathbf{C}}$ is bounded between -4 and -1. As the real dimension of $\mathbb{H}^2_{\mathbf{C}}$ is equal to four, Theorem 2.16 yields $\dim_N \mathbb{H}^2_{\mathbf{C}} = 4$. □

2.6. Lower cohomological bounds. There are several equivalent definitions of cohomological dimension. Consider the following, based on K.S. Brown's book [7, Section VIII], where the definition of group cohomology $H^n(\Gamma, \mathbf{Z})$ may also be found.

DEFINITION 2.18. The cohomological dimension of a group Γ, denoted by $\operatorname{cd}(\Gamma)$, is defined as

$$\operatorname{cd}(\Gamma) = \sup\{n \mid H^n(\Gamma, \mathbf{Z}) \neq 0\}.$$

The dimension of an aspherical manifold provides an upper bound for the cohomological dimension of its fundamental group.

PROPOSITION 2.19 [7, Proposition 8.1]. *Suppose that Y is a d-dimensional $K(\Gamma, 1)$-manifold (possibly with boundary). Then:*

(1) *$\operatorname{cd}(\Gamma) \leq d$, with equality if and only if Y is closed (that is, compact and without boundary); and*
(2) *if Y is compact, then Γ has a finite classifying space $B\Gamma$.*

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As a consequence of this proposition, if M is an aspherical manifold, then

$$\mathrm{cd}(\pi_1(M)) = \dim M.$$

Dranishnikov showed the following proposition.

PROPOSITION 2.20 [19, Proposition 5.10]. *Let Γ be a finitely presented discrete group such that its classifying space $B\Gamma$ is dominated by a finite complex. Then*

$$\operatorname{asdim} \Gamma \geq \mathrm{cd}(\Gamma).$$

In the case of aspherical manifolds, we obtain the next lemma.

LEMMA 2.21. *Let M be an aspherical manifold. Then*

$$\operatorname{asdim} \pi_1(M) \geq \mathrm{cd}(\pi_1(M)) = \dim M. \tag{2-5}$$

PROOF. This is mentioned by Gromov in his asymptotic invariants of infinite groups essay [28, page 33]. A proof also follows by observing that, for a fundamental group $\pi_1(M)$ of a compact aspherical manifold, M itself is a finite model for the classifying space $B\pi_1(M)$. Therefore, by Proposition 2.20, inequality (2-5) holds. □

2.7. Properties of Alexandrov spaces of dimension three. A metric space (X, d) is called a *length space* if, for every $x, y \in X$, $d(x, y) = \inf\{L(\gamma) \mid \gamma(a) = x,\ \gamma(b) = y\}$. Here, the infimum is taken over all continuous curves $\gamma : [a, b] \to X$, and $L(\gamma)$ denotes the length of the curve γ, defined as

$$L(\gamma) = \sup_F \left\{ \sum_{i=1}^{n-1} d(\gamma(t_i),\ \gamma(t_{i+1})) \right\},$$

where the supremum runs over all finite partitions F of $[a, b]$.

Observe that if the length metric space (X, d) is complete and locally compact, then there exists at least one geodesic between each pair of points $x, y \in X$.

Let k be a real number. We call a complete, simply connected two-dimensional Riemannian manifold of constant curvature k a *model space*, and denote it by M_k^2. Depending on the sign of k, the space M_k^2 is isometric to one of the following [35, 47].

(1) If $k > 0$, then it is a sphere of constant curvature k, $\mathbb{S}_k^2$.
(2) If $k = 0$, then it is the Euclidean plane of null curvature, $\mathbb{E}_k^2$.
(3) If $k < 0$, then it is a hyperbolic plane of constant curvature k, $\mathbb{H}_k^2$.

Let $|\cdot, \cdot|$ be the usual length metric on the corresponding model space. Consider a geodesic triangle pqr in the length space (X, d). That is, pqr is a collection of three points $p, q, r \in X$ and the segments connecting them, $[pq]$, $[qr]$ and $[rp]$, are geodesics. Given a geodesic triangle pqr in X, the geodesic triangle $\bar{p}, \bar{q}, \bar{r}$ in the model space M_k^2 is a comparison triangle for pqr if $d(p, q) = |p, q|$, $d(q, r) = |q, r|$, and $d(r, p) = |r, p|$.

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A length space (X, d) is said to have curvature bounded below by $k \in \mathbf{R}$ if, for every $x \in X$, there exists an open neighborhood $U \subset X$ of x such that, for every geodesic triangle pqr in X and every comparison triangle $\bar{p}, \bar{q}, \bar{r}$ in the model space M_k^2, for all $s \in [p, q]$ and $\bar{s} \in [\bar{(}p), \bar{(}q)]$ such that $d(p, s) = |\bar{p}, \bar{s}|$, we have that $d(r, s) \geq |\bar{r} - \bar{s}|$.

An Alexandrov space is a complete and locally compact length space (X, d) with curvature bounded below by some $k \in \mathbf{R}$. The following geometrization theorem for Alexandrov 3-spaces was shown by F. Galaz-García and Guijarro.

THEOREM 2.22 [25]. *A closed three-dimensional Alexandrov space admits a geometric decomposition into geometric three-dimensional Alexandrov spaces.*

Moreover, they showed that an Alexandrov 3-space Y may be presented as the quotient of a smooth 3-manifold Y^*, under the action of an isometric involution $\iota : Y^* \to Y^*$ [25, Lemma 1.8]. The fixed points of ι descend under the quotient to the singular points $\mathcal{S}(Y)$ of the Alexandrov structure on Y.

We need the following Lemma, included here for completeness.

LEMMA 2.23. *The universal coverings $\widetilde{Y^*}$ and $\widetilde{Y}$ of the spaces Y^* and Y, mentioned immediately above (and in the same order), are the same. This universal covering space is unique up to covering isomorphism.*

Although this is likely to be evident to experts, we include a brief proof.

PROOF. The space Y, being presented as a global quotient of the 3-manifold Y^*, is a very good orbifold. In this light, the quotient map $Y^* \to Y$ is an orbifold covering map. Now observe the fact, first recorded by Thurston in his notes [57] (with further details also provided by Choi [13, Proposition 8]), that there exists a universal covering orbifold, and that it is unique up to covering isomorphism. A standard argument proves uniqueness, and extensive details are also available [13, Proposition 9]. □

2.8. Coarse Baum–Connes conjecture. Let M be a manifold and let $\Gamma = \pi_1(M)$. Recall that a metric space is called proper if closed, bounded sets are compact. The group Γ endowed with the word metric is a proper metric space. Consider the C^*-algebra $C^*(\Gamma)$. The coarse assembly map is defined as

$$\mu_X : KX_*(\Gamma) \longrightarrow K_*(C^*(\Gamma)),$$

where $K_*(C^*(\Gamma))$ denotes the K-theory of the C^*-algebra and $KX_*(\Gamma)$ is the limit of the K-homology groups (see [59]). A metric space is said to have bounded geometry if, for every $r > 0$, the cardinality of balls of radius r is uniformly bounded. The coarse Baum–Connes conjecture states that if a proper metric space has bounded geometry, then the coarse assembly map is an isomorphism. Yu [63] proved the following: theorem.

THEOREM 2.24 [63, Theorem 7.1]. *The coarse Baum–Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

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Therefore, combining Theorems 1.1, 1.3, and 2.24, we obtain a proof of the following result.

COROLLARY 2.25.

(i) *Let X be an oriented closed 4-manifold that is either geometric or admits a geometric decomposition, as in Theorem 1.1. Then the coarse Baum–Connes conjecture holds for $\pi_1(X)$.*
(ii) *Let Y be a closed three-dimensional Alexandrov space. Then the coarse Baum–Connes conjecture holds for $\pi_1(Y)$.*

2.9. Novikov conjecture. Let M be a manifold and let $\Gamma = \pi_1(M)$. If M is oriented, then a rational cohomology class $x \in H^*(B\Gamma, \mathbf{Q})$ defines a rational characteristic number, called a higher signature (see [63]): that is,

$$\sigma_x(M, u) = \langle \mathcal{L}(M) \cup u^*(x), [M] \rangle \in \mathbf{Q}.$$

Here, $\mathcal{L}(M)$ is the Hirzebruch $\mathcal{L}$-genus and $u : M \longrightarrow B\Gamma$ is the classifying map. The Novikov conjecture posits that all higher signatures are invariants of oriented homotopy equivalences over $B\Gamma$.

As a way of explaining how a good picture of a metric space could be 'drawn' inside a Hilbert space, Gromov [28] introduced the following concept.

DEFINITION 2.26. Let (H, d_H) be a Hilbert space and let (X, d) be a metric space. A map $f : X \to H$ is a coarse embedding into H if there exist nondecreasing functions ρ_1 and ρ_2 on $[0, \infty)$ such that:

(1) $\rho_2(d(x, y)) \leq d_H(f(x), f(y)) \leq \rho_1(d(x, y))$, for all x, y in X; and
(2) $\lim_{r\to+\infty} \rho_1(r) = +\infty$.

Coarse embeddability of a countable group into a Hilbert space is independent of the choice of proper length metrics [28]. Crucially for us, groups with finite asymptotic dimensions are coarsely embeddable into a Hilbert space [64]. Moreover, the next result follows from Yu [64], Higson [30], and Skandalis *et al.* [53] (see [65, Section 3]).

THEOREM 2.27. *The Novikov conjecture holds if the fundamental group of a manifold is coarsely embeddable into a Hilbert space.*

The following result was established by Yu [65] (see also Bartels [3, Theorems 1.1 and 7.2] and Carlsson–Goldfarb [10, Main Theorem]).

COROLLARY 2.28 [65, Corollary 7.2]. *Let Γ be a finitely generated group whose classifying space has the homotopy type of a finite CW complex. If Γ has finite asymptotic dimension (as a metric space with a word-length metric), then the Novikov conjecture holds for Γ.*

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COROLLARY 2.29.

(i) *Let X be an oriented closed 4-manifold that is either geometric or admits a geometric decomposition, as in Theorem 1.1. Then the Novikov conjecture holds for X.*
(ii) *Let Y be a closed three-dimensional Alexandrov space. Then the Novikov conjecture holds for Y.*

To the best of our knowledge, the second item in Corollary 2.29 above is new.

PROOF. Combining Theorems 1.1, 1.3, and 2.27, we obtain the first item.

For the second item, we have already shown in Theorem 1.3 that Y has finite asymptotic dimension. Moreover, as Y is aspherical, it serves as its own classifying space. Hence, by Proposition 2.28, it remains to show that a closed three-dimensional Alexandrov space Y is a finite CW complex.

Recall that we may decompose Y into a 3-manifold with boundary Y^0 together with a finite and pairwise disjoint collection of cones over RP^2, one for each singular point s in the singular set $\mathcal{S}_Y$ (see [25, Section 1]). Therefore, to describe an explicit finite CW-complex structure on Y, we will describe one on a cone over RP^2 and explain how this may be made compatible with a CW structure on Y^0. Repeating this process for each s in $\mathcal{S}_Y$ will then exhibit a finite CW-complex structure on Y.

First, we describe a CW structure on RP^2 as follows.

(1) 0-cell: a single point, denoted e^0.
(2) 1-cell: a single 1-cell, denoted e^1, with its boundary attached to e^0.
(3) 2-cell: a single 2-cell, denoted e^2, with its boundary attached to e^1.

The attachment map for e^2 is the map that identifies antipodal points on the boundary of the 2-cell with the 1-cell.

Second, from the CW structure on RP^2, we describe a CW structure on $C(RP^2)$, the cone over RP^2.

(1) 0-cells:
e^0_1: the original 0-cell of RP^2.
e^0_2: the apex.
(2) 1-cells:
e^1_1: the original 1-cell of RP^2.
e^1_2: the 1-cell connecting e^0_1 and e^0_2.
e^1_3: a 1-cell connecting e^0_2 to a point on e^1_1. (Note: This 1-cell can be identified with the interval [0,1], and its boundary is attached to e^0_2 and e^1_1.)
(3) 2-cells:
e^2_1: the original 2-cell of RP^2.
e^2_2: a 2-cell connecting e^0_2 to the boundary of e^2_1. (Note: This 2-cell can be identified with a cone over a disk, and its boundary is attached to e^1_2 and e^2_1.)

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Third, as Y^0 is a 3-manifold with boundary, it admits a finite CW structure. Consider a CW structure on Y^0 such that its restriction to each boundary component gives the same CW structure on RP^2 as described in the first step.

Collecting the previous steps, we construct a finite CW structure on Y. □

2.10. Zero in the spectrum. Recall that the Laplace–Beltrami operator Δ_p, with $0 \leq p \leq n$, of a complete oriented Riemannian n-manifold acts on square-integrable forms. It is an essentially self-adjoint positive operator, so its spectrum is a subset of the positive reals. A space X is said to be uniformly contractible if, for each $R > 0$, there exists some $S > R$ such that, for all $x \in X$, the ball $B(x, R)$ is contractible within $B(x, S)$. Gromov's zero-in-the-spectrum conjecture asks whether the spectrum of Δ_p of a uniformly contractible Riemannian n-manifold contains zero, for any $0 \leq p \leq n$ (see [38]). As a consequence of Theorem 2.24, Yu showed the following corollary.

COROLLARY 2.30 [63, Corollary 7.4]. *Gromov's zero-in-the-spectrum conjecture holds for uniformly contractible Riemannian manifolds with finite asymptotic dimension.*

Recall that the universal cover of an aspherical manifold is not only contractible but also uniformly contractible. This means that the contraction can be performed in a controlled manner, independent of the starting point. Therefore, the following corollary holds.

COROLLARY 2.31. *Let Z be either an aspherical manifold from Theorem 1.1 or a closed aspherical three-dimensional Alexandrov space. Then there exists a $p \geq 0$, such that zero belongs to the spectrum of the Laplace–Beltrami operator Δ_p acting on square-integrable p-forms of the universal cover $\widetilde{Z}$ of Z.*

PROOF. Observe that, by Corollary 2.30, the result holds for an aspherical manifold X from Theorem 1.1.

Let Y be a closed, aspherical, three-dimensional Alexandrov space. Then, by the universal property in Lemma 2.23, its universal covering space $\widetilde{Y}$ is also the universal cover of the manifold Y^* that is a (potential) double branched cover of Y. So $\widetilde{Y}$ is a smooth manifold, and, moreover, the Alexandrov structure on Y lifts to a Riemannian metric g on Y^* such that Y is the quotient of (Y^*, g) with respect to an isometric involution (see [25, Section 1]). Then we lift g to $\tilde{g}$ on the universal covering $\widetilde{Y}$ of Y. Consider the Laplace–Beltrami operator Δ_p acting on square-integrable p-forms, on the smooth Riemannian manifold $(\widetilde{Y}, \tilde{g})$. Observe that the asymptotic dimension of $\widetilde{Y}$ is finite, because it is equal to that of $\pi_1(Y)$, which is at most three by Theorem 1.3. Therefore, again by Corollary 2.30, Gromov's zero-in-the-spectrum conjecture holds for $\widetilde{Y}$. □

2.11. Yamabe invariant. Obtaining bounds, or exact computations, of the Yamabe invariant is a notoriously difficult problem. Schoen [51] showed that:

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(i) M has $\mathcal{Y}(M) > 0$ if and only if it admits a positive scalar curvature smooth metric; and
(ii) if M admits a volume collapsing sequence of metrics with bounded curvature, then $\mathcal{Y}(X) \geq 0$.

A notable result by Petean states that every simply connected smooth compact manifold of dimension greater than four has nonnegative Yamabe invariant [46].

As previously mentioned, Yu showed that an aspherical manifold with fundamental group of finite asymptotic dimension does not admit a metric of positive scalar curvature [63].

We now recall a notion, first introduced by Gromov [27], that generalizes the effect of having a circle action in terms of vanishing of various invariants of smooth manifolds (compare with [42]).

An $\mathcal{F}$-structure on a closed manifold M is given by:

(1) a finite open cover $\{U_1, \dots, U_N\}$;
(2) $\pi_i\colon\ \widetilde{U}_i \to U_i$ a finite Galois covering with group of deck transformations Γ_i, $1 \leq i \leq N$;
(3) a smooth torus action with finite kernel of the k_i-dimensional torus,
$\phi_i\colon\ T^{k_i} \to \mathrm{Diff}(\widetilde{U}_i)$, $1 \leq i \leq N$;
(4) a homomorphism $\Psi_i\colon\ \Gamma_i \to \mathrm{Aut}(T^{k_i})$ such that

$$\gamma(\phi_i(t)(x)) = \phi_i(\Psi_i(\gamma)(t))(\gamma x)$$

for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$ and $x \in \widetilde{U}_i$; and
(5) for any finite sub-collection $\{U_{i_1}, \dots, U_{i_l}\}$ such that $U_{i_1\cdots i_l} := U_{i_1} \cap \cdots \cap U_{i_l} \neq \emptyset$ the following compatibility condition holds: let $\widetilde{U}_{i_1\cdots i_l}$ be the set of points $(x_{i_1}, \dots, x_{i_l}) \in \widetilde{U}_{i_1} \times \cdots \times \widetilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \cdots = \pi_{i_l}(x_{i_l})$. The set $\widetilde{U}_{i_1\cdots i_l}$ covers $\pi_{i_j}^{-1}(U_{i_1\cdots i_l}) \subset \widetilde{U}_{i_j}$ for all $1 \leq j \leq l$. Then we require that ϕ_{i_j} leaves $\pi_{i_j}^{-1}(U_{i_1\cdots i_l})$ invariant and it lifts to an action on $\widetilde{U}_{i_1\cdots i_l}$ such that all lifted actions commute.

The second-named author showed that the manifolds in Corollary 1.6 admit an $\mathcal{F}$-structure.

THEOREM 2.32 [55, Theorems A and B]. *Let X be a manifold that is either geometric or admits a geometric decomposition into pieces modeled on one of the following geometries.*

$$\begin{array}{cccccccc} \mathbb{S}^4, & \mathbb{CP}^2, & \mathbb{S}^3\times\mathbb{E}, & \mathbb{H}^3\times\mathbb{E}, & \widetilde{\mathbb{SL}}_2\times\mathbb{E}, & \mathbb{N}il^3\times\mathbb{E}, & \mathbb{N}il^4, & \mathbb{S}ol_1^4, \\ \mathbb{S}^2\times\mathbb{E}^2, & \mathbb{H}^2\times\mathbb{E}^2, & \mathbb{S}ol_{m,n}^4, & \mathbb{S}ol_0^4, & \mathbb{S}^2\times\mathbb{S}^2, & \mathbb{S}^2\times\mathbb{H}^2, & \mathbb{E}^4, & \mathbb{F}^4. \end{array}$$

Then X admits an $\mathcal{F}$-structure.

The connection between the existence of $\mathcal{F}$-structures and bounds for the Yamabe invariant is given by the next theorem of Paternain and Petean.

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THEOREM 2.33 [42, Theorem 7.2]. *If a closed smooth manifold X admits an $\mathcal{F}$-structure,* $\dim X > 2$, *then* $\mathcal{Y}(X) \geq 0$.

We are now ready to present a proof of Corollary 1.6.

PROOF OF COROLLARY 1.6. By Theorem 2.32, the manifolds with geometric pieces modeled on these geometries admit an $\mathcal{F}$-structure. Then, by Paternain and Petean's Theorem 2.33, its Yamabe invariant is nonnegative. From Theorem 1.1 and Yu's celebrated result [63] it follows that such a manifold X does not admit a metric of positive scalar curvature. Then Schoen's result mentioned above implies that $\mathcal{Y}(X) \leq 0$. Therefore, $\mathcal{Y}(X) = 0$.

Now consider the connected sums of X with $S^3 \times S^1$. We appeal to a result of Petean, who showed that performing zero-dimensional surgery on X leaves the Yamabe invariant unchanged [45, Proposition 3]. Iterating this last argument yields the result for any finite number of connected sums with $S^3 \times S^1$, as claimed. □

We now include a proof of Lemma 1.7.

PROOF OF LEMMA 1.7. By the work of Paternain and Petean on collapsing of compact complex surfaces, X admits an $\mathcal{F}$-structure [43, Theorems A and B]. Hence, Theorem 2.33 yields $\mathcal{Y}(X) \geq 0$. Now, by Chodosh and Li [12] and Gromov [29], X does not admit a metric of positive scalar curvature. Thus, by the previously mentioned results, we obtain $\mathcal{Y}(X) \leq 0$. Therefore, we conclude that $\mathcal{Y}(X) = 0$. □

3. Proofs of our main results

We start with the following lemma, which is needed for the proofs of our main results.

LEMMA 3.1. *Let Y be a compact* 3*-manifold that is geometric in the sense of Thurston. Then* $\operatorname{asdim} \pi_1(Y) \leq 3$.

PROOF. This can be verified for each of the model geometries, which we group as follows.

(1) $\mathbb{E}^3$ this case follows from Lemma 2.2, item (i).
(2) $\mathbb{H}^3$ this case follows from Lemma 2.2, item (ii).
(3) $\mathbb{S}^3$ these groups are finite, which follows from Lemma 2.2, item (iii).
(4) $Nil^3, Sol^3, \widetilde{\mathrm{SL}}_2$ these cases are covered by Lemma 2.4. The geometries $\mathbb{E}^3, Nil^3$ and Sol^3 are all Lie groups. Notice that the Lie group $\widetilde{\mathrm{SL}}_2$ is the universal cover of the 3-dimensional Lie group SL_2 of all 2×2 matrices with determinant 1.
(5) $\mathbb{S}^2 \times \mathbb{E}, \mathbb{H}^2 \times \mathbb{E}$ for these geometries the proof follows from the previously mentioned result for $\mathbb{S}^2$ and $\mathbb{H}^2$ in Lemma 2.3 in combination with the bound for products of spaces found in Proposition 2.9.

Therefore, in all of the possible cases, we obtain that $\operatorname{asdim} \pi_1(Y) \leq 3$. □

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3.1. Proof of Theorem 1.1.

PROOF OF THEOREM 1.1. First, we prove the upper bound for asdim $\pi_1(M)$ for geometric manifolds, and then for manifolds with geometric decomposition. □

3.1.1. Geometric manifolds. Here, we prove the statement for manifolds modeled on a single model Thurston geometry.

Finite fundamental groups: $\mathbb{S}^4, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{C}\mathrm{P}^2$. Let Γ be a finite group. Then Γ is finitely generated. Hence, by item (3) in Lemma 2.2, its asymptotic dimension asdim $\Gamma = 0$. Therefore, as the fundamental groups of geometric manifolds modeled on $\mathbb{S}^4, \mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{C}\mathrm{P}^2$ are finite, they have asymptotic dimension zero.

Quotients of Lie groups. The asymptotic dimension of quotients of simply connected Lie groups can be effectively bounded. By Lemma 2.4 a cocompact lattice Γ in a connected Lie group G with maximal compact subgroup K satisfies asdim $\Gamma = \dim(G/K)$. Hence, we obtain that asdim $\pi_1(X) \leq 4$ for geometric manifolds X modeled on the geometries $\mathbb{N}il^3 \times \mathbb{E}$, $\mathbb{N}il^4$, $\mathbb{S}ol^4_1$, $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$, or $\mathbb{E}^4$.

Product geometries: $\mathbb{S}^3 \times \mathbb{E}$, $\mathbb{H}^3 \times \mathbb{E}$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$. By Proposition 2.9, we know that the asymptotic dimension of a product of coarse spaces is bounded by the sum of the asymptotic dimensions of each space. Therefore, for all of the product geometries $\mathbb{S}^3 \times \mathbb{E}$, $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, and $\mathbb{H}^2 \times \mathbb{H}^2$, we have that their asymptotic dimension is bounded above by the sum of the asymptotic dimensions of their factors. Therefore, by Lemmas 2.3 and 3.1 and Proposition 2.9, the asymptotic dimension of each of these product geometries is at most four.

$\mathbb{H}^4$ *and* $\mathbb{H}^2_{\mathbf{C}}$ *manifolds*. Observe that compact $\mathbb{H}^4$ or $\mathbb{H}^2_{\mathbf{C}}$ manifolds have hyperbolic fundamental groups, so, by Theorem 2.12, above their asymptotic dimension is finite. Finite volume manifolds modeled on these geometries, truncated to be used as pieces of a geometric decomposition, are relatively hyperbolic with respect to their peripheral structure, that is, the systems of fundamental groups of their boundary components. Such groups have finite asymptotic dimension, by Theorem 2.13. Moreover, we showed in Corollary 2.17 that complex hyperbolic pieces have Nagata dimension four. That real hyperbolic pieces have asymptotic dimension four follows from item *(ii)* in Lemma 2.2.

$\mathbb{F}^4$ *manifolds.* For the case of $\mathbb{F}^4$, the extension result of Bell–Dranishnikov in Theorem 2.7, applied to a short exact sequence of the fundamental group, yields the desired bound. Let X be a manifold modeled on $\mathbb{F}^4$. Then $\pi_1(X)$ is isomorphic to a lattice in $\mathbf{R}^2 \ltimes \mathrm{SL}(2, \mathbf{R})$ [31]. Let $\overline{\pi_1(X)}$ be the image of $\pi_1(X)$ in $\mathrm{SL}(2, \mathbf{R})$. Recall that $X = \mathbb{F}^4/\pi_1(X)$ as an elliptic surface over the base $B = \mathbb{H}^2/\overline{\pi_1(X)}$, where B is a noncompact orbifold [61, page 150].

The identity component of $\mathrm{Iso}(\mathbb{F}^4)$ is the semidirect product $\mathbf{R}^2 \ltimes_\alpha \mathrm{SL}(2, \mathbf{R})$, where α is the natural action of $\mathrm{SL}(2, \mathbf{R})$ on $\mathbf{R}^2$. Let $p\colon\ \mathbf{R}^2 \ltimes_\alpha \mathrm{SL}(2, \mathbf{R}) \to \mathrm{SL}(2, \mathbf{R})$ be the projection homomorphism. The manifold X is diffeomorphic to the quotient of $T^2 \times \mathbb{H}^2$ under the action of $p(\pi_1(X))$, acting on T^2 through

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$$\psi : p(\pi_1(X)) \to \mathrm{Aut}(T^2 = \mathbf{R}^2/(\pi_1(X) \cap \mathbf{R}^2)),$$

and on $\mathbb{H}^2$ in the usual way.

The quotient $B := \mathbb{H}^2/p(\pi_1(X))$ is a finite volume hyperbolic 2-orbifold, and hence X is an orbifold bundle over B. If B is smooth, that is, $p(\pi_1(X))$ acts without fixed points, then M is a torus bundle over B with structure group $\mathrm{SL}(2, \mathbf{Z})$ and ψ is precisely its holonomy.

The manifold X is a T^2 fibration over a noncompact, finite area, hyperbolic orbifold B (see [61, page 150] and [55, Section 10.1]). Therefore, the fundamental group $\pi_1(X)$ can be written as an extension, $\pi_1(T^2) \to \pi_1(X) \to \pi_1(B)$. Thus, (2-2) of the extension Theorem 2.7 implies that $\mathrm{asdim}\, \pi_1(X) \leq \mathrm{asdim}\, \pi_1(T^2) + \mathrm{asdim}\, \pi_1(B) \leq 2 + 2$.

Therefore, our arguments have now covered all the possible cases and we conclude that all the four-dimensional geometric manifolds have asymptotic dimension at most four.

3.1.2. Manifolds with a geometric decomposition. As we explained in Section 2.2 above, the fundamental group $\pi_1(X)$ of a geometrizable 4-manifold X with a proper and π_1-injective geometric decomposition is isomorphic to a graph of groups. By Theorem 2.10, the asymptotic dimension of a graph of groups is finite provided each vertex group has finite asymptotic dimension. Moreover, we have computed the explicit bound we need.

We now cover each of the possible geometric decompositions, in the same order of Hillman's Theorem 2.11 above.

X is the total space of an orbifold bundle with general fiber S^2 over a hyperbolic 2-orbifold. Notice that, by Theorem 2.6, the relevant fiber and base orbifold groups have asymptotic dimension at most two. Consider the decomposition of X into its geometric pieces X_i, $i \in \{1, \ldots, k\}$ (see [32]). Then the arguments explained above for the geometric cases yield $\mathrm{asdim}\, \pi_1(X_i) \leq 4$. As X is compact for $k < \infty$, the finite union Theorem 2.8 implies that $\mathrm{asdim}\, \pi_1(X) \leq 4$.

Manifolds that decompose into $\mathbb{H}^2 \times \mathbb{H}^2$ pieces. These manifolds have two kinds of decompositions: they are called irreducible if the boundary inclusion into each piece is π_1-injective and called reducible otherwise.

In the irreducible case, we obtain a decomposition of the fundamental group into a graph of groups, and the result follows as in other similar cases.

In the reducible case, we use the finite union Theorem 2.8. First, we apply it to a couple of contiguously glued $\mathbb{H}^2 \times \mathbb{H}^2$-pieces. Then we perform induction over the number of pieces of the geometric decomposition to obtain the desired upper bound. Hence, in both the reducible and irreducible cases we have that $\mathrm{asdim}\, \pi_1(X) \leq 4$.

Manifolds that decompose into $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$, or $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}^2$ pieces. First, observe that a manifold X that decompose into pieces modeled on the hyperbolic geometry $\mathbb{H}^4$ are relatively hyperbolic. Their ends are either flat or nilpotent, and therefore the fundamental group of each geometric piece has finite asymptotic dimension by Theorem 2.13, because they are relatively hyperbolic. Notice that the fundamental groups $\pi_1(Y)$ of flat or nilpotent 3-manifolds Y have asdim $\pi_1(Y) = 3$, by Lemma 2.4.

Now, for the case where the manifold X decomposes into pieces modeled on the geometries $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}^2$. We have already proved that the asymptotic dimension of all of these product geometries is at most four. Therefore, in both cases, by Theorem 2.10 the asymptotic dimension of $\pi_1(X)$ is bounded above by four.

Manifolds that decompose into $\mathbb{H}^2_{\mathbf{C}}$ or $\mathbb{F}^4$ pieces. If a manifold X decomposes into pieces modeled on the hyperbolic geometry $\mathbb{H}^2_{\mathbf{C}}$, then, again, it is relatively hyperbolic. So, as in the case of $\mathbb{H}^4$, the fundamental group of each geometric piece has finite asymptotic dimension.

For the case where X decomposes into pieces modeled on the geometry $\mathbb{F}^4$, we know that the asymptotic dimension of each piece is bounded above by four. By Theorem 2.10, the asymptotic dimension of $\pi_1(X)$ is bounded above by four.

Therefore, we have shown that asdim $\pi_1(M) \leq 4$ when X is a closed orientable 4-manifold that is geometric or admits a geometric decomposition in the sense of Thurston.

3.1.3. Equality for aspherical manifolds. Now we prove that the lower bound asdim $\pi_1(M) \geq 4$ in the case of aspherical manifolds, which will imply the equality we claim. By Theorem 2.11, we know that a geometric manifold modeled on $\mathbb{H}^3 \times \mathbb{E}$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^4$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2_{\mathbf{C}}$, $\widetilde{\mathrm{SL}}_2 \times \mathbb{E}$, $\mathbb{N}il^3 \times \mathbb{E}$, $\mathbb{N}il^4$, $\mathbb{S}ol^4_1$, $\mathbb{S}ol^4_{m,n}$, or $\mathbb{S}ol^4_0$ is aspherical. Hence, Lemma 2.21 implies that the asymptotic dimension of such a fundamental group is bounded below by its cohomological dimension. Observe that the cohomological dimension of $\pi_1(M)$ is equal to the dimension of M, so asdim $\pi_1(M) \geq \dim M = 4$. This concludes the proof for geometric manifolds.

In the cases of manifolds with a geometric decomposition, by items (4) and (5) of Theorem 2.11, we know that if the pieces of the geometric decomposition have geometries $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^2$, $\mathbb{H}^2_{\mathbf{C}}$, or $\mathbb{F}^4$, then the manifold is aspherical. Using Lemma 2.21 again, we obtain that the lower bound for their asymptotic dimension is four.

Therefore the equality follows for both cases, covering all the possible aspherical manifolds, as we claimed.

Finally, we mention the effect of connected sums on the asymptotic dimension. Observe that taking connected sums of manifolds corresponds to performing free products at the level of fundamental groups. So, if the pieces of the connected sum have finite asymptotic dimension, then the resulting connected sum also has finite asymptotic dimension. Moreover, the upper bound in this case remains the same, according to Theorem 2.10. This concludes our proof.

3.2. Proof of Theorem 1.2.

We now present a proof of Theorem 1.2.

PROOF OF THEOREM 1.2. The success of Thurston's geometrization program implies that $\pi_1(Y)$ may be presented as a graph of groups $\mathcal{G}_Y$; each vertex group V_i is a discrete

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group of isometries of one of the eight model geometries, while the edge groups $E_{i,j}$, between the vertices V_i and V_j, are surface groups. By Lemma 3.1, the asymptotic dimension of the groups V_i is bounded above by three, that is, asdim $V_i \leq 3$.

Continuing with the proof, observe that a finite graph of groups is isomorphic to an interated amalgamated product. Therefore, by Theorem 2.10, and Lemmas 3.1 and 2.3, we obtain asdim $\mathcal{G}_Y \leq \max\{\text{asdim } V_i,\ \text{asdim } E_{i,j} + 1\} \leq 3$. In the nonorientable case, consider the orientation double cover to obtain the same result.

Finally, in the aspherical case, Lemma 2.21 yields three as the lower bound for the asymptotic dimension. □

3.3. Proof of Theorem 1.3.

Now we continue with a proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Let Y be a compact three-dimensional Alexandrov space. Then, as explained previously, there exist both a smooth Riemannian 3-manifold Y^* and an isometric involution ι of Y^* such that Y is homeomorphic to $Y^*/\{y \cong \iota(y)\}$, with $y \in Y^*$ [25]. We write $\widetilde{Y}$ for the universal covering of Y^*. Then the fundamental group $\pi_1(Y^*)$, seen as a group of deck transformations, acts on $\widetilde{Y}$. Moreover, observe that, as ι is an isometric involution acting on Y^*, it lifts to an action on $\widetilde{Y}$.

Denote by Γ the group formed by composing the action of $\pi_1 Y^*$ on $\widetilde{Y}$, with the action of ι on Y^*. The orbit equivalence classes of Γ, acting on $\widetilde{Y}$, present Y as a quotient space. Recall that, by Lemma 2.23, $\widetilde{Y}$ is the unique universal cover of both Y^* and Y. Therefore, Γ is isomorphic to the fundamental group of Y.

We claim that Γ acts properly on $\widetilde{Y}$, because the isotropy groups are of the following two kinds only.

(i) The trivial group, for the nonsingular points of Y, whose space of directions is homeomorphic to a ball.
(ii) Isomorphic to $\mathbf{Z}/2$, for the singular points $\mathcal{S}(Y)$, whose space of directions is homeomorphic to a projective plane.

As these two cases cover all possible types of isotropy groups, the action is proper, as claimed. Hence, the group acts properly, and also isometrically, on the proper metric space $\widetilde{Y}$. Therefore, Theorem 2.6 implies that asdim $\Gamma \leq$ asdim $\widetilde{Y}$, and by Equation (2-1) and Theorem 1.2, asdim $\widetilde{Y} \leq 3$. □

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