BULL. AUSTRAL. MATH. SOC. MOS 47B05
VOL. 6 (1972), 355-356.

A remark on a theorem of Caradus

J.A. Johnson

It is shown how a result of S.R. Caradus on the approximation problem can be obtained from known theorems.

Terms used here are standard (see [*1*] or [*3*]).

Let X denote a Banach space, S its unit ball in the weak topology, and X^* the dual of X. In [*1*], the following theorem is proved:

(I) *If* X *is reflexive and* X^* *(considered as a subspace of the continuous scalar-valued functions* $C(S)$ *in the canonical way) is complemented in* $C(S)$ *, then* X *has the approximation property.*

It is our purpose to point out that (I) is a corollary to some known theorems that yield the stronger conclusion (II) below.

(II) *Let* X *be any Banach space and* $C(S)$ *the space of bounded continuous functions on* S *. If* X^* *is complemented in* $C(\dot{S})$ *then* X^* *, and therefore* X *, has the approximation property.*

Proof. It is well known (see [*2*] for an elementary proof) that $C(K)$ has the approximation property for K compact Hausdorff. Thus $C(S)$ does also (use the Stone-Čech compactification of S). Now it is an easy exercise to show that complemented subspaces inherit the approximation property. It is also well known that if X^* has the approximation property, so does X (see [3, Remark, p. 113]). This completes the proof.

[Added 6 December 1971]. After this note had been accepted for publication, it was pointed out to me by Professor Caradus that H.E. Lacey had also made the observation that a more efficient proof exists. His proof is somewhat different from mine, however. It shows that if X is

Received 16 November 1971.

reflexive and X^* is complemented in $C(S)$ then X is finite dimensional.

References

[1] S.R. Caradus, "The approximation problem for compact operators", *Bull. Austral. Math. Soc.* 1 (1969), 397-401.

[2] Jesús Gil de Lamadrid, "On finite dimensional approximations of mappings in Banach spaces", *Proc. Amer. Math. Soc.* 13 (1962), 163-168.

[3] Helmut H. Schaefer, *Topological vector spaces* (The Macmillan Company, New York; Collier-Macmillan, London; 1966).

Oklahoma State University,
Stillwater,
Oklahoma,
USA.