Can. J. Math.Vol. 44 (2), 1992 pp. 357–367

CONTINUATIONS OF RIEMANN SURFACES

MAKOTO SAKAI

ABSTRACT. We shall show that if a Riemann surface is continuable, then it admits one of three types of continuations. Using this classification of continuations, we construct two nontrivial examples of two-sheeted unlimited covering Riemann surfaces of the unit disk one of which is continuable and the other is not.

Introduction. In 1924, Radó [5] constructed an example of a noncompact maximal Riemann surface. A Riemann surface, always assumed to be connected, is called *maximal* if it is not a proper subdomain of another Riemann surface. Every compact Riemann surface is maximal. Every noncompact Riemann surface of finite genus is not maximal and it is a proper subdomain of a compact Riemann surface of the same genus. Consequently, every noncompact maximal Riemann surface has infinite genus. Every noncompact Riemann surface is homeomorphic to a Riemann surface which is not maximal. This means that, when we try to show that a noncompact Riemann surface is maximal, we need an argument on the global conformal structure of the Riemann surface.

Radó's example is a two-sheeted unlimited covering surface of the complex plane with an infinite number of branch points. We can show that a two-sheeted unlimited covering surface of the complex plane is maximal if and only if the number of branch points of the surface is infinite. The boundary of the complex plane is just one point, the point at infinity, and it is very small. It is natural to ask about a covering Riemann surface such that the boundary of the basic domain is large. Oikawa [4, Chapter 5] actually proposed the following problem: How about a covering surface whose basic domain is the unit disk?

In this paper, we shall be concerned with the problem. Let R be a two-sheeted unlimited covering Riemann surface of the unit disk and let P be the projection to the unit disk of the set of branch points of R. It is apparent that if the derived set of P, the set of all accumulation points of P in the complex plane, is not equal to the unit circle, then R is not maximal. Hence we assume that the derived set of P is equal to the unit circle.

We shall construct two examples of such surfaces one of which is maximal and the other is not. By de Possel, Tamura and Renggli, see e.g. Sario-Oikawa [8, Chapter X, 5A], it is known that a Riemann surface R is maximal if and only if

(a) R has no planar ends

and

Received by the editors September 26, 1989; revised May 3, 1991.
AMS subject classification: Primary: 30F20; secondary: 30F15.

(b) R has no simply connected regular subdomain D of R which admits a one-to-one conformal mapping onto a proper subdomain of the unit disk such that the relative boundary of D in R corresponds to a relatively open subset of the unit circle.

Here the relative boundary of D in R means the boundary of D considered in the topology of R. To make clear the difference between two above examples, we divide (b) into the following two cases introducing new notions of "skirts" and "disks with crowded ideal boundaries":

(b-1) R has no simply connected skirts

and

(b-2) R has no disks with crowded ideal boundaries.

If R has a planar end, then we can embed the end into a larger planar domain. Figuratively speaking, we can "fill up holes". If R has a simply connected skirt, we shall show that the skirt is conformally equivalent to a rectangle and its ideal boundary corresponds to the left vertical side of the rectangle. R can be continued by pushing the left vertical side of the rectangle to the outside, we can say this as "pushing against walls". If R has a disk with a crowded ideal boundary, then we first take a one-to-one conformal mapping of the disk into itself and next enlarge the image. This is neither "filling up holes" nor "pushing against walls". It is like "crushing a disk and beating out a domain" and it is a very pathological continuation. It seems, however, that Radó [5] already realized the existence of this complex continuation.

In our cases, we have assumed that R is a two-sheeted unlimited covering Riemann surface of the unit disk and P is the projection of the set of branch points of R, we shall show that the derived set of P is equal to the unit circle if and only if R satisfies (a) and (b-1). Hence the above example which is not maximal can be continued only by "crushing a disk and beating out a domain". For the example which is maximal, we shall show that if a Riemann surface R does not satisfy (b-1) nor (b-2), then R can be continued into a bordered Riemann surface. For every interior point of a bordered Riemann surface, the HD-span does not vanish. Thus our example is a surface which has a point where the HD-span vanishes.

This paper consists of seven sections. We shall discuss three types of continuations in Sections 1, 2 and 3. In Section 4, we shall show that if a Riemann surface is not maximal, then the Riemann surface admits one of these continuations. Two-sheeted unlimited covering Riemann surfaces of the unit disk are discussed in Section 5 and an example which is not maximal is given. In Section 6, we show a sufficient condition for a Riemann surface to be maximal and give an example of a maximal Riemann surface. Concluding remarks are given in the final section.

The author would like to express his heartfelt gratitude to Professor Oikawa for a stimulating discussion about the problem. The author is grateful to the referees for their suggestions.

1. **A continuation of a planar end.** We say that a Riemann surface is *continuable* if it is not maximal, namely, if it is a proper subdomain of another Riemann surface. The latter Riemann surface is called a continuation of the former. We say that a subdomain of a Riemann surface is planar if it is conformally equivalent to a plane domain.

Let R be a noncompact Riemann surface and let $\{R_n\}$ be an exhaustion of R, namely, each R_n is a relatively compact regular subdomain of R and satisfies $\overline{R_n} \subset R_{n+1}$ and $\bigcup_{n=1}^{\infty} R_n = R$, where $\overline{R_n}$ denotes the closure of R_n. We call $\{R_n\}$ *normal* if every connected component of $R \setminus R_n$ is noncompact and its relative boundary in R is a dividing simple closed curve.

DEFINITION 1.1. Let $\{R_n\}$ be a normal exhaustion of a Riemann surface R. Each connected component of $R \setminus \overline{R_n}$ is called an *end* of R, or more precisely, an end of the exhaustion $\{R_n\}$.

The following lemma is well known, but for the sake of the definition of a continuation of a planar end, we shall give a proof.

LEMMA 1.1. *If there is a planar end of R, then R is continuable.*

PROOF. Let E be a planar end of R and let φ be a one-to-one conformal mapping of E onto a bounded plane domain G such that the relative boundary of E in R corresponds to the outer boundary of G. We denote by $\tilde{G}$ a bounded simply connected domain which is surrounded by the outer boundary of G. A continuation of R is constructed as the union of R and $\tilde{G}$ identifying points $z \in E$ and $\varphi(z) \in G$. ∎

DEFINITION 1.2. We call the continuation in the proof of Lemma 1.1 a continuation of a planar end.

2. **A continuation of a simply connected skirt.** We need some preparations to define "skirts". Let R_0 be a relatively compact regular subdomain of R such that $R \setminus \overline{R_0}$ is connected. Let u be the harmonic measure of the relative boundary ∂R_0 of R_0 in R defined in $R \setminus \overline{R_0}$. It is defined as the limit function of harmonic functions u_n. For large n, the function u_n is the solution of the Dirichlet problem in $R_n \setminus \overline{R_0}$: u_n is harmonic in $R_n \setminus \overline{R_0}$ and has boundary values 1 on ∂R_0 and 0 on ∂R_n. We note that u is identically equal to a constant 1 if and only if there are no Green's functions in R. We also note that $d(u + iu*)$, where $u*$ is a conjugate function of u, is a holomorphic differential in $R \setminus \overline{R_0}$. It has at most a countable number of zeros in $R \setminus \overline{R_0}$ if u is not constant.

Assume that u is not constant. Let p be a point in $R \setminus \overline{R_0}$ at which $d(u + iu*)$ does not vanish. Take a branch of $u*$ in a neighborhood of p and denote it again by $u*$. We take a level arc of $u*$ starting from p so that $u(z)$ decreases as z on the arc leaves p. We continue the level arc as long as possible. It reaches a zero of $d(u + iu*)$ or approaches the ideal boundary of R. We denote $J(p)$ the level arc and call it *the level arc of* $u*$ starting from p. We note that it does not depend on the choice of the branch of $u*$.

Now we take two distinct points p and q in $R \setminus \overline{R_0}$ so that

(i) p and q can be joined by a simple arc J_λ contained entirely in the level set $\{z \in R \setminus \overline{R_0} \,;\, u(z) = \lambda\}$ of u, where λ denotes a constant with $0 < \lambda < 1$.

https://doi.org/10.4153/CJM-1992-024-1 Published online by Cambridge University Press

(ii) There are no zeros of $d(u + iu*)$ on J_λ. J_λ contains its end points p and q.

(iii) $u*(z)$ decreases as z moves from p to q on J_λ.

(iv) The level arc $J(p)$ (resp. $J(q)$) of $u*$ starting from p (resp. q) approaches the ideal boundary of R.

We note that $J(p) \cap J_\lambda = \{p\}$. In fact, $u*(z)$ decreases as z moves from p to q on J_λ, whereas $u*(z)$ is constant on $J(p)$. By the same reason, $J(q) \cap J_\lambda = \{q\}$. Assume next that $J(p) \cap J(q) \neq \emptyset$. Then, by the definition of $J(p)$ and $J(q)$, it follows that $J(p)$ passes through q. This contradicts that $u(z)$ decreases when z moves on $J(p)$ from p toward the ideal boundary of R. Thus $J(p) \cap J(q) = \emptyset$, and so the union $J = J(p) \cup J_\lambda \cup J(q)$ is a simple arc in $R \setminus \overline{R_0}$. Since R is connected, $R \setminus J$ is connected or consists of two connected components. In the latter case, there is a unique component which does not contain R_0. We denote the component by $S(p, q)$.

DEFINITION 2.1. We call $S(p, q)$ a *skirt* of R, more precisely, a *skirt* of the harmonic measure u.

We note that the skirt is defined only if R has the Green function and only if $R \setminus J$ is not connected. There is a Riemann surface having the Green function such that $R \setminus J$ is connected for every J. We shall show an example in Section 5.

If a skirt S is simply connected, then we can choose a branch of $u + iu*$ defined in S. We shall next show that any branch is univalent in S.

LEMMA 2.1. *If a skirt $S = S(p, q)$ of a harmonic measure u is simply connected, then every branch f of $u + iu*$ is a one-to-one conformal mapping of S onto a rectangle $\{w \in \mathbb{C}\ ; 0 < \operatorname{Re} w < \lambda = u(p) = u(q)$ and $u*(q) < \operatorname{Im} w < u*(p)\}$.*

PROOF. Let $\zeta = \varphi(z)$ be a one-to-one conformal mapping of S onto the unit disk U. The boundary of S consists of the relative boundary J of S in R and a part of the ideal boundary of R. Each point of J determines boundary elements of S. If it has two boundary elements of S, then $S = R \setminus J$. This contradicts the assumption that $R \setminus J$ consists of two connected components. Hence we see that each point of J determines just one boundary element of S, and so J corresponds by φ to a circular arc C on the unit circle Γ.

Next we shall show that $\Gamma \setminus C$ consists of more than one point. If it consists of one point, we may assume that $\Gamma \setminus C = \{-1\}$. We note that φ can be extended homeomorphically onto J. We denote again by φ the extension. Set $g = f \circ \varphi^{-1}$. Then $0 < \operatorname{Re} g(\zeta) < \lambda$ in U, $\lim_{\zeta \to -1} g(\zeta) = \lambda_1 + iu*(p)$ as ζ with $|\zeta| = 1$ and $\operatorname{Im} \zeta > 0$ tends to -1, and $\lim_{\zeta \to -1} g(\zeta) = \lambda_2 + iu*(q)$ as ζ with $|\zeta| = 1$ and $\operatorname{Im} \zeta < 0$ tends to -1, where λ_1 and λ_2 denote numbers with $0 \leq \lambda_j < \lambda$, because $u(z)$ decreases as z moves on $J(p)$ or $J(q)$ toward the ideal boundary of R. Hence, by Lindelöf's asymptotic value theorem, we get $\lambda_1 + iu*(p) = \lambda_2 + iu*(q)$ which contradicts $u*(p) \neq u*(q)$. Thus $\Gamma \setminus C$ consists of more than one point.

Next we shall show that $u(z)$ tends to 0 as z tends to the ideal boundary of S; more precisely, we shall show that, for every $\varepsilon > 0$, we can find R_n such that $u(z) < \varepsilon$ on $S \setminus R_n$, where $\{R_n\}$ denotes an exhaustion of R. Let v be the solution of the Dirichlet problem in U with boundary values 0 on $\Gamma \setminus C$ and $(u \circ \varphi^{-1})(\zeta)$ on C. Then $v(\zeta) \leq (u \circ \varphi^{-1})(\zeta)$

in U. Set $s(z) = (v \circ \varphi)(z)$ in S and $s(z) = u(z)$ on $R \setminus \overline{R_0} \setminus S$, where R_0 is a relatively compact regular subdomain of R appeared in the definition of a skirt. Since $s(z) \leq u(z)$ in S and $s(z) = u(z)$ on J, s is superharmonic in $R \setminus \overline{R_0}$. Hence $u_n(z) \leq s(z)$ in $R_n \setminus \overline{R_0}$ for large n, where u_n denotes the solution of the Dirichlet problem in $R_n \setminus \overline{R_0}$ with boundary values 1 on ∂R_0 and 0 on ∂R_n. Letting n tend to ∞, we get $u(z) \leq s(z)$ in $R \setminus \overline{R_0}$, and so, together with the previous inequality, we get $u(z) = s(z)$ in $R \setminus \overline{R_0}$. Hence, $u \circ \varphi^{-1} = v$ in U, and so $\operatorname{Re} g = u \circ \varphi^{-1}$ is continuously equal to zero on $\Gamma \setminus C$ except its end points. We apply again Lindelöf's asymptotic value theorem at the end points of C and see that $(\operatorname{Re} g)(\zeta)$ tends to zero as $\zeta \in C$ tends to the end points of C. Hence, for every $\varepsilon > 0$, we can find R_n such that $u(z) = (\operatorname{Re} g)\big(\varphi(z)\big) < \varepsilon$ on $S \setminus R_n$.

Finally we shall show that f is a one-to-one conformal mapping of S onto $T = \{w \,;\, 0 < \operatorname{Re} w < \lambda \text{ and } u*(q) < \operatorname{Im} w < u*(p)\}$. To do so, it is sufficient to show that, for each fixed $w \in \mathbb{C} \setminus \bar{T}$, $(1/2\pi) \int_{\partial(S \cap R_n)} d \arg(f - w) = 0$ and, for each fixed $w \in T$, $(1/2\pi) \int_{\partial(S \cap R_n)} d \arg(f - w) = 1$ for large n. We have already seen that $0 < u(z) < \lambda$ in S. Hence, to prove the above equalities, we may assume that $0 < \operatorname{Re} w < \lambda$. By definition, $f(z) = u(z) + iu*(p)$ on $J(p)$, $f(z) = \lambda + iu*(z)$ on J_λ and $f(z) = u(z) + iu*(q)$ on $J(q)$. Now take n so large that $u(z) < (\operatorname{Re} w)/2$ on $S \setminus R_n$. If $\operatorname{Im} w < u*(q)$ or $\operatorname{Im} w > u*(p)$, then the image of $\partial(S \cap R_n)$ under f can not surround w, and so $(1/2\pi) \int_{\partial(S \cap R_n)} d \arg(f - w) = 0$. If $u*(q) < \operatorname{Im} w < u*(p)$, then $(1/2\pi) \int_{\partial(S \cap R_n)} d \arg(f - w) = 1$, because the image of $\partial(S \cap R_n)$ under f consists of a finite number of closed curves. ∎

Next we shall show the second type of continuation.

LEMMA 2.2. *If there is a simply connected skirt of R, then R is continuable.*

PROOF. Let $S = S(p,q)$ be a simply connected skirt of u. Then, by Lemma 2.1, a branch f of $u+iu*$ is a one-to-one conformal mapping of S onto a rectangle $T = \{w \in \mathbb{C} \,;\, 0 < \operatorname{Re} w < u(p) \text{ and } u*(q) < \operatorname{Im} w < u*(p)\}$. We denote by $\tilde{T}$ a continuation of T onto the left vertical side of T; for example, set $\tilde{T} = T \cup \{w \in \mathbb{C} \,;\, |w - i\big(u*(p)+u*(q)\big)/2| < \big(u*(p) - u*(q)\big)/2 \text{ and } \operatorname{Re} w \leq 0\}$. A continuation of R is constructed as the union of R and $\tilde{T}$ identifying points $z \in S$ and $f(z) \in T$. ∎

COROLLARY 2.3. *If there is a planar skirt of R, then R is continuable.*

PROOF. Let S be a planar skirt of a harmonic measure. Assume that S is conformally equivalent to a bounded plane domain G. We may assume that the relative boundary of S corresponds to a subset of the outer boundary of G. There are two possibilities: G is simply connected or not. If G is simply connected, then S is a simply connected skirt, and so, by Lemma 2.2, R is continuable. If G is not simply connected, then we can find a regular real analytic simple closed curve in G such that some points in ∂G are contained inside of the curve. The intersection of G and the inside of the curve corresponds to a planar end of R, and so R is continuable by Lemma 1.1. ∎

DEFINITION 2.2. We call the continuation in the proof of Lemma 2.2 a *continuation of a simply connected skirt*.

3. **A continuation of a disk with a crowded ideal boundary.** We shall define disks with crowded ideal boundaries. Let D be a simply connected regular subdomain of a Riemann surface R. It can be mapped conformally onto the unit disk U. The relative boundary of D in R corresponds to a relatively open subset of the unit circle Γ. We denote by I the complement of the set with respect to Γ and call it a *realization of the ideal boundary* of D. A closed subset K of the complex plane is said to be of class N_D if the complement $\mathbb{C}\setminus K$ of K is connected and there are no nonconstant holomorphic functions in $\mathbb{C}\setminus K$ with finite Dirichlet integrals.

DEFINITION 3.1. We call D a *disk with a crowded ideal boundary* if a realization I of the ideal boundary of D is totally disconnected and is not of class N_D.

We note that the definition does not depend on the choice of the conformal mapping of D onto U. For the existence of a closed subset of Γ which is totally disconnected and is not of class N_D, we refer to Sario-Oikawa [8, Chapter IX, 4G–4I].

We now discuss the third type of continuation.

LEMMA 3.1. *If there is a disk of R with a crowded ideal boundary, then R is continuable.*

PROOF. Let φ be a one-to-one conformal mapping of a disk D with a crowded ideal boundary onto the unit disk U and let I be the realization of the ideal boundary of D. Let ψ be a one-to-one conformal mapping of U onto a proper subdomain V of U such that $\Gamma\setminus I$ corresponds to a relatively open subset of Γ. Such a mapping ψ exists, because I is not of class N_D, see e.g. Sario-Oikawa [8, Chapter X, 5B]. Take a domain $\tilde{V}$ so that $V \subset \tilde{V} \underset{\neq}{\subset} U$; for example, let $\tilde{V} = U$. A continuation of R is constructed as the union of R and $\tilde{V}$ identifying point $z \in D$ and $(\psi \circ \varphi)(z) \in V$. ∎

DEFINITION 3.2. We call the continuation in the proof of Lemma 3.1 a *continuation of a disk with a crowded ideal boundary.*

4. **Continuable Riemann surfaces.** In this section we shall show the following theorem:

THEOREM 4.1. *Let R be a continuable Riemann surface. Then R admits at least one of the following continuations: a continuation of a planar end, a continuation of a simply connected skirt and a continuation of a disk with a crowded ideal boundary.*

PROOF. Assume that R is a proper subdomain of a Riemann surface R'. Take a relative boundary point of R in R' and a local coordinate disk U' in R' centered at the point. By taking smaller U' if necessary, we may assume that $R\cap(\partial U') \neq \emptyset$, where $\partial U'$ denotes the relative boundary of U' in R'. Let V be a connected component of $R \cap U'$. We note that $V \neq U'$. We have two possibilities: V is simply connected or not.

If V is not simply connected, then there is a regular real analytic simple closed curve in V which surrounds a part of ∂V contained entirely in the relative boundary of R in R'.

The intersection of V and the inside of the curve is a planar end of R. Hence R admits a continuation of a planar end.

Assume next that V is simply connected. We note that every connected component of $(\partial V) \cap U'$ consists of more than one point, and so every point on $(\partial V) \cap U'$ is a regular boundary point of V with respect to the Dirichlet problem. Let h be the solution of the Dirichlet problem with boundary values 0 on $(\partial V) \setminus \big(R \cap (\partial U')\big)$ and 1 on $(\partial V) \cap R \cap (\partial U')$. Let $\{R_n\}$ be an exhaustion of R and let R_0 be a relatively compact regular subdomain of R such that $R \setminus \overline{R_0}$ is connected. We may assume that $R_0 \cap U' = \emptyset$. Then $u_n(z) \leq h(z)$ in $V \cap R_n$ for every n, where u_n denotes the solution of the Dirichlet problem in $R_n \setminus \overline{R_0}$ with boundary values 1 on ∂R_0 and 0 on ∂R_n. Letting n tend to ∞, we get $u(z) \leq h(z)$ in V. Hence the harmonic measure u of ∂R_0 is not constant, because h is continuously equal to zero on $(\partial V) \cap U'$.

Let φ be a one-to-one conformal mapping of V onto the unit disk U. The set $(\partial V) \cap R \cap (\partial U')$ corresponds to a relatively open subset of the unit circle Γ. We denote by I the complement of the set with respect to Γ. We have two cases: I contains a relatively open circular arc or not.

If I contains a circular arc A, then the harmonic function $v = u \circ \varphi^{-1}$ in U is continuously equal to zero on A. Hence v can be extended in a neighborhood of A, and the holomorphic function $v + iv*$ is also extended and univalent in a neighborhood of a point on A. Take $T = \{ w \in \mathbb{C} \ ; 0 < \operatorname{Re} w < \lambda \text{ and } \alpha < \operatorname{Im} w < \beta \}$ so that the closure $\bar{T}$ of T is contained in the image of the neighborhood under the mapping $v + iv*$. Then $\big(\varphi^{-1} \circ (v + iv*)^{-1}\big)(T) = (u + iu*)^{-1}(T)$ is a simply connected skirt of u contained in V. Hence R admits a continuation of a simply connected skirt.

If I contains no circular arcs, then I is totally disconnected. The set I is not of class N_D, because V is a proper subdomain of U' and $(\partial V) \cap R \cap (\partial U')$ corresponds to a relatively open subset of Γ, see e.g. Sario-Oikawa [8, Chapter X, 5B]. Hence V is a disk with a crowded ideal boundary and R admits a continuation of a disk with a crowded ideal boundary. ∎

If V is simply connected and if I contains no circular arcs, then there are no simply connected skirts contained in V. In fact, assume that there is a simply connected skirt contained in V and let G be the image of the skirt under φ. Then $\bar{G} \cap \Gamma$ consists of just one point. We apply the Lindelöf asymptotic value theorem to $v + iv*$ in G and get a contradiction. This is why a continuation of a simply connected skirt and a continuation of a disk with a crowded ideal boundary are distinct.

If a Riemann surface has a planar end E, then, for every normal exhaustion $\{R_n\}$, there is a planar end of $\{R_n\}$ contained in E. The same holds for simply connected skirts. If a Riemann surface has a simply connected skirt S, then for every harmonic measure u, there is a simply connected skirt of u contained in S. Therefore a Riemann surface admits a continuation of a simply connected skirt of any given harmonic measure if it admits a continuation of a simply connected skirt of some harmonic measure. We shall apply the fact in the proof of Lemma 5.1.

5. **Two-sheeted unlimited covering Riemann surfaces of the unit disk.** Let R be a two-sheeted unlimited covering Riemann surface of the unit disk U. Let π be the projection of R onto U and let P be the image of the set of branch points of R under π. We note that P is a nonempty discrete set and R does not admit a continuation of a planar end if and only if P is an infinite set. We shall show

LEMMA 5.1. *A two-sheeted unlimited covering Riemann surface of the unit disk does not admit a continuation of a planar end nor a continuation of a simply connected skirt if and only if the derived set of the projection P of the set of branch points is the whole unit circle; namely, every point of the unit circle is an accumulation point of P.*

PROOF. Take ρ with $0 < \rho < 1$ so that $\{w \in \mathbb{C} \; ; \; |w| = \rho\} \cap P = \emptyset$ and set $R_0 = \pi^{-1}(\{w \in \mathbb{C} \; ; \; |w| < \rho\})$. Then the harmonic measure u of ∂R_0 is equal to $(\log|\pi(z)|)/(\log\rho)$ and $u*$ is equal to $\big(\arg\pi(z)\big)/(\log\rho)$, except for an additive constant.

Assume that the derived set of P is not equal to the unit circle Γ. Then we can find a small set $F = \{w = re^{i\theta} \; ; \; r_1 < r < 1 \text{ and } \theta_1 < \theta < \theta_2\}$ such that $F \cap P = \emptyset$. A connected component of $\pi^{-1}(F)$ is a simply connected skirt of u. Hence R admits a continuation of a simply connected skirt.

Assume next that the derived set of P is equal to Γ. If there is a simply connected skirt S of u, then $\pi(S) = \{re^{i\theta} \; ; \; r_1 < r < 1 \text{ and } \theta_1 < \theta < \theta_2\}$ for some r_1, θ_1 and θ_2, and so $\pi(S)$ contains points of P. Since there are no branch points on the relative boundary ∂S of S in R, this implies that $R \setminus \partial S$ is connected, a contradiction. Thus R does not admit a continuation of a simply connected skirt. ∎

Next we shall construct an example of a two-sheeted unlimited covering Riemann surface of the unit disk such that the derived set of the projection of the set of branch points is the whole unit circle but it is not maximal. By Theorem 4.1, it admits a continuation of a disk with a crowded ideal boundary.

EXAMPLE 1. Let I be a totally disconnected closed subset of the unit circle Γ which is not of class N_D. Let ψ be a one-to-one conformal mapping of U onto a proper subdomain V of U such that $\Gamma \setminus I$ corresponds to a relatively open subset of Γ. Let R_1 be a two-sheeted unlimited covering Riemann surface of V such that the derived set of the projection P_1 of the set of branch points of R_1 is equal to $(\partial V) \cap \Gamma$. The Riemann surface R_1 is not maximal and it is conformally equivalent to a two-sheeted covering Riemann surface R of the unit disk by the conformal mapping which is induced by ψ^{-1}. The derived set of the projection $P = \psi^{-1}(P_1)$ of branch points of R is equal to Γ, because it contains $\Gamma \setminus I$ and I is totally disconnected.

After the author completed the paper, Naondo Jin constructed a more concrete example. The author appreciates his kindness in agreeing to include the example here.

EXAMPLE 1′ (N. JIN). Let I be a set as in Example 1. For fixed θ with $0 < \theta < \pi/2$, let $\Omega(\theta)$ be a domain surrounded by a line segment joining 1 and $(\sin\theta)e^{i(\pi/2-\theta)}$, a circular arc $\{w \in \mathbb{C} \; ; \; |w| = \sin\theta \text{ and } \pi/2 - \theta \le \arg w \le 3\pi/2 + \theta\}$ and a line segment joining $(\sin\theta)e^{i(3\pi/2+\theta)}$ and 1. Set $\Omega = \bigcup\{\zeta\Omega(\theta) \; ; \; \zeta \in I\}$, where $\zeta\Omega(\theta) = \{\zeta w \; ; \; w \in \Omega(\theta)\}$.

Let R be a two-sheeted unlimited covering Riemann surface of U such that the projection P of the set of branch points of R is contained in $U\setminus\Omega$ and the derived set of P is equal to Γ. Then R is not maximal, because a connected component D of $\pi^{-1}(\Omega)$ is a disk with crowded ideal boundary. In fact the restriction of π onto D is a one-to-one conformal mapping of D onto Ω and the relative boundary of D in R corresponds to the relative boundary of Ω in U. The boundary of Ω in $\mathbb{C}$ is Lipschitz continuous, and so a one-to-one conformal mapping of Ω onto U can be extended to a quasiconformal mapping of the whole plane $\mathbb{C}$, see e.g. Lehto-Virtanen [3, Chapter II, § 8]. Thus the image of I under the extended quasiconformal mapping of $\mathbb{C}$ is a realization of the ideal boundary of D. Since whether a closed set belongs to class N_D or not is invariant under a quasiconformal mapping of $\mathbb{C}$, the realization of the ideal boundary of D is totally disconnected and is not of class N_D.

We note the following interesting fact: For small θ, $U\setminus\Omega$ is large and we can construct a continuable Riemann surface R such that the projection P of the set of branch points of R is thick and is contained in large $U\setminus\Omega$.

6. **A sufficient condition for a Riemann surface to be maximal.** First we recall the classes HD(R) and KD(R) of functions in a Riemann surface R. We denote by HD(R) the class of functions in R which are harmonic and have finite Dirichlet integrals on R. We denote by KD(R) the class of functions $h \in$ HD(R) such that the conjugate $dh*$ of the differential dh of each h has vanishing periods along all dividing cycles.

DEFINITION 6.1. Let p be a point in a Riemann surface R and let $\tau = x+iy$ be a local coordinate about p. The square of

$$\sup\{\frac{\partial h}{\partial x}(p)\ ;\ h \in \mathrm{HD}(R) \text{ and } D_R[h] \leq \pi\}$$

is called an HD-*span* for p and τ, where $D_R[h]$ denotes the Dirichlet integral of h on R. We denote by $\mathcal{S}^1_{\mathrm{HD}}$ the class of Riemann surfaces R such that the HD-span vanishes for some point p in R and some local coordinate $\tau = x+iy$ about p. We also define a KD-span for p and τ and the class $\mathcal{S}^1_{\mathrm{KD}}$ replacing HD(R) by KD(R).

We note that $R \in \mathcal{S}^1_{\mathrm{HD}}$ (resp. $\mathcal{S}^1_{\mathrm{KD}}$) if and only if there are a point p in R and a local coordinate $x + iy$ about p satisfying $\partial h/\partial x(p) = 0$ for every $h \in$ HD(R) (resp. KD(R)). We shall show

PROPOSITION 6.1. *Let R be a Riemann surface having no planar ends. If R is of class $\mathcal{S}^1_{\mathrm{KD}}$, then R is maximal.*

PROOF. Assume that R is continuable and let R', U' and V be as in the proof of Theorem 4.1. Since R has no planar ends, V is simply connected. We note that every connected component of $(\partial V) \cap U'$ consists of more than one point. Take a connected component of $U'\setminus V$ and let W be a simply connected domain which is the complement of the component with respect to U'. We note that V is contained in W. Let φ be a one-to-one conformal mapping of W onto a half unit disk $H = \{w \in \mathbb{C}\ ;\ |w| < 1$ and

$\operatorname{Re} w > 0\}$ such that $(\partial W) \cap (\partial U')$ corresponds to $\{w \ ; \ |w| = 1 \text{ and } \operatorname{Re} w \geq 0\}$. Let R_1 be a continuation of R which is the union of R and $\tilde{H} = \{w \in \mathbb{C} \ ; \ |w| < 1$ and $|w + 1/2| > 1/4\}$ identifying $z \in V$ and $\varphi(z) \in \varphi(V) \subset H$. Let B_1 be a border corresponding to $\{w \in \mathbb{C} \ ; \ |w + 1/2| = 1/4\}$. Then (R_1, B_1) is a bordered Riemann surface such that the interior R_1 is a continuation of R.

Lemma 1.6 of Sakai [7] asserts that the interior of a bordered Riemann surface is not of class $\mathcal{S}^1_{\mathrm{KD}}$. Hence we see that a subdomain R of R_1 is not of class $\mathcal{S}^1_{\mathrm{KD}}$, either. ■

We note that the proposition also holds for the class of Riemann surfaces R such that a KD-span of higher order vanishes for some point in R and some local coordinate about the point, see Sakai [7]; since $O_{\mathrm{HD}} \subset O_{\mathrm{KD}} \subset \mathcal{S}^1_{\mathrm{KD}}$, where O_{HD} (resp. O_{KD}) denotes the class of Riemann surfaces R such that $\mathrm{HD}(R)$ (resp. $\mathrm{KD}(R)$) consists of constant functions, the proposition holds if $\mathcal{S}^1_{\mathrm{KD}}$ is replaced by O_{HD} or O_{KD}. The proof on O_{HD} was given by Oikawa and that on O_{KD} was given by Jurchescu [2].

Now we show

EXAMPLE 2. Let us consider an example of a two-sheeted unlimited covering Riemann surface of the unit disk which is of class $\mathcal{S}^1_{\mathrm{HD}}$ (see Sakai [7, Example 1.5]); since $\mathcal{S}^1_{\mathrm{HD}} \subset \mathcal{S}^1_{\mathrm{KD}}$, by Proposition 6.1, we see that it is maximal.

7. **Concluding remarks.** 1. Let G be a plane domain and set $K = \mathbb{C} \setminus G$. Let R be a two-sheeted unlimited covering Riemann surface of G. Tamura [10] proved that if R has no planar ends and K is of class N_{SD}, then R is maximal. Here we say that a closed set K is of class N_{SD} if $\mathbb{C} \setminus K$ is connected and there are no functions which are holomorphic and univalent in $\mathbb{C} \setminus K$ and have finite Dirichlet integrals on $\mathbb{C} \setminus K$. Our argument in Section 5 shows that the result is the best one in the sense that we can not replace N_{SD} by a larger class of closed sets. In fact, let K be a totally disconnected subset of the unit circle Γ which is not of class N_{SD}. It is known that a subset of Γ is of class N_{SD} if and only if it is of class N_D, see e.g. Sario-Oikawa [8, Chapter XI, 3E]. Hence K is not of class N_D. Let ψ be a one-to-one conformal mapping of U onto a subset V of U such that V does not contain the origin and $\Gamma \setminus K$ corresponds to a subset of Γ. The mapping can be extended by the reflection with respect to $\Gamma \setminus K$ onto $G = \mathbb{C} \setminus K$. We denote again by ψ the extended conformal mapping and set $G_1 = \psi(G)$, which is contained in $\mathbb{C}$ because V does not contain the origin. We take a two-sheeted unlimited covering Riemann surface R_1 of G_1 such that the derived set of the projection P_1 of branch points is equal to $(\partial G_1) \cap \Gamma$. Let R be a two-sheeted unlimited covering Riemann surface of G obtained from R_1 by the conformal mapping induced by ψ^{-1}. Then R is a continuable Riemann surface having no planar ends.

2. It seems to be an interesting problem to replace the condition (b-2) by a more concrete condition. To examine (b-2), we have to take all regular simply connected subdomains and determine whether their ideal boundaries are crowded or not. Is it possible to replace all regular simply connected subdomains by a "canonical" regular simply connected subdomain? Can we describe the condition for the ideal boundary of a disk to be crowded or not by not using a realization of the ideal boundary of the disk? If a Riemann

surface has no simply connected skirts, then the realization I of the ideal boundary of every disk is totally disconnected. It is known that I is of class N_D if and only if the inner logarithmic capacity of $\Gamma\backslash I$ is equal to the logarithmic capacity of Γ; see, e.g., Sario-Oikawa [8, Chapter XI, 3F]. Since $\Gamma\backslash I$ corresponds to the relative boundary of the disk in the Riemann surface, there is a possibility that we can describe (b-2) not using the realization of the ideal boundary of the disk.

3. There are many results on continuation. For example, Bochner [1] proved that there is a maximal continuation for every Riemann surface. Renggli [6] gave a condition for a Riemann surface to admit a unique maximal continuation. It is very important to determine or describe all continuations of a given Riemann surface. An exhaustive study was begun by Shiba [9] and others.

REFERENCES

1. S. Bochner, *Fortsetzung Riemannscher Flächen*, Math. Ann., **98**(1928), 406–421.
2. N. Jurchescu, *Bordered Riemann surfaces*, Math. Ann., **143**(1961), 264–292.
3. O. Lehto and K. I. Virtanen, *Quasiconformal Mapping in the Plane*, Sec. Ed. Springer-Verlag, Berlin, 1973.
4. K. Oikawa, *Riemann Surfaces*, to be translated from Japanese into English.
5. T. Radó, *Über eine nicht fortsetzbare Riemannsche Mannigfaltigkeit*, Math. Z., **20**(1924), 1–6.
6. H. Renggli, *Structural instability and extensions of Riemann surfaces*, Duke Math. J., **42**(1975), 211–224.
7. M. Sakai, *On the vanishing of the span of a Riemann surface*, Duke Math. J., **41**(1974), 497–510.
8. L. Sario and K. Oikawa, *Capacity Functions*, Springer-Verlag, Berlin, 1969.
9. M. Shiba, *The Riemann-Hurwitz relation, parallel slit covering map, and continuation of an open Riemann surface of finite genus*, Hiroshima Math. J., **14**(1984), 371–399.
10. J. Tamura, *On a theorem of Tsuji*, Japan J. Math., **29**(1959), 138–140.

Tokyo Metropolitan University
Tokyo, Japan