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ATOMLESS LATTICE-ORDERED GROUPS

In Memoriam—C. S. Milloy

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ABSTRACT. We show the existence of atomless lattice-ordered groups which have doubly transitive representations. In so doing, we answer a question of M. Giraudet from 1981 [4].

Let G be a lattice-ordered group with identity e. A strictly positive element of G is called an *atom* if it cannot be written as the join of two disjoint strictly positive elements of G. Note that every strictly positive element of any linearly ordered group is an atom.

If $(\Omega, \leq)$ is an infinite chain (linearly ordered set), we write $A(\Omega)$ for $\mathrm{Aut}(\Omega, \leq)$. $A(\Omega)$ is a group under composition and a lattice under the pointwise ordering. An important sublattice subgroup of this lattice-ordered group is $B(\Omega)$, the subset of all elements of $A(\Omega)$ whose support is bounded both above and below ($\mathrm{supp}(g) = \{\alpha \in \Omega : \alpha g \neq \alpha\}$).

A subgroup G of $A(\Omega)$ is said to be *doubly transitive* on Ω if for all $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ in $(\Omega, \leq)$, there is a $g \in G$ such that $\alpha_j g = \beta_j$ $(j = 1, 2)$. (Sublattice subgroups of $A(\Omega)$ that are doubly transitive are m-transitive for all positive integers m—see [2, Lemma 1.10.1]).

In 1981, M. Giraudet [4] asked (Problem 10.16) if for some infinite chain $(\Omega, \leq)$, there is an atomless doubly transitive sublattice subgroup H of $B(\Omega)$; the other question of [4], Problem 10.15, was referred to and partially answered in [1]. The purpose of this short note is to observe that a construction due to Keith R. Pierce (see [5] or [2, Chapter 10]) provides a positive answer. Indeed

THEOREM. *For every lattice-ordered group G, there is a lattice-ordered group H containing G as a sublattice subgroup and such that every pair of strictly positive elements of H are conjugate in H. Moreover, we can find such an H that is a doubly transitive sublattice subgroup of $B(\Omega)$ for some infinite chain $(\Omega, \leq)$.*

The proof we give assumes the Generalized Continuum Hypothesis; this dependence can be avoided, see [2, p. 205].

PROOF. All but the last sentence of the theorem is established in Theorem 10.8 of [2]. Indeed, by [2, Corollary 2L], it suffices to prove the theorem for $G \subseteq B(T)$, G doubly transitive on T and $|T| = |G|$, a regular uncountable cardinal. Now the proof of

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[2, Lemma 10.9] shows that if F is a sublattice subgroup of $B(T_1)$, then $F\psi \subseteq B(T_1^b)$. Similarly, the proof of [2, Lemma 10.10] establishes that $F\psi \subseteq B(T_1^\#)$ for such F. Hence, as noted in [2], for the chain $(\Omega_1, \leq)$ obtained on p. 203 of [2], $G \subseteq B(\Omega_1)$. Moreover, any two strictly positive elements of G are conjugate in $B(\Omega_1)$. The construction ensures that $|\Omega_1| = |G|$ and that $B(\Omega_1)$ is doubly transitive (since Ω_1 is an α-set—see [2, pp. 203 and 187]). For each pair of strictly positive elements of the image of G, choose an element of $B(\Omega_1)$ conjugating the first to the second. Also for each pair of strictly increasing pairs of elements of Ω_1, choose an element of $B(\Omega_1)$ mapping the former to the latter. Let $G^\dagger$ be the sublattice subgroup of $B(\Omega_1)$ generated by the image of G and these $|G| + |G|$ elements of $B(\Omega_1)$. Then $|G^\dagger| = |G|$, $G^\dagger \subseteq B(\Omega_1)$ and $G^\dagger$ is doubly transitive on Ω_1. We can therefore proceed by induction: $G_0 = G$, $G_{m+1} = (G_m)^\dagger$, for each natural number m, to obtain $G_{m+1} \subseteq B(\Omega_{m+1})$, a sublattice subgroup acting doubly transitively on Ω_{m+1}, $|G_{m+1}| = |G|$ and G_{m+1} containing an image of G_m. Consequently, $H = \bigcup_{m=0}^{\infty} G_m$ acts doubly transitively on $\Omega = T \cup \bigcup_{m=1}^{\infty} \Omega_m$ and satisfies the conclusion of the theorem. ∎

COROLLARY 1. *Every lattice-ordered group G can be embedded in an atomless lattice-ordered group H. Moreover, H can be chosen to be a doubly transitive sublattice subgroup of $B(\Omega)$ for some suitable infinite chain* $(\Omega, \leq)$.

PROOF. Let H be as in the theorem. Let $h_1 \in H$ be strictly positive. Let $\alpha, \beta \in \Omega$ with $\alpha < \mathrm{supp}(h_1) < \beta$. Since H is transitive on Ω (indeed, doubly transitive), there is $b \in H$ such that $\alpha b = \beta$. Let $h_2 = b^{-1}h_1 b \in H$. Then $h_1 \wedge h_2 = e$ and $h_1, h_2 \neq e$. Hence $h = h_1 \vee h_2$ is not an atom. If $f \in H$ is strictly positive, then for some $a \in H$, $f = a^{-1}ha$. Now f is the join of the disjoint strictly positive elements $a^{-1}h_1 a$ and $a^{-1}h_2 a$, and so is not an atom. ∎

As noted in [2, Theorem 12E], some algebraically closed lattice-ordered groups are doubly transitive. By [3, Proposition 0.2.3], they are not completely distributive; so any doubly transitive algebraically closed lattice-ordered group is not contained in the set of elements of bounded support of that chain [2, Theorems 8D and 8.2.1]. However, by Corollary 1, we immediately obtain another rich non-trivial class of atomless lattice-ordered groups.

COROLLARY 2. *Every algebraically closed lattice-ordered group is atomless.*

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