A NOTE ON THE BORSUK CONJECTURE

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1. According to the still unproved conjecture of Borsuk [1] a bounded subset A of the Euclidean n-space E^n is a union of $n + 1$ sets of diameters less than the diameter $D(A)$ of A. Since A can be imbedded in a set of constant width $D(A)$, [2], it may be assumed that A is already of constant width. If in addition A is smooth, i.e., if through every point of its boundary ∂A there passes one and only one support plane of A, then the truth of Borsuk's conjecture can be proved very easily [3]. The question arises whether Borsuk's conjecture holds also for arbitrary smooth convex bodies, not merely for those of constant width. Since it is not known whether a smooth convex body K can be imbedded in a smooth set of constant width $D(K)$, the answer is not immediate. In this note we show that the answer is affirmative.

THEOREM 1. A smooth convex body K in E^n is a union of $n + 1$ sets of diameters $< D(K)$.

The theorem is not particularly surprising and the proof is elementary, but the method of proof is novel and may be of some interest. Our main tool is visibility sets; roughly speaking, these are subsets of ∂K, visible from a point outside K. Small Latin letters o, p, q, ... will denote points, xy will stand for the straight closed segment joining x to y, and $|xy|$ for its length.

2. Let K be any convex body in E^n, that is, a compact convex subset of E^n with nonempty interior. Let x be a point outside K and put

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$$V(x, K) = \{y : y \in \partial K ,\ xy \cap K = \{y\}\} ,$$

$$U(x, K) = \{y : y \in \partial K ,\ xy \cap (K - \partial K) = \emptyset\} ;$$

on account of the obvious physical analogy, these may be called the sets of visibility and of semivisibility, of K from x.

To prove Theorem 1 it suffices to represent ∂K as a union of $n + 1$ sets, say $B_1, \ldots, B_{n+1}$, of diameter $< D(K)$. For if that is done, let o be any point in the interior of K and let F_i be the closed convex hull of $B_i \cup \{o\}$. It follows then that

$$D(F_i) = \max \left(\sup_{x, y \in B_i} |xy| ,\ \sup_{x \in B_i} |ox| \right) < D(K) ,$$

so that

$$K = \bigcup_{i=1}^{n+1} F_i ,\ D(F_i) < D(K) \quad (i = 1, \ldots, n+1).$$

To obtain the desired decomposition of ∂K, inscribe K into a simplex with the vertices $x_1, \ldots, x_{n+1}$, and let U_i be the i-th semivisibility set $U(x_i, K)$. If x is any point in ∂K and N a plane supporting K at x, then the vertices $x_1, \ldots, x_{n+1}$ cannot all lie strictly on the same side of N as K. Therefore there is a vertex, say x_1, such that either $x_1 \in N$ or x_1 is strictly separated from K by N. In either case it follows that $x \in U_i$; hence $\partial K = \bigcup_{i=1}^{n+1} U_i$.

To complete the proof we show that the hypothesis of smoothness of K implies $D(U_i) < D(K)$ $(i = 1, \ldots, n+1)$. Suppose to the contrary that $D(U_i) = D(K)$ for some i. Then U_i contains points p and q, such that $|pq| = D(U_i) = D(K)$, and the planes P and Q, passing through p and q and orthogonal to pq, both support K. Suppose, without loss of generality, that $|x_i p| \geq |x_i q|$, so that $x_i p$ is not contained in P and lies on the same side of P as K. The sets $x_i p$ and K are convex and have no interior points in common, they can therefore be separated by a plane R supporting K. As p lies in ∂K, R supports K at p. Since $x_i p$ lies on the same side of P as K, P and R are distinct. However, this contradicts the smoothness of K because P and R are two distinct planes supporting K at p, and the proof is complete.

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