Canad. Math. Bull. Vol. **28** (1), 1985

ON THE EXISTENCE OF RESTRICTED K-LIMITS

BY
URBAN CEGRELL

ABSTRACT. The purpose of this paper is to generalize the Lindelöf-Čirka theorem.

1. **Introduction and notation**. Denote by B^n the unit ball in $\mathbb{C}^n$ and by $H(B^n)$ ($H^\infty(B^n)$) the (bounded) analytic functions on B^n.

A continuous curve $\Gamma: [0,1] \to \overline{B^n}$ is called *special* at $\xi \in \partial B^n$ if $\Gamma(t) \in B^n$, $t \in [0,1[$; $\Gamma(1) = \xi$ and if

$$\frac{|\Gamma(t) - \langle \Gamma(t), \xi\rangle \xi|^2}{1 - |\langle \Gamma(t), \xi\rangle|^2} \to 0, \qquad t \to 1.$$

For $A > 0$ we define an approach region

$$D_A(\xi) = \left\{ z \in B^n; |1 - \langle z, \xi\rangle| < \frac{A}{2}(1 - |z|^2) \right\}.$$

DEFINITION. A function f defined on B^n is said to have a K-limit at $\xi \in \partial B^n$ if $\lim_{z \to \xi} \{f(z): z \in D_A(\xi)\}$ exists for all A.

Koranyi [4] showed that if $f \in H^\infty(B^n)$ then f has K-limit for almost all $\xi \in \partial B^n$ (with respect to $d\sigma$, the Lebesgue measure on ∂B^n).

On the other hand, we have the following example:

EXAMPLE 1. (Čirka [2, p. 631]). Put $f(z, \omega) = \omega^2/(1 - z^2)$, then $f \in H^\infty(B^n)$ and $f(z, 0) = 0$ so the radial limit of f at $\xi = (1, 0)$ equals zero. But f has no K-limit at 1. Choose $c \in]0, 1[$ and take $\Gamma(t) = (t, c\sqrt{1 - t^2})$. Then $f(\Gamma(t)) = c^2$ and since $|1 - \langle \Gamma(t), \xi\rangle| = |1 - t|$, $1 - |\Gamma(t)|^2 = 1 - t^2 - c^2(1 - t^2)$ we have

$$\frac{|1 - \langle \Gamma(t), \xi\rangle|}{1 - |\Gamma(t)|^2} = \frac{(1 - t)}{(1 - t^2)(1 - c^2)} = \frac{1}{(1 + t)(1 - c^2)}$$

so

$$\Gamma(t) \in D_{2/(1-c^2)}(1), \qquad \forall t \in [0, 1[.$$

To avoid this we follow Čirka [2], and introduce restricted approach regions and limits.

Let $A > 0$ and $g(t)$ a positive decreasing function with

Received by the editors February 9, 1982 and, in final revised form, April 4, 1984.
AMS Subject Classification (1980): 32 A 40.
Research partially supported by the Swedish Natural Science Research Council.

https://doi.org/10.4153/CMB-1985-002-6 Published online by Cambridge University Press

$$\lim_{t\to 1} g(t) = 0 \qquad (*)$$

Put

$$R_{A,g}(\xi) = \left\{ z \in B^n; \frac{|1 - \langle z, \xi\rangle|}{1 - |z|^2} < \frac{A}{2}, \frac{|z - \langle z, \xi\rangle|^2}{1 - |\langle z, \xi\rangle|^2} < g(|z|) \right\}.$$

Each set $R_{A,g}(\xi)$ is then called a restricted approach region and if f is a function defined on B^n so that

$$\lim_{z\to\xi} \{f(z): z \in R_{A,g}(\xi)\}$$

exists for every restricted approach region $R_{A,g}(\xi)$, then f is said to have a restricted K-limit at ξ.

The Lindelöf-Čirka theorem [5], [2] now reads as follows.

THEOREM 1. *If $f \in H^\infty(B^n)$ and if $\lim_{t\to 1} f(\Gamma(t))$ exists for a special curve $\Gamma(t)$ with $\Gamma(1) = \xi$ then f has a restricted K-limit at ξ.*

(In particular, $\lim_{z\to\xi} \{f(z): z \in \mathrm{K}\}$ exists where K is any cone in B^n with vertex at ξ).

We shall generalize this theorem for $f \in H(D_A(\xi))$ in the case where $\Gamma(t) = t\xi$, i.e., we shall assume that f has a radial limit at ξ.

Observe that, if $n \geq 2$ and if f has a restricted K-limit, then f has a *tangential* limit in certain directions. The curve

$$\Gamma(t) = (t, (1 - t^2)^{2/3}), \qquad t \in [0, 1]$$

shows this.

2. **The extremal function**. Let Ω be a bounded and open subset of $\mathbb{C}^n$ and put

$$F = \{\varphi \in PSH(\Omega); \varphi \leq 0, \lim_{z'\to z} \varphi(z') \text{ exists } \forall z \in \partial\Omega\}.$$

If E is any subset of $\bar{\Omega}$ we define the extremal function h_E by

$$h_E(z) = \sup\{\varphi(z); \varphi \in F, \varphi \leq -1 \text{ on } E\}, \qquad z \in \bar{\Omega}.$$

Denote by M_z, $z \in \bar{\Omega}$, the class of positive measures μ on $\bar{\Omega}$ such that

$$\varphi(z) \leq \int \varphi(\xi)\, d\mu(\xi), \forall \varphi \in F.$$

LEMMA 1. *For every compact set* K *in $\bar{\Omega}$ and every $z \in \bar{\Omega}$ there is a $\mu_z \in M_z$ so that*

$$-h_{\mathrm{K}}(z) = \mu_z(\mathrm{K}).$$

Proof. Proposition 2:1 in Cegrell [1].

LEMMA 2. *Assume that Ω is convex and that $\mu_\nu \in M_{z_\nu}$ where $z_\nu \to \xi$, $\nu \to +\infty$ and that ξ is a strictly convex boundary point. Then $\lim_{\nu\to+\infty} \mu_\nu(\mathrm{K}) = 0$ for every compact subset of $\bar{\Omega}$ not containing ξ.*

Proof. The assumption implies that there is a convex function ψ on Ω, continuous up to the boundary such that $\psi(\xi) = 0$ but $\psi(z) < 0$ for $z \in \bar{\Omega} \setminus \{\xi\}$. Since convex

https://doi.org/10.4153/CMB-1985-002-6 Published online by Cambridge University Press

functions are plurisubharmonic, the inequalities $\psi(z_\nu) \le \int \psi(\xi)\, d\mu_\nu(\xi)$, $\nu \in N$ proves Lemma 2.

3. **The case $n = 1$**. If Ω is an open and bounded subset of the plane, then the extremal function h_E ($E \subset \bar{\Omega}$) is harmonic on $\Omega \setminus \bar{E}$.

PROPOSITION 1. *Let Ω be an open, bounded, convex and symmetric subset of the plane and let l denote the segment of symmetry. If E is the part of $\partial\Omega$ that lies on one given side of l then $h_E|_l = -\frac{1}{2}$.*

If K is any closed cone in Ω with vertex $\xi \in l \cap \partial\Omega$ such that $K \setminus \{\xi\} \cap B(\xi, r) \subset \Omega$ for some $r > 0$ then there is an $\epsilon > 0$ so that

$$K \cap B(\xi, \epsilon) \subset \{z \in \bar{\Omega};\ h_l(z) \le -\epsilon\}.$$

($B(\xi, \epsilon)$ is the ball with center ξ and radius ϵ.)

Proof. The first part follows easily from the symmetry and the second part is proved by repeated use of symmetry.

PROPOSITION 2. *Let K be a closed convex cone with vertex ξ and l a line segment in K^0 with endpoint ξ. If $f \in H^\infty(K^0 \cap B(\xi, r))$ for some $r > 0$ and if $\lim_{z\to\xi} \{f(z) : z \in l\}$ exists then $\lim_{z\to\xi} \{f(z) : z \in K'\}$ exists for every closed cone K' in $K^0 \cup \{\xi\}$ with vertex at ξ.*

Proof. (By a wellknown method). We can assume that $|f| \le 1$ and that $\lim_{z\to\xi} \{f(z) : z \in l\} = 0$. Let K′ be given and choose $\epsilon > 0$ as in Proposition 1. For $z \in K' \cap B(\xi, \epsilon)$ there is a $\mu_z \in M_z$ with $\mu_z(l) \ge \epsilon$. If we let z tend to ξ in K′ we get by Lemma 2 $\int \log |f(\eta)|\, d\mu_z(\eta) \to -\infty$, $z \to \xi$. But since $\log |f(z)| \le \int \log |f(\eta)|\, d\mu_z(\eta)$ so $\lim_{z\to\xi} \{f(z) : z \in K'\} = 0$ which completes the proof.

EXAMPLE 2. Proposition 2 cannot be generalized to bounded harmonic functions. Let E be the part of the unit circle in the lower halfplane. Then $h_E = -\frac{1}{2}$ on the real axes (by the observation in the beginning of this section). The restriction of h_E to segment of $y = x - 1$ in B^1 does not exceed $-\frac{3}{4}$. Thus h_E has a radial but not non-tangential limit.

As pointed out by J. C. Taylor, the next lemma is a consequence of Harnack's inequality.

LEMMA 3. *Let h be a positive harmonic function on $D_A(\xi) = \{z \in B^1; |\xi - z| < (A/2)(1 - |z|^2)\}$ ($\xi \in \partial B^1$). If $\sup_{0<r<1} h(r\xi) < +\infty$ then $\sup_{z \in D_{A'}(\xi)} h(z) < +\infty$ for every $A' < A$.*

4. **Statement of the theorems**.

THEOREM 2. *Let $A > 0$ and g with property (*) be given. Assume that*

(1) $f \in H(D_A(\xi))$

(2) $\sup\left\{|f(\omega\xi)| : \omega \in B^1, \dfrac{|1 - \omega|}{1 - |\omega|^2} < \dfrac{A}{2}\right\} < +\infty$

(3) $|f|$ *has a plurisuperharmonic majorant* ψ *such that*

https://doi.org/10.4153/CMB-1985-002-6 Published online by Cambridge University Press

$$\lim_{\omega\to 1}\left\{\psi(\omega\xi)g(|\omega|)^{1/2}:\omega\in B^1,\ \frac{|1-\omega|}{1-|\omega|^2}<\frac{A}{2}\right\}=0.$$

If $\lim_{r\to 1} f(r\xi)$ *exists, then* $\lim_{z\to\xi}\{f(z):z\in R_{A',g}\}$ *exists for every* $A' < A$.

REMARK. Related results have been obtained by Cima and Krantz in [3]. However, our results also apply to non-normal functions, e.g.

$$z_2^2(1-z_1)^{-1}\log(1/1-z_1).$$

COROLLARY 1. *Assume that* $f\in H^1(B^n)$ *(i.e.,* $|f|$ *has a harmonic majorant). Then* $\lim_{r\to 1} f(r\xi)=f^*(\xi)$ *exists a.e.* $(d\sigma)$ *on* ∂B^n *and* $f^*\in L^1$ $(d\sigma)$. *If*

$$\sup_{0<r<1}\int P(r\xi,\eta)|f^*(\eta)|\,d\sigma(\eta)<+\infty$$

where

$$P(z,\eta)=\frac{(1-|z|^2)^n}{|(1-\langle z,\eta\rangle)|^{2n}}$$

and if $\lim_{r\to 1} f(r\xi)$ *exists, then f has a restricted K-limit at* ξ.

Proof. It follows from Koranyi [4] and Rudin [6, Theorem 5.4.12] that $|f|$ is bounded in every $D_A(\xi)$. Thus, Theorem 2 applies.

THEOREM 3. *Assume that*

(1) $f\in H(D_A(\xi))$

(2) $\sup\left\{|f(\omega\xi)|:\omega\in B^1,\ \frac{|1-\omega|}{1-|\omega|^2}<\frac{A}{2}\right\}<+\infty$

(3) $|f|$ *has a plurisuperharmonic majorant* ψ *such that*

$$\lim_{t\to 1}\{g(t+\delta(t-1))^{1/2}\psi(t\xi):t\in\mathbb{R}\}=0$$

for some $0<\delta\le 1$.

If $\lim_{r\to 1} f(r\xi)$ *exists then* $\lim_{z\to\xi}\{f(z):z\in R_{A',g}\}$ *exists for every* $A'<A$.

COROLLORY 2. *Assume that* $f\in H(D_A(\xi))$ *and that* $|f|$ *has a plurisuperharmonic majorant* ψ *on* $D_A(\xi)$ *such that*

$$\sup_{0<r<1}\psi(\xi r)<+\infty.$$

If $\lim_{r\to 1} f(r\xi)$ *exists, then* $\lim_{z\to\xi}\{f(z):z\in R_{A',g}\}$ *exists for every* $A'<A$.

Proof. Assumptions (1) and (3) in Theorem 3 are clearly fulfilled. It remains to prove that (2) holds. Denote by U the part of the complex line through zero and ξ that is contained in $D_A(\xi)$. The restriction of ψ to U is superharmonic (and not identically $+\infty$). Hence, there is a harmonic function h on U so that

$$|f(z)|\le h(z)\le\psi(z),\qquad \forall z\in U.$$

An application of Lemma 3 gives (2).

5. **Proof of Theorem 2**. Let $0 < A' < A$ and a sequence $z_\nu \in R_{A',g}$, $\lim_{\nu\to+\infty} z_\nu = \xi$ be given. To prove the theorem it is enough to prove that $\lim_{\nu\to+\infty} f(z_\nu) = \lim_{r\to 1} f(r\xi)$. Consider for $\lambda \in \mathbb{C}$, $(1-\lambda)\langle z_\nu, \xi\rangle\xi + \lambda z_\nu$. This point is in $D_A(\xi)$ if and only if

$$|1 - \langle z_\nu, \xi\rangle| < \frac{A}{2}(1 - |(1-\lambda)\langle z_\nu, \xi\rangle\xi + \lambda z_\nu|^2)$$

$$\Leftrightarrow |\langle z_\nu, \xi\rangle\xi + \lambda(z_\nu - \langle z_\nu, \xi\rangle\xi)|^2 < 1 - \frac{2}{A}|1 - \langle z_\nu, \xi\rangle|$$

$$\Leftrightarrow |\langle z_\nu, \xi\rangle|^2 + |\lambda|^2|z_\nu - \langle z_\nu, \xi\rangle\xi|^2 < 1 - \frac{2}{A}|1 - \langle z_\nu, \xi\rangle|$$

$$\Leftrightarrow |\lambda|^2 < \frac{-\frac{2}{A}|1 - \langle z_\nu, \xi\rangle| - |\langle z_\nu, \xi\rangle|^2 + 1}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2}.$$

But since $z_\nu \in R_{A',g}$ we have

$$|1 - \langle z_\nu, \xi\rangle| < \frac{A'}{2}(1 - |\langle z_\nu, \xi\rangle|^2)$$

so the right hand side above is not smaller than

$$\frac{-\frac{A'}{A}(1 - |\langle z_\nu, \xi\rangle|^2) - |\langle z_\nu, \xi\rangle|^2 + 1}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2} = \frac{\left(1 - \frac{A'}{A}\right)(1 - |\langle z_\nu, \xi\rangle|^2)}{|z_\nu - \langle z_\nu, \xi\rangle\xi|^2}$$

which in turn are not smaller than

$$\left(1 - \frac{A'}{A}\right)\cdot\frac{1}{g(|z_\nu|)}$$

again because $z_\nu \in R_{A',g}(\xi)$.

Hence, if

$$|\lambda|^2 \leq \left(1 - \frac{A'}{A}\right)\cdot\frac{1}{g(|z_\nu|)}$$

it follows that

$$\tau(\lambda) = (1-\lambda)(z_\nu, \xi)\xi + \lambda z_\nu \in D_A(\xi)$$

so $f(\tau(\lambda))$ is analytic in

$$|\lambda|^2 < \left(1 - \frac{A'}{A}\right)\cdot\frac{1}{g(|z_\nu|)}$$

and Cauchy's integral formula gives

$$f(\tau(1)) - f(\tau(0)) = \frac{1}{2\pi i}\int_{|\lambda| = [(1-\frac{A'}{A})\cdot\frac{1}{g(|z_\nu|)}]^{1/2}} \frac{f(\tau(\lambda))}{\lambda(\lambda - 1)}\, d\lambda.$$

Hence

$$(i)\quad |f(z_\nu) - f(\langle z_\nu, \xi\rangle\xi)| \leq \frac{1}{2\pi}\int_{|\lambda| = \left[\left(1-\frac{A'}{A}\right)\cdot\frac{1}{g(|z_\nu|)}\right]^{1/2}} \frac{|f(\tau(\lambda))|}{|\lambda||\lambda - 1|}$$

$$\leq \frac{\psi(\langle z_\nu, \xi\rangle\xi)}{\left|\left[\left(1 - \frac{A'}{A}\right)\frac{1}{g(|z_\nu|)}\right]^{1/2} - 1\right|}.$$

The right hand side of (i) equals

$$\frac{g(|z_\nu|)^{1/2}\psi(\langle z_\nu, \xi\rangle\xi)}{\left(1 - \frac{A'}{A}\right)^{1/2} - g(|z_\nu|)^{1/2}} \to 0, \quad \nu \to +\infty.$$

by assumption 3. Thus,

$$f(z_\nu) - f(\langle z_\nu, \xi\rangle\xi) \to 0, \qquad \nu \to +\infty.$$

Now, by assumption 2, we can apply Proposition 2 and we have that $\lim_{\nu\to+\infty} f(\langle z_\nu, \xi\rangle\xi)$ exists and equals $\lim_{r\to 1} f(r\xi)$. Thus, $\lim_{\nu\to+\infty} f(z_\nu) = \lim_{r\to 1} f(r\xi)$ and the proof is complete.

6. **Proof of theorem 3**. Let $1 < A' < A$ be given and choose A'', $A' < A'' < A$ and let K_1 and K_2 be two cones with vertex at 1, symmetrical with respect to the real axis and so that

$$\left\{\omega \in \mathbb{C};\ |1 - \omega| < \frac{A'}{2}(1 - |\omega|^2)\right\} \subset K_1$$

$$\subsetneq K_2 \subset \left\{\omega \in \mathbb{C};\ |1 - \omega| < \frac{A''}{2}(1 - |\omega|^2)\right\}.$$

From now on, we think of $z \in B^n$ to be close to ξ. For $z \in R_{A',g}$ denote by t_2 the non-negative number such that $\langle z, \xi\rangle - t_z \perp \partial K_2$. Consider, for $\eta \in \mathbb{C}$,

$$L(\eta) = z + (\eta + t_z - \langle z, \xi\rangle)\xi.$$

There is a real number $K > 1$ (K independent of z) so that if $|\eta| \leq K|\langle z, \xi\rangle - t_z|$ then $\eta + t_z \in K_2$ and a calculation shows that then $L(\eta) \in R_{A''',g}$ for every fixed A''', $A'' < A''' < A$.

Let now P be the Poisson kernel for some smooth simply connected domain $D \subset \{\eta \in \mathbb{C};\ |\eta| < K|\langle z, \xi\rangle - t_z|\}$ containing $\langle z, \xi\rangle - t_z$ and zero. Then

$$|f(L(\langle z, \xi\rangle - t_z) - f(\langle L(\langle z, \xi\rangle - t_z), \xi\rangle\xi)|$$

$$\leq \int_{\partial D} |f(L(\eta)) - f(\langle L(\eta), \xi\rangle\xi)|\mathrm{P}(\langle z, \xi\rangle - t_z, \eta)\, d\sigma(\eta)$$

$\leq$ (Harnack's inequality)

$$\leq C\int_{\partial D} |f(L(\eta)) - f(\langle L(\eta), \xi\rangle\xi)\, \mathrm{P}(0, \eta)\, d\sigma(\eta) \leq (i)$$

https://doi.org/10.4153/CMB-1985-002-6 Published online by Cambridge University Press

$$\leq C \int_{\partial D} \frac{\psi(\langle L(\eta), \xi\rangle \xi)\, \mathrm{P}(0, \eta)}{\left|\left(1 - \frac{A'''}{A}\right) \frac{1}{g(|L(\eta)|)}\right|^{1/2} - 1} d\sigma(\eta)$$

$$\leq C \cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2} \int_{\partial D} \psi(\langle L(\eta), \xi\rangle \xi)\, \mathrm{P}(0, \eta)\, d\sigma(\eta)$$

$$\leq C \cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2} \psi(\langle L(0), \xi\rangle \xi)$$

$$= C \cdot C_1 \sup_{\partial D} g(|L(\eta)|)^{1/2} \psi(t_z \xi).$$

The constant C (that comes from the Harnack inequality) depends on the shape of D, and not on the scale. We can thus take ∂D to be an ellipse with focus 0 and $\langle z, \xi\rangle - t_z$ so that

$$|\eta + t_z| \geqslant t_z + \delta(t_z - 1), \quad \forall \eta \in \partial D.$$

We find that

$$|f(z) - f(\langle z, \xi\rangle \xi)| = |f(L(\langle z, \xi\rangle - t_z) - f(L(\langle\langle z, \xi\rangle - t_z), \xi\rangle)|$$
$$\leq C \cdot g(t_z + \delta(t_z - 1))^{1/2} \psi(t_z \xi) \to 0, \qquad z \to \xi,$$

and the proof is now finished in the same way as the end of the proof of Theorem 2.

References

1. U. Cegrell, *Capacities and extremal plurisubharmonic functions on subsets of* $\mathbb{C}^n$. Ark. Mat. **18** (1980), 199–206.

2. E. M. Čirka, *The theorems of Lindelöf and Fatou in* $\mathbb{C}^n$. Math. USSR Sb. **21** (1973), 619–641.

3. J. A. Cima and S. G. Krantz, *The Lindelöf principle and normal functions of several complex variables*. Duke J. **50** (1983), 303–328.

4. A. Koranyi, *Harmonic functions on Hermitian hyperbolic space*. Trans. Amer. Math. Soc. **135** (1969), 507–516.

5. E. Lindelöf, *Sur une principe générale de l'analyse et ses applications à la théorie de la représentation conforme*. Acta Soc. Sci. Fennicae **46** (1915), 1–35.

6. W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$. Springer Verlag 1980.

McGill University
Department of Mathematics
Montreal, Canada

Uppsala University
Department of Mathematics
Uppsala, Sweden