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A NOTE ON THE FIXED SUBRING OF AN *FPF* RING

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An associative ring R with identity is called a left (right) FPF ring if given any finitely generated faithful left (right) R-module A and any left (right) R-module M then M is the epimorphic image of a direct sum of copies of A. Faith and Page have asked if the subring of elements fixed by a finite group of automorphisms of an FPF ring need also be FPF. Here we present examples showing the answer to be negative in general.

1. Introduction

An associative ring R with identity is said to be left (right) FPF (short for finitely pseudo-Frobenius) if every finitely generated faithful left (right) R-module generates the category of all left (right) R-modules, while R is FPF if it is both left and right FPF. Quasi-Frobenius rings, Prufer domains and self-injective commutative rings are all FPF. A recent monograph by Faith and Page [**4**] on FPF rings contains a list of fifteen open problems. Problem fourteen discusses the action of a finite group G of automorphisms on a right FPF ring R and asks if in general the fixed ring $R^G = \{r \in R \colon \forall g \in G(g(r) = r)\}$ is also right FPF. Faith has shown in [**2**] that R^G is FPF when R is commutative FPF and finitely generated projective as a module over R^G. Here we provide two simple examples of FPF rings having commutative fixed rings which are not FPF.

2. The Examples

For our first example we begin by letting Q be the quaternion group of order eight, that is $Q = \langle a, b \colon a^4 = b^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle$. Then (see, for example Thomas and Wood [**6**]) Q has automorphism group $S_4 = \langle g, h \colon g^4 = h^2 = (gh)^3 = 1 \rangle$ where $g(a) = a$, $g(b) = ab$, $h(a) = b$, and $h(b) = a$. Now form the group ring $R = K[Q]$ where K is the field of two elements. Then (see, for example, p.79 of Passman [**5**]), since Q is finite, R is self-injective. Thus, since R is Artinian, R is quasi-Frobenius and so FPF. Now let G denote the group of automorphisms of R obtained by extending linearly to R the action of S_4 on Q. Then a straightforward calculation shows that

$$R^G = \{0, 1, a^2, 1 + a^2, w, 1 + w, a^2 + w, 1 + a^2 + w\}$$

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where $w = a + a^3 + b + ab + a^2b + a^3b$. Moreover R^G is commutative and its proper nonzero ideals are just $I = \{0, 1 + a^2\}$, $J = \{0, w\}$, $K = \{0, 1 + a^2 + w\}$ and the unique maximal ideal $M = \{0, 1 + a^2, w, 1 + a^2 + w\}$. Also $M^2 = 0$ and, since R^G is Artinian, R^G is its own classical ring of quotients.

Now Theorem A of [**3**] implies that the classical quotient ring of a commutative FPF ring is self-injective. Thus to show that R^G is not FPF it suffices to show that it is not self-injective. To this end we define $f\colon I \to J$ by $f(1 + a^2) = w$. Then f is an R^G-homomorphism, well-defined since $M^2 = 0$, which cannot be extended to an endomorphism on R^G since $I \cap J = 0$. Thus, by Baer's criterion for injectivity, R^G is not self-injective and so not FPF.

Our second example is commutative and of arbitrary prime characteristic p. Let K be a field of characteristic p and form

$$R = K[[x_1, \dots, x_p]]/J$$

where $x_1, \dots, x_p$ are commuting indeterminates and J is the ideal of the power series ring generated by $x_i^2 - x_j^2$ and $x_i x_j$ for $i \neq j$ and $i, j \in \{1, \dots, p\}$. Then R is a commutative local quasi-Frobenius ring and an arbitrary element of R has expression

$$(*) \qquad a + b_1x_1 + b_2x_2 + \dots + b_px_p + cx_1^2, \text{ where } a, b_1, b_2, \dots, b_p, c \in K.$$

Now let g be the automorphism of R determined by $g(x_i) = x_{i+1}$ for $i = 1, \dots, p-1$ and $g(x_p) = x_1$. Then $G = \langle g \rangle$ is of order p and R^G consists of elements of the form (*) where $b_1 = b_2 = \dots = b_p$. Noting that the units of R are those with nonzero a, straightforward arguments show that, as for our first example, R^G is commutative local Artinian but not FPF, having the property that the nonzero ideals properly contained in the maximal ideal are all simple. Indeed, taking $p = 2$ and K as the field of two elements gives the same R^G (but smaller R, of order 16) as before. In fact this is a minimal counterexample.

Finally we remark that the ring R of our first example was used in [**1**] as an example of an FPF ring whose centre C, of order 32, is not FPF. Replacing $G = S_4$ by the inner automorphism group H on Q and identifying this with its linear extension to R gives $R^H = C$, thus providing yet another example.

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