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On Best Proximity Points in Metric and Banach Spaces

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Abstract. In this paper we study the existence and uniqueness of best proximity points of cyclic contractions as well as the convergence of iterates to such proximity points. We take two different approaches, each one leading to different results that complete, if not improve, other similar results in the theory. Results in this paper stand for Banach spaces, geodesic metric spaces and metric spaces. We also include an appendix on CAT(0) spaces where we study the particular behavior of these spaces regarding the problems we are concerned with.

1 Introduction

Let A and B be two nonempty closed subsets of a complete metric space X. Consider a mapping $T\colon A \cup B \to A \cup B$ such that

$$T(A) \subseteq B \quad \text{and} \quad T(B) \subseteq A$$

with the additional condition that there exists $k \in (0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \text{ for all } x \in A \text{ and } y \in B,$$

then $A \cap B \neq \varnothing$ and T has a unique fixed point in $A \cap B$.

In [3–5, 18] a generalization of this situation was studied under the assumption of $A \cap B = \varnothing$. More precisely, in [3, 18] it was assumed that there exists $k \in (0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y) + (1-k)\operatorname{dist}(A, B) \tag{1.1}$$

for all $x \in A$ and $y \in B$ to obtain existence, uniqueness, and convergence of iterates to the so-called best proximity points; that is, a point x in either A or B such that $d(x, Tx) = \operatorname{dist}(A, B)$. This was first studied in [3] for uniformly convex Banach spaces; see also [12] for more on related topics. Then, in [18], the UC property (defined in Section 3) was introduced for a pair (A, B) of subsets of a metric space so a result on existence, uniqueness, and convergence of iterates stands (Theorem 2.6) in general metric spaces. Since, as it is also proved in [18], the UC property occurs in a large collection of pairs of subsets of uniformly convex Banach spaces, Theorem 2.6 actually contains the main theorem of [3] ([3, Theorem 3.10]) as a particular case.

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The UC property was even proved, in [18], to occur outside the setting of uniformly convex Banach spaces. In fact, this was obtained for UCED (uniformly convex in every direction) Banach spaces and strictly convex Banach spaces, but in both cases under the very strong condition (see Theorem 3.7) that one of the sets be of compact closure. In this work we first introduce a new property, the so-called WUC property, which is proved to occur under far less restrictive conditions than the UC property, and then an existence, uniqueness, and convergence theorem is proved for pairs of sets verifying property WUC. Second, we focus on the same problem from the new approach suggested by one of the authors in [5] to obtain more new results. As a result, a partial answer in the positive is given to a question raised in [3].

The work is organized as follows. In Section 2 we introduce most of the definitions, notations, and previous results we will need. In Section 3 we look for weaker conditions than the UC property. We introduce properties WUC and W-WUC and show that similar results to those in [18] hold under conditions that are easier to verify. In Section 4, we approach the same problem by the introduction of a semimetric. This is applied in a successful way by showing that the mappings verifying the contractive condition (1.1), under suitable assumptions, are contractions with respect to a certain semimetric. We finish this work with a remark on CAT(0) spaces. In [4] it was shown that when the ambient space is a Hilbert space, then the kind of mappings we are dealing with actually behave as nonexpansive ones. In [5] it was shown that the semimetric there defined coincides with the metric of the ambient space when this is a Hilbert space. Our remark on CAT(0) spaces, in a certain sense the nonlinear counterparts of Hilbert spaces, states that something similar happens in these spaces.

2 Preliminaries

In this section we compile the main concepts and results we will work with throughout this paper. We begin with some basic definitions and notations that are needed. Let (X, d) be a metric space and let A and B be two subsets of X. Define

$$\begin{aligned}\operatorname{dist}(x, A) &= \inf\{d(x, y) : y \in A\};\\ P_A(x) &= \{y \in A : d(x, y) = \operatorname{dist}(x, A)\};\\ \operatorname{dist}(A, B) &= \inf\{d(x, y) : x \in A, y \in B\};\\ \operatorname{diam}(A) &= \sup\{d(x, y) : x, y \in D\}.\end{aligned}$$

Recall that the set A is said to be *a Chebyshev set with respect to B* if $P_A(x)$ is a singleton for any $x \in B$.

A metric space (X, d) is said to be a *geodesic space* (*D-geodesic space*, respectively) if every two points x and y of X (with $d(x, y) \le D$) are joined by a geodesic, *i.e.*, a map $c \colon [0, l] \subseteq \mathbb{R} \to X$ such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. Moreover, (X, d) is called *uniquely geodesic* (*D-uniquely geodesic*) if there is exactly one geodesic joining x and y for each $x, y \in X$ (with $d(x, y) \le D$). When the geodesic between two points is unique, its image (called geodesic segment) is denoted by $[x, y]$. The midpoint m in between two points x and y in a uniquely

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geodesic metric space is the only point in $[x, y]$ such that $d(x, m) = d(y, m)$. Any Banach space is a geodesic space with usual segments as geodesic segments. Throughout this work we will just use geodesic metric space to refer to a uniquely geodesic space, since all our geodesic spaces will be uniquely geodesic.

A very important class of geodesic metric spaces are the CAT(k) spaces, that is, metric spaces of curvature uniformly bounded above by k. These spaces have been the object of a lot of interest by many researchers and we will get back to them, especially to CAT(0) spaces. For a very thorough treatment on CAT(k)-spaces, the reader can consult [2].

A subset A of a geodesic metric space (X, d) is said to be *convex* if the geodesic joining each pair of points x and y of A is contained in A.

We will need the notion of uniformly convex geodesic metric space (see also [9, p. 107]).

Definition 2.1 A geodesic metric space (X, d) is said to be *uniformly convex* if for any $r > 0$ and any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in X$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$ it is the case that

$$d(m, a) \leq (1 - \delta)r,$$

where m stands for the midpoint of the geodesic segment $[x, y]$. A mapping

$$\delta\colon (0, +\infty) \times (0, 2] \to (0, 1]$$

providing such a $\delta = \delta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is called a *modulus of uniform convexity*. If, moreover, δ decreases with r (for a fixed ε), we say that δ is a monotone modulus of uniform convexity of X.

The notion of a monotone modulus of uniform convexity seems to have been studied for first time in [14]. Of course, the usual modulus of convexity of a uniformly convex Banach space is monotone in this sense. For more on geometry of Banach spaces the reader can check [1, 8, 13].

Remark 2.2 If in the above definition we drop the uniformity conditions, then we find the notion of strict convexity. More precisely, if X is a Banach space and such a δ exists for each a, x and y as above with $d(x, y) > 0$, then we will say that X is a strictly convex Banach space. If the same condition is imposed on a geodesic metric space X, then we meet a wider class of geodesic metric spaces than those of nonpositive curvature in the sense of Busemann; see [16] for a detailed study on them.

Definition 2.3 Let A and B be two nonempty subsets of a metric space X. A map $T\colon A \cup B \to A \cup B$ is a *cyclic contraction map* if it satisfies:

(i) $T(A) \subseteq B$ and $T(B) \subseteq A$;
(ii) there is some $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y) + (1 - k)\ \operatorname{dist}(A, B)$, for all $x \in A$ and $y \in B$.

Remark 2.4 Notice that condition (ii) implies that T is a relatively nonexpansive mapping, *i.e.,* T satisfies that $d(Tx, Ty) \le d(x, y)$ for all $x \in A$ and $y \in B$, which were the main object of study in [4, 5].

Definition 2.5 Let A and B be two nonempty subsets of a metric space X. Let $T\colon A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x \in A \cup B$ is said to be a *best proximity point* for T if $d(x, Tx) = \operatorname{dist}(A, B)$.

Existence, uniqueness, and convergence of iterates to a best proximity point for cyclic contractions have recently been studied in [3, 18]. The goal of this work is to find improvements of the main results in these works. Next we state the main result from [18] (the definition of the UC property is in Section 3).

Theorem 2.6 *Let (X, d) be a metric space, and let A and B be nonempty subsets of X such that (A, B) satisfies the UC property. Assume that A is complete. Let T be a cyclic contraction on $A \cup B$. Then T has a unique best proximity point z in A, and $\{T^{2n}x\}$ converges to z for every $x \in A$.*

The main result in [3] basically states the same thing but with A and B nonempty closed and convex, X a uniformly convex Banach space, and no mention of the UC property.

In [5] a new approach to relatively nonexpansive mappings leads to the fact that such mappings, under suitable conditions, are actually nonexpansive with respect to an adequate semimetric. In Section 4 we apply this new approach to cyclic contractions. Next we introduce the main notions and results on semimetric spaces that we will need.

Definition 2.7 Let M be a nonempty set. A function $d\colon M \times M \to [0, \infty)$ is said to be a *semimetric* on M if

(i) $d(x, y) = 0$ if, and only if, $x = y$;
(ii) $d(x, y) = d(y, x)$ for any $x, y \in X$.

In this case, (M, d) is said to be a *semimetric space*.

Contractions with respect to semimetrics are defined in a similar way to contractions with respect to metrics.

Definition 2.8 Let (X, d) be a semimetric space. A mapping $T\colon X \to X$ is said to be a *contraction* if there is a constant $k \in (0, 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \le kd(x, y).$$

The next definition will make it easier to state some of our results.

Definition 2.9 Let X be a nonempty set. Let d and d_1 be a metric and a semimetric on X, respectively. We say that d and d_1 are compatible on X if for every $\varepsilon > 0$ and $x \in X$ there exist $f_x(\varepsilon) > 0$ and $g_x(\varepsilon) > 0$ such that

$$B_d(x, f_x(\varepsilon)) \subseteq B_{d_1}(x, \varepsilon) \quad \text{and} \quad B_{d_1}(x, g_x(\varepsilon)) \subseteq B_d(x, \varepsilon),$$

where $B_d(x, r)$ and $B_{d_1}(x, r)$ stand, respectively, for the closed balls of center x and radius r with respect to the metric and the semimetric.

In [10], different counterparts of Banach's contraction theorem are given for semimetric spaces. We state next a particular case of those results that is more appropriate to our context (see [10, Theorem 1]).

Theorem 2.10 *Let X, d, and d_1 be as in the definition above with d and d_1 compatible. Let T be a contraction on X for the semimetric d_1, then T has a unique fixed point x_0. Moreover, for any $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x_0.*

We finish this section by introducing two geometrical properties for Banach spaces. We begin describing property (H).

Definition 2.11 Let X be a Banach space. X is said to have the property (H) if for any sequence on the unit sphere of X, weak and norm convergence coincide.

Remark 2.12 This property has been very extensively studied in the literature and is closely related to the so-called Kadec–Klee property (KK property). For more on this topic, see [1, 8, 13, 15].

We will also need the following uniform version of the KK property.

Definition 2.13 Let X be a Banach space. X is said to have the property UKK (uniform Kadec-Klee property) if for any $\varepsilon > 0$ the number

$$\eta(\varepsilon) = \inf\{1 - \|x\|\} > 0,$$

where the infimum is taken over all points x such that x is a weak limit for some sequence $\{x_n\}$ in the unit ball of X with $\|x_n - x\| \geq \varepsilon$ for all n.

Different properties of UKK Banach spaces as well as the connection among all these geometrical notions can be found in the above-mentioned references. Let us just note here, as a matter of fact, that uniformly convex Banach spaces are UKK spaces and so they also have property (H). Both notions, uniform convexity and the UKK property, have to do with a certain rotoundity of the balls of the space. This is obvious for uniform convexity and far less obvious for the UKK property as there exist Banach spaces which are UKK and not even strictly convex.

3 The UC and WUC Properties

The UC property was defined in [18] in the following way.

Definition 3.1 Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the UC property if for $\{x_n\}$ and $\{x'_n\}$ sequences in A and $\{y_n\}$ a sequence in B such that $\lim_n d(x_n, y_n) = \lim_n d(x'_n, y_n) = \operatorname{dist}(A, B)$, then $\lim_n d(x_n, x'_n) = 0$.

The following proposition shows the uniform nature of the UC property.

Proposition 3.2 *For A and B nonempty subsets of a metric space X, the following are equivalent:*

(i) *(A,B) has the UC property;*
(ii) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{diam}(A \cap B(y, \operatorname{dist}(A,B)+\delta)) \leq \varepsilon$ for any $y \in B$.*

Proof First we see (i) $\Rightarrow$ (ii). Supposing the contrary implies that there is $\varepsilon_0 > 0$ such that for every $\delta = 1/n$ there exist $y_n \in B$ and $x_n, x_n' \in A$ satisfying

$$d(y_n, x_n) \leq \operatorname{dist}(A,B) + \tfrac{1}{n}, \quad d(y_n, x_n') \leq \operatorname{dist}(A,B) + \tfrac{1}{n} \quad \text{and} \quad d(x_n, x_n') > \varepsilon_0,$$

which obviously contradicts the UC property.

Now we prove (ii) $\Rightarrow$ (i). Let $x_n, x_n' \in A$ and $y_n \in B$ such that $d(y_n, x_n)$ and $d(y_n, x_n')$ both converge to $\operatorname{dist}(A,B)$ as $n \to \infty$. Then, given $\varepsilon > 0$, there is $\delta > 0$ such that $\operatorname{diam}(A \cap B(y, \operatorname{dist}(A,B)+\delta)) \leq \varepsilon$ for any $y \in B$. Now, it is enough to take $n_0 \in \mathbb{N}$ such that $d(x_n, y_n), d(x_n', y_n) \leq \operatorname{dist}(A,B) + \delta$ for any $n \geq n_0$ to deduce that $d(x_n, x_n') \leq \varepsilon$ for $n \geq n_0$. ■

In [18] it was shown that any pair of nonempty subsets (A,B) of uniformly convex Banach spaces with A convex enjoys the UC property. Next we show that something similar can be said for uniformly convex geodesic spaces under adequate conditions on the modulus of convexity.

Proposition 3.3 *Let (X,d) be a uniformly convex geodesic metric space with a monotone modulus of convexity $\delta(r,\varepsilon)$. Let A and B be two nonempty subsets of X with A convex. Then the pair (A,B) has the UC property.*

Proof Suppose on the contrary that there exist $\{x_n\}$ and $\{x_n'\}$ sequences in A, $\{y_n\}$ in B, and $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$, there exist $n_k \geq k$ for which $d(x_{n_k}, x_{n_k}') \geq \varepsilon_0$, while $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n) = \operatorname{dist}(A,B)$.

There is no loss of generality in assuming that $\delta(r,\varepsilon) < 1$ for $r, \varepsilon > 0$ and that $\operatorname{dist}(A,B) > 0$, since otherwise the result follows in a trivial way. For $\gamma > \operatorname{dist}(A,B)$ and $\varepsilon_1 = \varepsilon_0/\gamma$, choose $\varepsilon > 0$ such that

$$\varepsilon < \min\left\{\gamma - \operatorname{dist}(A,B), \frac{\operatorname{dist}(A,B)\delta(\gamma,\varepsilon_1)}{1-\delta(\gamma,\varepsilon_1)}\right\}.$$

Then there exists $N_0 \in \mathbb{N}$ such that if $n_k \geq N_0$, then $d(x_{n_k}, y_{n_k}) \leq \operatorname{dist}(A,B) + \varepsilon$ and $d(x_{n_k}', y_{n_k}) \leq \operatorname{dist}(A,B) + \varepsilon$. Let m_{n_k} be the mid-point of the geodesic segment $[x_{n_k}, x_{n_k}']$. Using the uniform convexity of X, we have that

$$\begin{aligned} d(y_{n_k}, m_{n_k}) &\leq \big(1 - \delta(\operatorname{dist}(A,B)+\varepsilon, \varepsilon_1)\big)(\operatorname{dist}(A,B)+\varepsilon) \\ &\leq \big(1 - \delta(\gamma, \varepsilon_1)\big)(\operatorname{dist}(A,B)+\varepsilon) < \operatorname{dist}(A,B). \end{aligned}$$

Then for $n_k \geq N_0$

$$d(y_{n_k}, m_{n_k}) < \operatorname{dist}(A,B),$$

which contradicts the fact that $m_{n_k} \in A$ by convexity of A. ■

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Remark 3.4 Notice that the same result remains true if the condition on the monotonicity of the modulus of convexity is replaced by the condition of being lower semi-continuous from the right.

As it was pointed in the introduction, the UC property was also shown in [18] to happen in UCED Banach spaces and strictly convex Banach spaces but requesting A is relatively compact. Regarding the assumption on the compactness of A, the following result from [3] is relevant.

Theorem 3.5 *Let A and B be nonempty closed subsets of a metric space (X, d), and let $T\colon A \cup B \to A \cup B$ be a cyclic contraction. If either A or B is boundedly compact, then there exists x in $A \cup B$ with $d(x, Tx) = \operatorname{dist}(A, B)$.*

Notice that (unlike Theorem 2.6) this theorem lacks uniqueness and convergence of iterates. We see next that this is easy to obtain by adding the very mild condition (see the remark below that supports this idea) of A being a Chebyshev set for proximinal points with respect to B.

Definition 3.6 Given A and B, two nonempty subsets of a metric space, we say that A is a Chebyshev set for proximinal points with respect to B if for any $x \in B$ such that $\operatorname{dist}(x, A) = \operatorname{dist}(A, B)$ we have that $P_A(x)$ is a singleton.

Then we can prove the following.

Theorem 3.7 *If in the above theorem, A is supposed to be boundedly compact and a Chebyshev set for proximinal points with respect to B, then the best proximity point $z \in A$ is unique, and the sequence $\{T^{2n}x\}$ converges to z for any $x \in A$.*

Proof We first show it is unique. Suppose z and z' are two best proximity points in A with $z \neq z'$. Then the Chebyshev condition on A implies that $Tz \neq Tz'$. Now, the relative nonexpansivity of T implies that

$$d(T^2 z, Tz) \leq d(z, Tz) = \operatorname{dist}(A, B),$$

and so, the Chebyshev condition on A also implies that z and z' are fixed points for T^2. If we write $d^*(x, y) = d(x, y) - \operatorname{dist}(A, B)$, then

$$d^*(z, Tz') = d^*(T^2 z, Tz') \leq k d^*(z', Tz) = k d^*(T^2 z', Tz) \leq k^2 d^*(z, Tz').$$

Hence, $d^*(z, Tz') = 0$ and so $z = z'$.

Finally the convergence of the iterates follows directly from the facts that A is boundedly compact, the sequences $\{T^{2n}x\}$ are bounded for any $x \in A$, and that $\lim d(T^{2n}x, Tz) = \operatorname{dist}(A, B)$ for any $x \in A$. ∎

Remark 3.8 Notice that the condition of being Chebyshev is a very natural one in this kind of problem. Think otherwise on the sets $A = \{(x, 0) : x \in [0, 1]\}$ and $B = \{(x, 1) : x \in [0, 1]\}$ as subsets of the plane with the maximum norm. Then any mapping $T\colon A \cup B \to A \cup B$ with $T(A) \subseteq B$ and $T(B) \subseteq A$ is a cyclic contraction.

We suggest replacing the UC property with the weaker one (WUC), which we define next.

Definition 3.9 Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the WUC property if for any $\{x_m\} \subseteq A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_m, y) \leq \operatorname{dist}(A, B) + \varepsilon$ for $m \geq m_0$, then it is the case that $\{x_m\}$ is convergent.

Remark 3.10 Another alternative for the above definition is to ask that the sequence $\{x_m\}$ be Cauchy instead of convergent. It is worthwhile to note here that this is a detail of a formal nature, since in all our main results we always assume A to be complete.

The next proposition gives the relation between the two mentioned properties.

Proposition 3.11 *Let A and B be nonempty subsets of a metric space (X, d) such that A is complete. Suppose the pair (A, B) has the UC property. Then (A, B) has the WUC property.*

Proof Let $\{x_m\} \subseteq A$ be such that for every $\delta > 0$ there exists $y \in B$ satisfying that $d(x_m, y) \leq \operatorname{dist}(A, B) + \delta$ for $m \geq m_0$. It suffices to show that $\{x_m\}$ is a Cauchy sequence. This follows directly from Proposition 3.2(ii). ∎

Next we show that the WUC property implies a nonuniform version of the equivalence given by Proposition 3.2 for the UC property. We omit its proof.

Proposition 3.12 *Let A and B be nonempty subsets of a metric space (X, d). Suppose (A, B) has the WUC property, then*

$$\lim_{\varepsilon \to 0} \operatorname{diam}(A \cap B(y, \operatorname{dist}(A, B) + \varepsilon)) = 0$$

for any $y \in B$.

The next propositions show that the WUC property is likely to happen in more situations than the UC property.

Proposition 3.13 *Let X be a UKK reflexive and strictly convex Banach space. Then, for $A, B \subseteq X$ nonempty and convex, it is the case that (A, B) has the WUC property.*

Proof Let $\{x_n\} \subseteq A$ be as in the above proof. Suppose $\{x_n\}$ is not convergent. First we show that this sequence needs to have a separated subsequence. Consider two convergent subsequences $\{x_{n_k}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$ with respective limits x and x' in the closure of A. For each $n \in N$ choose $y_n \in B$ such that the tales of both subsequences are in $B(y_n, \operatorname{dist}(A, B) + 1/n)$. Then it is clear that

$$x, x' \in \bigcap_{n \in \mathbb{N}} B(y_n, \operatorname{dist}(A, B) + 1/n).$$

Since $\{y_n\}$ is bounded, we can assume it is weakly convergent to a point y in the closure of B. Then it must be the case that $x, x' \in B(y, \operatorname{dist}(\mathrm{A}, \mathrm{B}))$ from where, since X is strictly convex, $x = x'$.

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Therefore we can assume that $\{x_n\}$ does not have any convergent subsequence, and so it is a separated sequence. Let $\varepsilon > 0$ such that $d(x_n, x_m) \geq \varepsilon$ for every $n \neq m$. Since this sequence is bounded and X is reflexive, we can also assume $\{x_n\}$ is weakly convergent to a point x. Now we only have to apply the UKK property in a similar way as the uniform convexity was applied in the previous proposition to deduce that x is in the closure of A but $\operatorname{dist}(x, B) < \operatorname{dist}(A, B)$, contradicting the definition of $\operatorname{dist}(A, B)$. ■

The UKK property for Δ-convergent sequences has been recently studied in [6,11] for CAT(k) spaces. If we assume that X is a geodesic space such that bounded sequences have a unique asymptotic center that belongs to the convex hull of the sequence (see [6, 11] for definitions), the previous proposition finds a metric counterpart that we state next and that can be proved in exactly the same way.

Proposition 3.14 *Let X be a geodesic metric space with the UKK property for Δ-convergent sequences and the above-mentioned property for bounded sequences. Suppose also that the function $\mu(r, \varepsilon)$ given by the UKK property decreases with respect to the radius, then, for $A, B \subseteq X$ nonempty and convex, it is the case that (A, B) has the WUC property.*

Remark 3.15 Although the situation for Banach spaces is clear in the sense that uniform convexity implies the UKK property, the same seems to be far from the case for geodesic spaces as defined for Δ-convergent sequences. Actually the UKK property for CAT(k) spaces as shown in [6, 11] seems to be more connected with the so-called Opial condition than with the uniform convexity. It is worth recalling at this point that only Hilbert spaces and the spaces of sequences ℓ_p are known to enjoy the Opial property. For more on this and related topics the interested reader can consult [13, Chapters 3, 4, 5 and 16] or [1, p. 102].

Next we show that WUC is enough to lead to a best proximity point for a cyclic contraction. For notation purposes, we will denote r as the contractive constant in the definition of cyclic contraction for the remainder of this section.

Theorem 3.16 *Let (X, d) be a metric space, and let A and B be two nonempty subsets of X such that (A, B) satisfies the WUC property. Assume that A is complete. Let T be a cyclic contraction on $A \cup B$. Then T has a unique best proximity point z in A, and the sequence $\{T^{2n}x\}$ converges to z for every $x \in A$.*

Proof As in [18] we consider $d^*(x, y) = d(x, y) - \operatorname{dist}(A, B)$. Then $d^*(Tx, Ty) \leq rd^*(x, y)$ for $x \in A$ and $y \in B$. In consequence, $d^*(T^2x, Tx) \leq rd^*(x, Tx)$ and $d^*(Ty, T^2y) \leq rd^*(Ty, y)$ for any $x \in A$ and $y \in B$.

Fix $x \in A$, $n \in N$, and let $m = n + k$ with $k \in \mathbb{N}$. Then

$$d^*(T^{2m}x, T^{2n+1}x) \leq r^{2n} d^*(T^{2k}x, Tx) \leq r^{2n} \sup\{d(Tx, T^{2k}x) : k \in \mathbb{N}\} = r^{2n} M(x).$$

Proposition 3.3 in [3] guarantees that $M(x)$ is finite for each x. Hence, given $\varepsilon > 0$ and taking n such that $r^{2n}M(x) < \varepsilon$, we have that

$$T^{2m}x \in B(T^{2n+1}x, \operatorname{dist}(A, B) + \varepsilon)$$

for $m \geq n$ and so, by the WUC property, $\{T^{2n}x\}$ is convergent. Now the proof follows the same patterns as the proof of [18, Theorem 3]. Let $z \in A$ be the limit of $\{T^{2n}x\}$, then

$$\begin{aligned} d^*(z, Tz) &= \lim_{n\to\infty} d^*(T^{2n}x, Tz) \leq \lim_{n\to\infty} rd^*(z, T^{2n-1}x) \\ &\leq \lim_{n\to\infty} r(d(z, T^{2n}x) + d^*(T^{2n}x, T^{2n-1}x)) \\ &\leq \lim_{n\to\infty} r(d(z, T^{2n}x) + r^{2n-2}d^*(T^2x, Tx)) = 0. \end{aligned}$$

Thus, z is a best proximity point. Now, we note that

$$d(T^2z, Tz) \leq d(z, Tz) = \operatorname{dist}(A, B),$$

and so Tz is a best proximity point in B. Moreover, it follows from the WUC property that $T^2z = z$, and so z is a fixed point for T^2.

Let z' be another proximity point in A, which will also be a fixed point for T^2. Then

$$\begin{aligned} d^*(z, Tz') &= d^*(T^2z, Tz') \leq rd^*(z', Tz) \\ &= rd^*(T^2z', Tz) \leq r^2d^*(z, Tz'). \end{aligned}$$

Hence, $d^*(z, Tz') = 0$ and so $d(z, z') = 0$. ■

Still one further weakening of the WUC property is possible to obtain a best proximity point result.

Definition 3.17 Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property W-WUC if for any $\{x_n\} \subseteq A$ such that for every $\varepsilon > 0$ there exists $y \in B$ satisfying that $d(x_n, y) \leq \operatorname{dist}(A, B) + \varepsilon$ for $n \geq n_0$, then there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

After this definition the following theorem is possible.

Theorem 3.18 *Under the same conditions as Theorem* 3.16 *with* (A, B) *satisfying the W-WUC property, assume that* A *is a Chebysev set with respect to* B. *It follows that* T *has a unique best proximity point* z *in* A, *and the sequence* $\{T^{2n}x\}$ *converges to* z *for every* $x \in A$.

Proof Following the same steps as in the previous proof we get that every subsequence of $\{T^{2n}x\}$ has a convergent subsequence. Consider therefore a convergent subsequence $\{T^{2n_k}x\}$ of $\{T^{2n}x\}$. Proceeding in a similar way, consider $z \in A$ as the limit of $\{T^{2n_k}x\}$. Then

$$\begin{aligned} d^*(z, Tz) &= \lim_{k\to\infty} d^*(T^{2n_k}x, Tz) \leq \lim_{k\to\infty} rd^*(z, T^{2n_k-1}x) \\ &\leq \lim_{k\to\infty} r(d(z, T^{2n_k}x) + d^*(T^{2n_k}x, T^{2n_k-1}x)) \\ &\leq \lim_{k\to\infty} r(d(z, T^{2n_k}x) + r^{2n_k-2}d^*(T^2x, Tx)) = 0. \end{aligned}$$

Thus, z is a best proximity point. In the same way as above, but using the Chebyshev condition instead of the WUC property, we obtain that z is a fixed point of T^2, and so uniqueness follows as in the previous proof. Also, since T is relatively nonexpansive, we have that $\{d(T^{2n}x, Tz)\}$ is a decreasing sequence with

$$\lim_{k\to\infty} d(T^{2n_k}x, Tz) = \operatorname{dist}(A, B).$$

Therefore the tales of $\{T^{2n}x\}$ are contained in $B(Tz, \operatorname{dist}(A, B) + \varepsilon)$ for $\varepsilon > 0$. Finally, the Chebyshev character of A and the above equality imply that any convergent subsequence of $\{T^{2n}x\}$ must converge to z, which complete the proof of the theorem. ∎

Remark 3.19 The condition of A being Chebysev with respect to B can be weakened to that one Chebyshev for proximinal points as introduced at the beginning of this section.

Remark 3.20 The contractive condition imposed in the main result of [18] is actually different than the one we have worked with in this paper. More precisely, in [18] it is supposed that there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(Ty, y)\} \; + \; (1-k) \; \operatorname{dist}(A, B)$$

for every $x \in A$ and $y \in B$. The proofs of our results in this section can be written, however, under this more general contractive condition without major modifications.

4 Reflexivity and Convergence

In this section we study the relation between cyclic contractions and the semimetric d_1 defined in [5] for certain Banach spaces. As a result we partially answer a question raised in [3] in the positive.

Definition 4.1 A pair (A, B) of subsets of a metric space is said to be *proximinal* if for each $(x, y) \in A \times B$ there exists $(x', y') \in A \times B$ such that

$$d(x, y') = d(x', y) = \operatorname{dist}(A, B).$$

If, additionally, we impose the condition that the pair of points $(x', y') \in A \times B$ is unique for each $(x, y) \in (A, B)$, then we say that the pair (A, B) is a *sharp proximinal pair.*

It was shown in [5] that when a pair of subsets (A, B) of a strictly convex Banach space is proximinal, then the sets A and B also satisfy the following definition.

Definition 4.2 Let A and B be nonempty subsets of a Banach space X. We say that A and B are *proximinal parallel sets* if the following two conditions are fulfilled:

(i) (A, B) is a sharp proximinal pair;

(ii) $B = A + h$ for a certain $h \in X$ such that $\|h\| = \text{dist}(A, B)$.

Remark 4.3 The notation introduced in this remark will be used throughout this section. Notice that, in the case of proximinal parallel sets, $a' = a + h$ for $a \in A$ and $b' = b - h$ for $b \in B$ are the proximinal points from B and A, respectively, to a and b. Given A and B proximinal parallel sets in a Banach space X, we will consider the set $C = A + 2h$, where $h \in X$ is such that $B = A + h$. It is immediate to see that A, B, and C are pairwise proximinal parallel sets.

From now on, we will say that a pair (A, B) satisfies a property if each of the sets A and B has that property. We will need the following technical result.

Lemma 4.4 *Let A and B be proximinal parallel subsets of a Banach space, and let T be a cyclic contraction map defined on $A \cup B$. Then $T(a + h) = Ta - h$ for any $a \in A$ and $T(b - h) = Tb + h$ for any $b \in B$.*

Proof Given $a \in A$, let $a' = a + h$ be its proximinal point in B. Since T is relatively nonexpansive,

$$\|Ta - Ta'\| \leq \|a - a'\| = \text{dist}(A, B).$$

Hence, by the uniqueness of the proximinal points, $T(a + h) = Ta - h$. ∎

Next we define the semimetric d_1 introduced in [5].

Definition 4.5 Let A, B, and C be as in Remark 4.3. We define the function $d_1 \colon B \times B \to [0, \infty)$ by

$$d_1(x, y) = \inf\{r > 0 : y \in B(x - h, d + r) \cap B(x + h, d + r)\},$$

where $d = \text{dist}(A, B)$.

The following proposition follows in an immediate way.

Proposition 4.6 (i) *d_1 defines a semimetric on B.*
(ii) *For every $x, y \in B$, $d_1(x, y) \leq \|x - y\|$.*

The next corollary immediately follows from Proposition 4.6(ii).

Corollary 4.7 *If $B(x, r)$ and $B_1(x, r)$ denote respectively the closed balls with respect to the norm and the semimetric d_1 in B, then $B(x, r) \subseteq B_1(x, r)$ for any $x \in B$ and $r \geq 0$.*

The next proposition gives sufficient conditions for the reverse contention to happen in a nice way.

Proposition 4.8 *Let X be a reflexive and strictly convex Banach space that has property (H). Suppose that A, B, and C are subsets of X as in Remark 4.3. If B is nonempty, closed, and convex, then there exists $f \colon (0, \infty) \to (0, \infty)$ such that $\lim_{r \to 0} f(r) = 0$ and $B_1(x, r) \subseteq B(x, f(r))$.*

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Proof The existence of $f(r) > 0$ such that $B_1(x, r) \subseteq B(x, f(r))$ is immediate because $B_1(x, r)$ is bounded in X. We need to prove that f can be chosen so that the limit condition holds. Since the balls B_1 are monotone with respect to the radius, it is enough to see that

$$\lim_{n\to\infty} f\left(\tfrac{1}{n}\right) = 0.$$

Consider the set $B_1(x, 1/n)$, and let $x_n, y_n \in B$ such that

$$d(x_n, y_n) = \operatorname{diam}(B_1(x, \tfrac{1}{n})),$$

where the diameter is taken with respect to the norm metric (we assume the diameter is reached for simplicity). Obviously, these points must belong to $\partial B_1(x, 1/n)$ where this border is with respect to the topology induced in B by the norm of X. Since $\{B_1(x, 1/n)\}_n$ is a decreasing sequence of sets, we have that x_n and y_n are in $B_1(x, 1)$ for any n. Now, since $B_1(x, 1)$ is bounded closed and convex in a reflexive space X, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $x_{n_k} \rightharpoonup z$ and $y_{n_k} \rightharpoonup w$, for some $z, w \in B_1(x, 1)$. Moreover, since we know that $x_n \in B_1(x, 1/n_0)$ for all $n \geq n_0$, we get that $z, w \in B_1(x, 1/n)$ for all $n \in \mathbb{N}$. Hence it must be the case $z = w = x$, and so the sequences $\{x_n\}$ and $\{y_n\}$ are weakly convergent themselves to x. Let $B^+ = B(x+h, d)$, where $d = \|h\| = \operatorname{dist}(A, B)$. Now consider the sequence $\{P_{B^+}(x_n)\}$ of the radial projections of $\{x_n\}$ onto B^+. Thus,

$$\|(x+h) - x_n\| = \|(x+h) - P_{B^+}(x_n)\| + \|P_{B^+}(x_n) - x_n\| = d + \|P_{B^+}(x_n) - x_n\|.$$

Since $x_n \in B_1(x, 1/n)$, it is the case that $\|(x+h) - x_n\| \leq d + 1/n$. Hence, we obtain that $P_{B^+}(x_n) \rightharpoonup x$. Now, property (H) allows us to assure that this last sequence is also convergent in norm, and, therefore, we also get that $x_n \to x$. We can proceed analogously to prove that $y_n \to x$. This implies that $d(x_n, y_n) = \operatorname{diam}(B_1(x, \frac{1}{n})) \to 0$, and so the theorem follows by setting $f(1/n) = \operatorname{diam}(B_1(x, 1/n))$. ■

Corollary 4.9 *Under the assumptions of the above proposition, the metric induced by the norm on B and the semimetric d_1 are compatible in the sense of Definition* 2.9.

Now we prove the main result of this section. The proof is inspired by those that first appeared in [4, 5].

Theorem 4.10 *Let A and B be nonempty, closed, and convex subsets of a reflexive and strictly convex Banach space X. Let $T\colon A\cup B \to A \cup B$ be a cyclic contraction map. Suppose that X has the property (H). Then there exists a unique point $b_0 \in B$ such that $d(b_0, Tb_0) = dist(A, B)$. Moreover, there exists $h \in X$ and $B_0 \subseteq B$ such that, if $T'(b) = Tb + h$ for $b \in B_0$, then*

(i) $T'\colon B_0 \to B_0$,
(ii) $b_0 = Tb_0 + h$, *and*
(iii) $(T')^n(b) \to b_0$ *for each* $b \in B_0$.

Proof Given the pair (A, B), let A_0 and B_0 be the subsets defined as follows:

$$A_0 = \{x \in A : d(x, y') = \operatorname{dist}(A, B) \text{ for some } y' \in B\},$$
$$B_0 = \{y \in B : d(x', y) = \operatorname{dist}(A, B) \text{ for some } x' \in A\}.$$

From the reflexivity of the space and the fact that (A, B) is a closed and convex pair, the pair (A_0, B_0) is nonempty, closed, and convex itself. It follows from their definitions that the pair (A_0, B_0) is proximinal with $\operatorname{dist}(A_0, B_0) = \operatorname{dist}(A, B)$.

Now we see that the mapping T is still a cyclic contraction on $A_0 \cup B_0$. Given $x_0 \in A_0 \subseteq A$, we have that there exists $y' \in B_0$ such that $d(x_0, y') = \operatorname{dist}(A, B)$. Since T is a cyclic contraction on $A \cup B$, $d(Tx_0, Ty') = \operatorname{dist}(A, B)$.

Since $Ty' \in A$, we get that $Tx_0 \in B_0$, and therefore $T(A_0) \subseteq B_0$. In the same way we prove that $T(B_0) \subseteq A_0$. Then $T\colon A_0 \cup B_0 \to A_0 \cup B_0$. Moreover, since $\operatorname{dist}(A_0, B_0) = \operatorname{dist}(A, B)$,

$$d(Tx, Ty) \le kd(x, y) + (1 - k) \operatorname{dist}(A_0, B_0),$$

for all $x \in A_0$, $y \in B_0$.

From [5, Lemma 3.1] we also have that A_0 and B_0 are proximinal parallel sets. Let, therefore, $h \in X$ such that $B_0 = A_0 + h$. This equality of sets directly implies (i). As was shown in Lemma 4.4, we have that for any $a \in A_0$ and $b \in B_0$,

$$T(a + h) = Ta - h \quad \text{and} \quad T(b - h) = Tb + h.$$

Now we prove that the mapping T' is a contraction on B_0 with respect to the semimetric d_1. Let $x, y \in B_0$. Denote $r = d_1(x, y)$ and $d = \operatorname{dist}(A_0, B_0)$. To see that T' is a contraction with constant k with respect to d_1 it will be enough to show that

$$T'y \in B(T'x - h, d + kr) \cap B(T'x + h, d + kr).$$

To see this, we proceed as follows:

$$\begin{aligned} T'y \in B(T'x - h, d + kr) &\Leftrightarrow \|T'y - (T'x - h)\| \le d + kr \\ &\Leftrightarrow \|Ty + h - (Tx + h - h)\| \le d + kr \\ &\Leftrightarrow \|Tx - (Ty + h)\| \le d + kr \\ &\Leftrightarrow \|Tx - T(y - h)\| \le d + kr. \end{aligned} \tag{4.1}$$

Since T is a cyclic contraction,

$$\|Tx - T(y - h)\| \le k\|x - (y - h)\| + (1 - k)d \le k(d + r) + (1 - k)d = d + kr,$$

and hence the chain of inequalities (4.1) holds. To show that $T'y \in B(T'x+h, d+kr)$ we need to introduce a new mapping. Let $\widehat{T}\colon B \cup C \to B \cup C$ the mapping defined by $\widehat{T}(b) = T(b) + 2h$ if $b \in B$ and $\widehat{T}(c) = T(c - 2h)$ when $c \in C$. It is immediate that $\widehat{T}(B) \subseteq C$ and $\widehat{T}(C) \subseteq B$. Moreover, since $T(b) + h = T(b - h)$ and $T(c - 2h) = T(c - h) + h$, we have

$$\begin{aligned} \|\widehat{T}(b) - \widehat{T}(c)\| &= \|T(b) + 2h - T(c - 2h)\| = \|T(b - h) - T(c - h)\| \\ &\le k\|b - c\| + (1 - k) \ \operatorname{dist}(A, B) = k\|b - c\| + (1 - k) \ \operatorname{dist}(B, C). \end{aligned}$$

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Thus, $\widehat{T}$ is a cyclic contraction. Now, repeating the same reasoning as above we finally obtain that T' is a d_1-contraction.

We conclude the proof of the theorem by applying Theorem 2.10 to the set B_0 with the mapping T'. The best proximity point b_0 is given by T', so it is immediate that (ii) and (iii) hold. The fact that b_0 is unique (as a best proximity point) even in B comes directly by following patterns similar to those in Theorem 3.16. ■

We state the convergence of the iterates for any $b \in B$ in the following corollary.

Corollary 4.11 *Let X, A, B, and T be as in the above theorem. Then, for every $b \in B$ the sequence of the iterates $\{T^{2n}(b)\}$ converges to b_0, the unique best proximity point of T in B.*

Proof Let b_0 be the unique best proximity point of T in B given by the above theorem and consider $b \in B$. We proceed as follows:

$$\operatorname{dist}(A,B) \leq \|T^{2n}(b) - T(b_0)\| = \|T^{2n}(b) - T^{2n-1}(b_0)\|$$

(by iterating the contractive condition)

$$\leq k^{2n-1}\|T(b) - b_0\| + (1-k)\ \operatorname{dist}(A,B)\left(\sum_{i=0}^{2n-2} k^i\right) \leq k^{2n-1}\|T(b) - b_0\| + \operatorname{dist}(A,B),$$

therefore

$$\|T^{2n}(b) - T(b_0)\| \to \operatorname{dist}(A,B) \quad \text{as } n \to \infty. \tag{4.2}$$

On the other hand, since B is closed and convex in a reflexive space X and $\{T^{2n}(b)\}$ is bounded, there exists a subsequence $\{T^{2n_i}(b)\}$ weakly converging to a point $x \in B$. Moreover, since

$$\|x - T(b_0)\| \leq \liminf \|T^{2n_i}(b) - T(b_0)\| = \operatorname{dist}(A,B),$$

we get that $x = b_0$. Therefore, since $\{T^{2n_i}b\}$ is any weakly convergent subsequence, we obtain that

$$T^{2n}(b) \rightharpoonup b_0. \tag{4.3}$$

Consider the closed ball $\widehat{B} = B(T(b_0), \operatorname{dist}(A,B))$, and the radial projection of $T^{2n}(b)$ onto $\widehat{B}$, which we denote by $P_{\widehat{B}}(T^{2n}(b))$. Then

$$\|T^{2n}(b) - T(b_0)\| = \|T(b_0) - P_{\widehat{B}}(T^{2n}(b))\| + \|P_{\widehat{B}}(T^{2n}(b)) - T^{2n}(b)\|,$$

together with (4.2), gives that

$$\|P_{\widehat{B}}(T^{2n}(b)) - T^{2n}(b)\| \to 0.$$

Then, by (4.3), we obtain that $P_{\widehat{B}}(T^{2n}(b)) \rightharpoonup b_0$. Now it is enough to apply property (H), following the same reasoning as was used in Proposition 4.8, to conclude that $T^{2n}(b)$ converges to b_0. ■

Remark 4.12 If $b \in B_0$ in the above proof, the convergence is in fact immediate because $T^{2n}(b) = (T')^{2n}(b)$.

Remark 4.13 In [3] Eldred and Veeramani raised the question of whether a best proximity point exists when A and B are nonempty closed and convex subsets of a reflexive Banach space. The conjunction of Theorem 4.10 and Corollary 4.11 partially answers this question, as they also request strict convexity and property (H). Remember, however, that the strict convexity assumption is a natural one in this kind of problems (see Remark 3.8).

A Appendix: The CAT(0) Case.

Among the most important examples of uniformly convex uniquely geodesic metric spaces are the so-called CAT(0) spaces. A CAT(0) space is a space of nonpositive curvature in the sense of Gromov, and they are characterized by having thinner triangles than the comparison ones in the 2-dimensional Euclidean space. For a proper definition and further properties, the reader can consult [2, Chapter II.1]. CAT(0) spaces can be viewed as a metric analog to the Hilbert spaces in the classical theory of nonlinear analysis. Properties studied in [2, Chapter II.2] support this idea very strongly. In [4] it was shown that relatively nonexpansive mappings with $T(A) \subseteq A$ and $T(B) \subseteq B$ are actually nonexpansive mappings with respect to the norm (*i.e.*, $\|Tx - Ty\| \leq \|x - y\|$) when the Banach space in their theorems happens to be a Hilbert space. It was also shown in [5] that, under similar assumptions to those of Theorem 4.10, the metric induced by the norm coincides with the d_1-semimetric on the set B_0. The purpose of this appendix is to show that the same stands for CAT(0) spaces.

Let, therefore, X, A, B, and T be as in Theorem 4.10 with no further assumption on X except for the fact of being a complete CAT(0) space (also called a *Hadamard space*). Suppose $\operatorname{dist}(A, B) > 0$, otherwise there is nothing to prove, and define the sets A_0 and B_0 as before by

$$A_0 = \{x \in A : d(x, y') = \operatorname{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B : d(x', y) = \operatorname{dist}(A, B) \text{ for some } x' \in A\}.$$

The properties of the pair (A_0, B_0) are summarized next.

Proposition A.1 *The pair (A_0, B_0) is a nonempty, closed, and convex pair in X. Furthermore, each point $b \in B_0$ can be joined through a geodesic segment of length $d = \operatorname{dist}(A, B)$ to its proximinal point $b - h$ in A_0 and vice versa.*

Proof That they are closed follows in a straightforward way from their definition and the fact that A and B are both closed. The fact that A_0 and B_0 are nonempty also follows in a similar way to the linear case under the assumption of reflexivity, since it is a very well-known fact (see [6, 11]) that decreasing sequences of nonempty, bounded, closed, and convex subsets of a CAT(0) space have nonempty intersection. Finally, the convexity of the sets A_0 and B_0 follows from the convexity of the metric of CAT(0) spaces (see [2, Proposition 2.2, Chapter II.2]). ■

The next thing we need to do is to define the semimetric d_1 on B_0. We will define it in such a way that a third set C_0 is not needed.

Definition A.2 We define the function $d_1 \colon B_0 \times B_0 \to [0, \infty)$ by

$$d_1(x, y) = \inf\{r > 0 : y \in B(x - h, d + r) \text{ and } y - h \in B(x, d + r)\},$$

where $d = \operatorname{dist}(A, B)$.

Remark A.3 Notice that Definition 4.5 and Definition A.2 coincide in linear spaces.

Theorem A.4 *The semimetric d_1 coincides with the metric d induced by X on B_0.*

Proof This result follows as an easy application of The Flat Quadrilateral Theorem ([2, p.181]). Indeed, consider the four point $x, y, x - h$, and $y - h$. Then x (respectively, y) is the proximinal point of $x - h$ (resp., $y - h$) in B_0, and vice versa. In consequence, the angles $\angle_x(x - h, y)$, $\angle_y(x, y - h)$, $\angle_{y-h}(x - h, y)$, and $\angle_{x-h}(y - h, x)$ are all greater than or equal to $\pi/2$. Therefore, the Flat Quadrilateral Theorem implies that the convex hull of the points $x, y, x-h$ and $y-h$ is isometric to a rectangle in the 2-dimensional Euclidean space. Now, by the Pythagorean theorem, it is immediate to deduce that

$$B_1(x, \sqrt{d^2 + r^2} - d) = B_0 \cap B(x - h, \sqrt{d^2 + r^2}) = B_0 \cap B(x, r),$$

as we wanted to prove. ∎

We close this appendix by observing that it is also possible to show that the mapping $T'(b) = Tb + h$ for $b \in B_0$ is actually a contraction. To see this we just need to proceed as in the proof of Theorem 4.10 and recall, at the proper moment, that the convex hull of the points $x, y, x - h$, and $y - h$ is actually a rectangle.

Added in proof. After this paper was submitted quite a large number of articles by several authors on cyclic contractions, best proximity points, and related topics have appeared in the literature. Among them we would like to mention [17] where an example of a pair of sets verifying the WUC property but lacking the UC one is given. Also, reference [7], by the second author, can be regarded as a continuation of this work.

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