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ON EXTENSIONS OF THE GENERALISED JENSEN FUNCTIONS ON SEMIGROUPS

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Abstract

Assume that $(G, +)$ is a commutative semigroup, τ is an endomorphism of G and an involution, D is a nonempty subset of G and $(H, +)$ is an abelian group uniquely divisible by two. We prove that if D is 'sufficiently large', then each function $g : D \to H$ satisfying $g(x + y) + g(x + \tau(y)) = 2g(x)$ for $x, y \in D$ with $x + y, x + \tau(y) \in D$ can be extended to a unique solution $f : G \to H$ of the generalised Jensen functional equation $f(x + y) + f(x + \tau(y)) = 2f(x)$.

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1. Introduction

The well-known and important Jensen functional equation (see, for example, [13])

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2}$$

can be written in the equivalent form

$$f(x + y) + f(x - y) = 2f(x) \tag{1.1}$$

(for example, for real functions f). It is enough to replace x and y by $x + y$ and $x - y$, respectively. So the functional equation

$$f_0(x + y) + f_0(x + \tau(y)) = 2f_0(x), \tag{1.2}$$

with a given function τ, is a natural generalisation of (1.1) which is (1.2) with $\tau(y) = -y$.

The equation (1.2) is considered in [14, 15]. In this form, with a suitable function τ (compare with [14] and [16, page 207]), it can be considered on semigroups. We examine the possibility of extensions of solutions of (1.2) on some 'large enough' subsets D of a semigroup to a solution of the equation on the whole semigroup. Our results have been motivated by somewhat analogous situations in [4, 9–12] (see also

[13] and [5]) and a program declared by a group of mathematicians of the University of Debrecen (Hungary) concerning the possibility of extensions of solutions of some functional equations from restricted domains (the program has been motivated by a problem of Aczél [1] and further discussions with him).

Let $(H,+)$ be an abelian group uniquely divisible by two, let $(G,+)$ be a commutative semigroup and let D be a nonempty subset of G. Let τ be an endomorphism of G and an involution, that is, $\tau(x+y)=\tau x+\tau y$ and $\tau(\tau x)=x$ for $x,y\in G$, where $\tau x:=\tau(x)$ for $x\in G$ (see [3]). We investigate the possibility of extensions of functions $f:D\to H$, satisfying the conditional equation

$$f(x+y)+f(x+\tau y)=2f(x),\quad x,y\in D, x+y, x+\tau y\in D, \tag{1.3}$$

to the solutions $f_0:G\to H$ of (1.2).

Note that if G is a group and $\tau x\equiv -x$, then we can replace x by $x+y$ in (1.2) and obtain the equation

$$f_0(x+2y)+f_0(x)=2f_0(x+y),$$

which can be written as

$$\Delta_y^2 f_0(x):=\Delta_{y,y}f_0(x)=0, \tag{1.4}$$

where Δ stands for the Fréchet difference operator given by

$$\Delta_y f_0(x)=\Delta_y^1 f_0(x):=f_0(x+y)-f_0(x),\quad x,y\in G.$$

Recurrently, we define

$$\Delta_{x_{n+1},x_n,\ldots,x_1}:=\Delta_{x_{n+1}}\circ\Delta_{x_n,\ldots,x_1},\quad x_1,\ldots,x_{n+1}\in G, n\in\mathbb{N}.$$

Therefore, somewhat analogous properties for (1.1) for real functions can be deduced from the quite involved results in [9, 13] proved for general polynomial equations, of which (1.4) is a particular case. Related results for the analogously generalised d'Alembert equation have been obtained in [2].

2. Preliminaries

Write

$$T+a:=\{x+a : x\in T\}\quad\text{and}\quad T-a:=\{x\in G : x+a\in T\},$$

for $a\in G$ and $T\subset G$. Clearly, if G is a group, then

$$T-a=\{x-a : x\in T\}.$$

Throughout the paper, we assume that $D\in\mathcal{L}$, where $\mathcal{L}$ is a family of subsets of G, satisfying:

(i) $\mathcal{L}\neq 2^G$;
(ii) $B\in\mathcal{L}$ for $B\in 2^G$ with $2^B\cap\mathcal{L}\neq\emptyset$;
(iii) $A\cap B\in\mathcal{L}$ for $A,B\in\mathcal{L}$;
(iv) $B-x, B+x\in\mathcal{L}$ for $x\in G$ and $B\in\mathcal{L}$;
(v) $\tau(B)\in\mathcal{L}$ for $B\in\mathcal{L}$.

Clearly, Conditions (i) and (ii) imply that $\emptyset\notin\mathcal{L}$.

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REMARK 2.1. Conditions (ii) and (iii) mean that $\mathcal{L}$ is a filter. It is well known that if $\mathcal{I} \subset 2^G$ is an ideal (that is, if $2^B \subset \mathcal{I}$ and $B \cup C \in \mathcal{I}$ for every $B, C \in \mathcal{I}$), then

$$\mathcal{L} := \{A \subset G : G \setminus A \in \mathcal{I}\}$$

is a filter. Moreover, if $\mathcal{I}$ has some suitable additional properties, then also conditions (i), (iv) and (v) are valid. Below we provide several examples of ideals $\mathcal{I} \subset 2^G$ having suitable properties (compare with [2, Remark 1]).

(a) G is cancellative and not of finite cardinality and $\mathcal{I} = \{A \subset G : \operatorname{card} A < \operatorname{card} G\}$.
(b) d is an invariant metric in G (that is, $d(x + y, z + y) = d(x, z)$ for $x, y, z \in G$), $\sup_{x,y\in G} d(x, y) = \infty$, the set $\tau(B)$ is bounded (that is, $\sup_{x,y\in\tau(B)} d(x, y) < \infty$) for each bounded set $B \in 2^G$ and $\mathcal{I}$ is the family of all bounded subsets of G.
(c) $G = \{z \in \mathbb{C} : \Re z > 0\}$ (with the usual addition of complex numbers), $\mathcal{I}$ is the family of all subsets A of G with $\sup_{z\in A} \Re z < \infty$ and $\tau z = \overline{z}$ for $z \in G$, where $\overline{z}$ is the complex conjugate and $\Re z$ is the real part of the complex number z.
(d) G is a topological group of the second Baire category, $\mathcal{I}$ is the family of all first category subsets of G and τ is continuous (which actually means that τ is a homeomorphism because $\tau^{-1} = \tau$).
(e) G is a locally compact topological group, μ is the Haar measure in G with $\mu(G) = \infty$, $\mathcal{I} = \{A \subset G : \mu(A) < \infty\}$ and τ is continuous.
(f) G is a Polish group, $\mathcal{I}$ is the σ-ideal of Haar zero subsets of G (see [6]) and τ is continuous.
(g) G is a Polish group, $\mathcal{I}$ is the σ-ideal of Christensen zero subsets of G (see [8]) and τ is continuous.
(h) G is an abelian Polish group, $\mathcal{I}$ is the σ-ideal of all Haar meagre subsets of G (see [7]) and τ is continuous.

Let us now recall some useful results from [2, 14].

THEOREM 2.2 [14, Theorem 2]. *A function $h : G \to H$ satisfies (1.2) on G if and only if there exist an additive function $A : G \to H$ and a constant $a \in H$ such that*

$$A(\tau x) = -A(x), \quad x \in G,$$
$$h(x) = A(x) + a, \quad x \in G.$$

LEMMA 2.3 [2, Lemma 1]. *Assume that $S \in \mathcal{L}$. Then*

$$\{z \in G : (S + z) \cap S \neq \emptyset\} = G. \tag{2.1}$$

LEMMA 2.4 [2, Lemma 2]. *Assume that $S \in \mathcal{L}$ and that $h_0 : S \to H$ satisfies*

$$h_0(x + y) = h_0(x) + h_0(y), \quad x, y \in S, x + y, x + \tau y \in S.$$

Then there exists a unique additive function $h : G \to H$ such that $h(x) = h_0(x)$ for $x \in S$.

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3. The main result

First, we prove a lemma.

LEMMA 3.1. *Let $D \in \mathcal{L}$ and $g : D \to H$ satisfy (1.3). Then there exist $\widehat{D} \in \mathcal{L}$ and $c \in H$ with $\widehat{D} \subset D$, $\tau(\widehat{D}) = \widehat{D}$ and*

$$g(\tau x) = -g(x) + c, \quad x \in \widehat{D}. \tag{3.1}$$

PROOF. Let

$$D_0 := D \cap \tau(D) \in \mathcal{L}.$$

Fix $u \in D_0$ and write

$$D_u := (D - u) \cap D \cap (D - \tau u) \in \mathcal{L}, \quad \widehat{D} := D_u \cap \tau(D_u).$$

Clearly, $\tau(\widehat{D}) = \widehat{D}$ and $\widehat{D} \in \mathcal{L}$.

Now take an arbitrary $w \in \widehat{D}$. Then

$$\tau u, \tau w, w + u, w + \tau u, u + \tau w, \tau u + \tau w \in D,$$

and hence

$$2g(u) = g(u + w) + g(u + \tau w),$$
$$2g(\tau u) = g(\tau u + w) + g(\tau u + \tau w).$$

Consequently,

$$\begin{aligned} 2g(u) + 2g(\tau u) &= g(u + w) + g(\tau u + w) + g(u + \tau w) + g(\tau u + \tau w) \\ &= 2g(w) + 2g(\tau w). \end{aligned}$$

Thus we have proved that there is $c_0 \in H$ with

$$2g(w) + 2g(\tau w) = c_0, \quad w \in \widehat{D},$$

and, setting $c := c_0/2$, we obtain (3.1). □

PROPOSITION 3.2. *Let $D \in \mathcal{L}$ and $g : D \to H$ satisfy (1.3). Then there exist $\widehat{D} \in \mathcal{L}$, a unique additive function $\mathbf{A} : G \to H$ and a unique constant $a \in H$ such that $\widehat{D} \subset D$, $\tau(\widehat{D}) = \widehat{D}$,*

$$\mathbf{A}(\tau x) = -\mathbf{A}(x), \quad x \in G, \tag{3.2}$$

$$g(x) = \mathbf{A}(x) + a, \quad x \in \widehat{D}. \tag{3.3}$$

PROOF. According to Lemma 3.1, there are $\widehat{D} \in \mathcal{L}$ and $c \in H$ such that (3.1) holds, $\widehat{D} \subset D$ and $\tau(\widehat{D}) = \widehat{D}$. Let

$$A(x) := \tfrac{1}{2}(g(x) - g(\tau x)), \quad x \in \widehat{D}.$$

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Then

$$A(\tau x) = \tfrac{1}{2}(g(\tau x) - g(\tau(\tau x))) = \tfrac{1}{2}(-g(x) + g(\tau x)) = -A(x), \quad x \in \widehat{D}. \tag{3.4}$$

Hence, by (3.1),

$$g(x) = \frac{1}{2}(g(x) - g(\tau x)) + \frac{1}{2}(g(x) + g(\tau x)) = A(x) + \frac{c}{2}, \quad x \in \widehat{D}. \tag{3.5}$$

Now, take $x, y \in \widehat{D}$ such that $x + y, x + \tau y \in \widehat{D}$. Then $y + \tau x \in \widehat{D}$ and, by (1.3) and (3.1),

$$\begin{aligned} 2A(x + y) &= 2g(x + y) - c \\ &= [g(x + y) + g(x + \tau y)] - g(x + \tau y) + [g(y + x) + g(y + \tau x)] - g(y + \tau x) - c \\ &= 2g(x) + 2g(y) - g(x + \tau y) - [-g(x + \tau y) + c] - c \\ &= 2g(x) - c + 2g(y) - c = 2A(x) + 2A(y). \end{aligned}$$

Thus, according to Lemma 2.4, there is a unique additive function $\mathbf{A} : G \to H$ such that $\mathbf{A}|_{\widehat{D}} = A$.

Take $z \in G$. In view of (2.1), there are $x, y \in \widehat{D}$ such that $z + x = y$; clearly, $\tau z + \tau x = \tau y$ and $\tau x, \tau y \in \tau(\widehat{D}) = \widehat{D}$. So, by the additivity of $\mathbf{A}$,

$$\mathbf{A}(z) + A(x) = \mathbf{A}(z) + \mathbf{A}(x) = \mathbf{A}(y) = A(y),$$
$$\mathbf{A}(\tau z) + A(\tau x) = \mathbf{A}(\tau z) + \mathbf{A}(\tau x) = \mathbf{A}(\tau y) = A(\tau y),$$

and hence, by (3.4),

$$\mathbf{A}(\tau z) = A(\tau y) - A(\tau x) = -A(y) + A(x) = -\mathbf{A}(z).$$

Moreover, in view of (3.5), (3.3) holds with $a := c/2$.

It remains to show the uniqueness of $\mathbf{A}$ and c. So let $\mathbf{A}_1 : G \to H$ be an additive function and let $c_1 \in H$ be such that $\mathbf{A}_1|_{\widehat{D}} + c_1 = g = \mathbf{A}|_{\widehat{D}} + c$. Let $x \in G$. Then, using (2.1), there exist $s, t \in \widehat{D}$ such that $x + t = s$ and hence

$$\begin{aligned} \mathbf{A}_1(x) &= \mathbf{A}_1(x) + \mathbf{A}_1(t) - \mathbf{A}_1(t) = \mathbf{A}_1(x + t) - \mathbf{A}_1(t) \\ &= \mathbf{A}_1(s) - \mathbf{A}_1(t) = \mathbf{A}(s) + c - c_1 - (\mathbf{A}(t) + c - c_1) \\ &= \mathbf{A}(x + t) - \mathbf{A}(t) = \mathbf{A}(x) + \mathbf{A}(t) - \mathbf{A}(t) = \mathbf{A}(x). \end{aligned}$$

Consequently, $c = c_1$, which completes the proof. □

We also need the following lemma.

Lemma 3.3. *Let $f, g : D \to H$ satisfy (1.3) and $S := \{x \in D : g(x) = f(x)\} \in \mathcal{L}$. Then $g(x) = f(x)$ for $x \in D$.*

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PROOF. Take an arbitrary $w \in D$ and choose $s \in (S - w) \cap \tau^{-1}(S - w) \in \mathcal{L}$. Then $w + s, w + \tau s \in S$ and hence

$$2g(w) = g(w + s) + g(w + \tau s) = f(w + s) + f(w + \tau s) = 2f(w).$$

Consequently, $g(w) = f(w)$. □

Now we are ready to prove the main result of this paper.

THEOREM 3.4. *Let $D \in \mathcal{L}$ and $g : D \to H$ satisfy (1.3). Then there is a unique solution $f : G \to H$ of (1.2) such that $g(x) = f(x)$ for $x \in D$.*

PROOF. On account of Proposition 3.2, there exist $\widehat{D} \in \mathcal{L}$, a unique additive function $\mathbf{A} : G \to H$ and a constant $a \in H$ such that $\widehat{D} \subset D$, $\tau(\widehat{D}) = \widehat{D}$ and conditions (3.2)–(3.3) are valid. Let

$$f(x) := \mathbf{A}(x) + a, \quad x \in G.$$

Clearly, f satisfies (1.2) for every $x, y \in G$ (see Theorem 2.2). Moreover, by (3.3),

$$\widehat{D} \subset \{x \in D : g(x) = f(x)\} \in \mathcal{L}.$$

So, according to Lemma 3.3, $g = f|_D$.

Finally, we show the uniqueness of f. To this end, suppose that $f_0 : G \to H$ is a solution of (1.2) with $g = f_0|_D$. Clearly,

$$D \subset \{x \in G : f_0(x) = f(x)\} \in \mathcal{L}.$$

Hence, again by Lemma 3.3 (with $D := G$ and $g := f_0$), $f_0 = f$. This completes the proof. □

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