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A NOTE ON THE $\sum(S)$-INJECTIVITY OF $R(S)$

BY
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1. Let R be a ring with 1. All modules considered are to be unital left R-modules unless otherwise noted.

DEFINITION. A *σ-set* for R is a nonempty set Σ of left ideals of R satisfying the following conditions:

(σ_1): If $I \in \Sigma$, J is a left ideal of I, and $J \supseteq I$, then $J \in \Sigma$.
(σ_2): If $I \in \Sigma$ and $r \in R$, then $Ir^{-1} = \{s \in R \mid sr \in I\} \in \Sigma$.
(σ_3): If I is a left ideal of R, $J \in \Sigma$, and $It^{-1} \in \Sigma$ for each $t \in J$, then $I \in \Sigma$.

Sanderson [4] defined an R-module M to be Σ-*injective* iff each $f \in \mathrm{Hom}_R(I, M)$ can be extended to an $\bar{f} \in \mathrm{Hom}_R(R, M)$ whenever $I \in \Sigma$.

A submodule N of a module M is Σ-*essential* in M (${}_RM$ is a Σ-*essential extension* of ${}_RN$) if for each $0 \neq x \in M$,

$$Nx^{-1} = \{r \in R \mid rx \in N\} \in \Sigma \quad \text{and} \quad (Nx^{-1})x \neq 0.$$

In [3] it was shown that ${}_RM$ is Σ-injective iff given ${}_RA$ and a Σ-essential extension ${}_RB$ of ${}_RA$, each $f \in \mathrm{Hom}_R(A, M)$ has an extension $\bar{f} \in \mathrm{Hom}_R(B, M)$.

Let S be a semigroup. If S has a two-sided zero, denote it by z; otherwise adjoin a two-sided zero z to S. Let M be an R-module and define

$$M(S) = \{f : S \to M \mid f(z) = 0 \quad \text{and} \quad f(s) = 0 \text{ for all but a finite number of } s \in S\}.$$

$M(S)$ is an abelian group under pointwise addition. A scalar multiplication $R(S) \times M(S) \to M(S)$ is defined for $r \in R(S)$ and $m \in M(S)$ by $(rm)(s) = \sum_{th=s} r(t)m(h)$ if $s \neq z$, and $(rm)(z) = 0$. If $M = R$, $R(S)$ with the above defined multiplication and addition is a ring called the (contracted) semigroup ring of S. When M is an R-module, $M(S)$ is an $R(S)$-module under the above defined operations. If S has an identity 1, M can be embedded in $M(S)$ by mapping $m \mapsto m'$ where $m'(s) = 0$ if $s \neq 1$, and $m'(1) = m$. Where $1 \in S$, we identify M with its image under the map $m \mapsto m'$. An element $m \in M(S)$ is often denoted by $m = \sum m(s)s$ ($s \neq z$). Scalar multiplication may then be written as $(\sum r(s)s)(\sum m(s)s) = \sum(\sum_{th=s} r(t)m(h))s$. Define

$$\Sigma(S) = \{T \mid T \text{ is a left ideal of } R(S) \text{ and } T \supset J(S) \text{ for some } J \in \Sigma\}.$$

It will be shown that $\Sigma(S)$ is a σ-set for $R(S)$ if S is a monoid (semigroup with 1)

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or if S is finite and regular. If S is a monoid and $R(S)$ is $\Sigma(S)$-injective, then R is Σ-injective. If S is a finite group and R is Σ-injective, then $R(S)$ is $\Sigma(S)$-injective. This generalizes Theorem 4.1 of Connell [2]. Finally, if S is a finite inverse semigroup, and $Z_{\Sigma(S)}(R(S))=0$, then $Q_{\Sigma(S)}(R(S))\approx Q_{\Sigma}(R)(S)$ (ring iso.).

In the course of proving several of our results, we require the following facts from [3]:

(1) Let R be a ring and Σ be a σ-set for R, then Σ is closed under finite intersections.

(2) Let R be a ring and Σ a nonempty collection of left ideals of R satisfying (σ_1) and (σ_2), then Σ satisfies property (σ_3) if and only if Σ satisfies property (σ_3'):

(σ_3'): If for some $J\in\Sigma$ there is associated to each $b\in J$ a $K_b\in\Sigma$, then $\sum K_b\, b\in\Sigma$.

2. Lemma. *Let S be a semigroup, R be a ring with* 1, *and Σ be a σ-set for R. If S is a monoid or a finite regular semigroup, then $\Sigma(S)$ is a σ-set for $R(S)$.*

Proof. (σ_1) is clearly satisfied.

(σ_2): Let $\sum r(s)s\in R(S)$ and $J(S)\in\Sigma(S)$ where $J\in\Sigma$. We must show that $J(S)(\sum r(s)s)^{-1}\in\Sigma(S)$. If

$$T=\bigcap\{Jr(s)^{-1}\mid r(s)\neq 0\},$$

then $T\in\Sigma$ since $R(S)$ consists of elements of finite support and Σ is closed under finite intersections. Thus $T(S)\in\Sigma(S)$ and $T(S)(\sum r(s)s)\subseteq J(S)$, so $J(S)(\sum r(s)s)^{-1}\in\Sigma(S)$.

(σ_3): Let K be a left ideal of $R(S)$ and suppose $K\alpha^{-1}\in\Sigma(S)$ for all $\alpha\in J(S)$ where $J\in\Sigma$.

If S is a monoid, then $J\subseteq J(S)$; thus $Ka^{-1}\in\Sigma(S)$ for all $a\in J$. Hence, for each $a\in J$, there is $I_a\in\Sigma$ with $I_a(S)a\subseteq K$. By (σ_3'), $\sum_{a\in J} I_a a\in\Sigma$ and $(\sum I_a a)(S)\subseteq K$. Thus $K\in\Sigma(S)$.

If S is regular, fix $s\in S$. For each $a\in J$, $as\in J(S)$ so there is $I_a\in\Sigma$ with

$$I_a(S)as=I_a a(S)s\subseteq K.$$

Hence

$$\textstyle\sum_{a\in J} I_a a\in\Sigma \quad\text{and}\quad (\sum I_a a)(S)s\subseteq K.$$

Since $\sum I_a a$ depends on s, let $\sum I_a a=K_s\in\Sigma$. Since we can find, for each $s\in S$, a $K_s\in\Sigma$ with $K_s(S)s\subseteq K$,

$$\textstyle\sum_{s\in S} K_s(S)s\subseteq K.$$

Since S is finite,

$$T=\textstyle\bigcap_{s\in S} K_s\in\Sigma.$$

Since S is regular, for each $s\in S$, there is an $a\in S$ for which $sas=s$, then

$$Ts\subseteq K_s s=K_s(sa)s\subseteq K_s(S)s\subseteq K$$

so

$$T(S)=\textstyle\sum_{s\in S} Ts\subseteq K \quad\text{and}\quad K\in\Sigma(S).$$

3. THEOREM. *Let S be a monoid, R be a ring with 1, Σ be a σ-set for R, and M be an R-module. Then M is Σ-injective if $M(S)$ is $\Sigma(S)$-injective.*

Proof. Let $J \in \Sigma$ and $f \in \mathrm{Hom}_R(J, M)$, then $\tilde{f}: J(S) \to M(S)$ defined by

$$\tilde{f}(\sum r(s)s) = \sum f(r(s))s$$

is an $R(S)$-homomorphism. Since $M(S)$ is $\Sigma(S)$-injective, there is a $t = \sum t(s)s \in M(S)$ with $\tilde{f}(\sum r(s)s) = (\sum r(s)s)(\sum t(s)s)$ for all $\sum r(s)s \in J(S)$. Thus for $r \in J$,

$$f(r)\cdot 1 = \tilde{f}(r\cdot 1) = (r\cdot 1)(\sum t(s)s) = \sum rt(s)s \in M\cdot 1,$$

and $rt(s)=0$ for $s \neq 1$. Since $t(1) \in M$ and $f(r)=rt(1)$ for $r \in J$, M is Σ-injective.

4. PROPOSITION. *Let M be a Σ-injective R-module and G be a finite group. Then $M(G)$ is a $\Sigma(G)$-injective $R(G)$-module.*

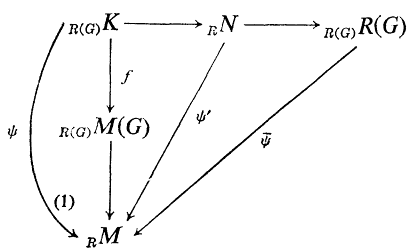
Proof. Let K be a $\Sigma(G)$-essential left ideal of $R(G)$ and $f \in \mathrm{Hom}_{R(G)}(K, M(G))$. Define $\psi: K \to M$ by $\psi(k)=f(k)(1)$, then $\psi \in \mathrm{Hom}_R(K, M)$. By Zorn's lemma, there is an R-module N, $K \subseteq N \subseteq R(G)$, and a $\psi' \in \mathrm{Hom}_R(N, M)$ extending ψ and maximal with respect to the extension of ψ. N is a Σ-essential submodule of $R(G)$ for if $0 \neq \sum r(g)g \in R(G)$, Then

$$N(\sum r(g)g)^{-1} \cap R \supseteq K(\sum r(g)g)^{-1} \cap R \supseteq J(G) \cap R \supseteq J$$

for some $J \in \Sigma$ since K is a $\Sigma(G)$-essential left ideal of $R(G)$. Moreover,

$$(N(\sum r(g)g)^{-1} \cap R)(\sum r(g)g) \neq 0$$

by the maximality of N. Hence we have the commutative diagram.



where the R-homomorphism $\bar{\psi}: R(G) \to M$ exists since M is Σ-injective and N is a Σ-essential submodule of $R(G)$. Define $\eta: R(G) \to M(G)$ by $\eta(r)(g) = \bar{\psi}(g^{-1}r)$ for all $r \in R(G)$, $g \in G$. Then η is an $R(G)$-homomorphism and for $k \in K$,

$$\eta(k)(g) = \bar{\psi}(g^{-1}k) = f(g^{-1}k)(1) = (g^{-1}f(k))(1) = f(k)(g)$$

for all $g \in G$. Thus $\eta(k)=f(k)$ for all $k \in K$ and so $M(G)$ is $\Sigma(G)$-injective.

REMARKS. The proof of this proposition depends on the fact that when G is a finite group, each $f \in \mathrm{Hom}_R(R(G), M)$ yields an $\bar{f} \in \mathrm{Hom}_{R(G)}(R(G), M(G))$ by

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defining $\bar{f}(r)(g)=f(g^{-1}r)$. If G is infinite, then $f \in \mathrm{Hom}_R(R(G), R)$ defined by $f(g)=1$ yields $\bar{f}(1)(g)=f(g^{-1})=1\neq 0$ for all $g\in G$, and so $\bar{f}\notin \mathrm{Hom}_{R(G)}(R(G), M(G))$. If G is not a group, then $\bar{f}(r)(g)$ is not necessarily defined for each $g\in G$. Hence, this proof depends strongly on the fact that G is a finite group.

THEOREM. *Let R be a ring with* 1, *Σ be a σ-set for R, and G be a finite group. Then $R(G)$ is $\Sigma(G)$-injective iff R is Σ-injective.*

Proof. Let $R=M$ in Proposition 4 and Theorem 3.

5. The proof of Proposition 4 would have been shortened if the following were true:

> If T is a $\Sigma(S)$-essential left ideal of $R(S)$, then ${}_RT$ is a Σ-essential submodule of ${}_RR(S)$ where S is a monoid.

Unfortunately, this is not true as shown by the following example.

EXAMPLE. Let R be a field of characteristic $p\neq 0$, G a finite p-group,

$$\Sigma = \{0, R\} \quad \text{and} \quad T = \{\textstyle\sum r(g)g \mid \sum r(g) = 0\},$$

the augmented ideal of R. Then $\Sigma(G)$ is the lattice of all left ideals of $R(G)$, and T is $\Sigma(G)$-essential in $R(G)$. (For any $\sum r(g)g\in R(G)$, $0\neq T(\sum r(g)g)\subseteq T$ and $T\in\Sigma(G)$.) However let $1\in G$, then

$$T1^{-1} = \{r\in R \mid r\cdot 1\in T\} = (0) \quad \text{and} \quad (T1^{-1})1 = 0.$$

Therefore, ${}_RT$ is not a Σ-essential submodule of ${}_RR(G)$.

The above example also shows that if T is a $\Sigma(S)$-essential left ideal of $R(S)$, $T\cap R$ need not be a Σ-essential left ideal of R.

PROPOSITION. *Let S be a monoid, R a ring with* 1, *and Σ be a σ-set for R. For each R-module M, N is a Σ-essential submodule of M if and only if $N(S)$ is a $\Sigma(S)$-essential $R(S)$-submodule of $M(S)$. Moreover, in this case $N(S)$ is a Σ-essential R-submodule of $M(S)$.*

Proof. ($\Rightarrow$) Let ${}_RN\subseteq{}_RM$ and $0\neq m\in M$. If ${}_{R(S)}N(S)$ is $\Sigma(S)$-essential in ${}_{R(S)}M(S)$, then $N(S)(m\cdot 1)^{-1}\in\Sigma(S)$ and for some $T\in\Sigma$, $T(S)(m\cdot 1)\subseteq N(S)$. Thus $Tm\subseteq N$ and $Nm^{-1}\in\Sigma$. Moreover, there is $\sum t(s)s\in N(S)(m\cdot 1)^{-1}$ with

$$0\neq (\textstyle\sum t(s)s)(m\cdot 1) = \sum (t(s)m)s\in N(S).$$

Since $M(S)$ is a free R-module, there is an $s\in S$ for which $0\neq t(s)m\in N$. Hence $(Nm^{-1})m\neq 0$ and ${}_RN$ is Σ-essential in ${}_RM$.

($\Leftarrow$) Let ${}_RN$ be Σ-essential in ${}_RM$ and $0\neq\sum m(s)s\in M(S)$. Then if

$$T = \bigcap \{Nm(s)^{-1} \mid m(s)\neq 0\},$$

$T\in\Sigma$ since $\sum m(s)s$ has finite support. Thus

$$T(S)(\textstyle\sum m(s)s)\subseteq N(S) \quad \text{and} \quad N(S)(\sum m(s)s)^{-1}\in\Sigma(S).$$

Let $\{s_1, \ldots, s_n\}$ be the support of $\sum m(s)s$. There is $y_1 \in Nm(s_1)^{-1}$ such that $y_1 m(s_1) \neq 0$, since ${}_R N$ is Σ-essential in ${}_R M$. If $y_1 m(s_2) = 0$, let $y_2 = 1$; otherwise choose $y_2 \in N(y_1 m(s_2))^{-1}$ such that $y_2 y_1 m(s_2) \neq 0$. Continuing in this way we obtain $y_1, \ldots, y_n \in R$ with

$$0 \neq (y_n \ldots y_1)(\sum m(s)s) \in N(S).$$

Since $R \subseteq R(S)$, ${}_{R(S)}N(S)$ is $\Sigma(S)$-essential in ${}_{R(S)}M(S)$. Moreover, since $y_n \ldots y_1 \in R$, ${}_R N(S)$ is Σ-essential in ${}_R M(S)$.

Given a ring R and a σ-set Σ for R, in [3] we defined the Σ-singular ideal of R as

$$Z_\Sigma(R) = \{r \in R \mid Sr = 0 \text{ for some } \Sigma\text{-essential left ideal } S \text{ of } R\}.$$

A Johnson maximal left Σ-quotient ring of R, $J_\Sigma(R)$, was constructed as

$$J_\Sigma(R) = \varinjlim \{\mathrm{Hom}_R\,(J, R) \mid J \text{ is a } \Sigma\text{-essential left ideal of } R\}.$$

When $Z_\Sigma(R) = 0$, $J_\Sigma(R)$ was shown to be the Σ-injective hull of ${}_R R$, and to be unique up to isomorphism over R.

LEMMA. *Let R be a ring with* 1, Σ *be a σ-set for R, and S be a monoid. Then* $Z_\Sigma(R) = 0$ *if* $Z_{\Sigma(S)}(R(S)) = 0$.

Proof. Suppose $r \in Z_\Sigma(R)$, and let J be a Σ-essential left ideal of R with $Jr = 0$. Fix $s(\neq z) \in S$. Then $J(S)(r \cdot s) = 0$ and $J(S)$ is a $\Sigma(S)$-essential left ideal of $R(S)$. Hence

$$r \cdot s \in Z_{\Sigma(S)}(R(S)) = 0$$

so that $r \cdot s = 0$. Thus $r = 0$ and $Z_\Sigma(R) = 0$.

THEOREM. *Let R be a ring with* 1, Σ *be a σ-set for R, and G be a finite group. If* $Z_{\Sigma(G)}(R(G)) = 0$, *then*

$$J_{\Sigma(G)}(R(G)) \approx (J_\Sigma(R))(G) \quad \text{(ring iso.)}.$$

Proof. Since $Z_{\Sigma(G)}(R(G)) = 0$, $Z_\Sigma(R) = 0$ and so $J_\Sigma(R)$ is Σ-injective. By Proposition 4, $(J_\Sigma(R))(G)$ is $\Sigma(G)$-injective, and by Proposition 5, $(J_\Sigma(R))(G)$ is a $\Sigma(G)$-essential extension of $R(G)$ since R is Σ-essential in $J_\Sigma(R)$. Thus $(J_\Sigma(R))(G)$ is the $\Sigma(G)$-injective hull of $R(G)$ and so is ring isomorphic to $J_{\Sigma(G)}(R(G))$.

6. Let R be a ring with 1 and Σ be a σ-set for R. For any subset T of R and positive integer n, let T_n denote the set of $n \times n$ matrices with entries from T. As usual, R_n denotes the $n \times n$ matrix ring with entries from R. In [3], we defined Σ_n as

$$\Sigma_n = \{K \mid K \text{ is a left ideal of } R_n \text{ and } K \supseteq J_n \text{ for some } J \in \Sigma\}$$

and showed that Σ_n is a σ-set for R_n. A maximal Utumi left Σ-quotient ring of R, $Q_\Sigma(R)$, was constructed and $(Q_\Sigma(R))_n$ was shown to be ring isomorphic with $Q_{\Sigma_n}(R_n)$. It was also proved that $J_\Sigma(R) \approx Q_\Sigma(R)$ (ring iso.) whenever $Z_\Sigma(R) = 0$.

Let S be a finite Brandt semigroup. Then $S=M^0(G;m;m;\Delta)$, an $m\times m$ Rees matrix semigroup over a group with zero G^0 and with the $m\times m$ identity matrix Δ as sandwich matrix [1, Theorem 3.9, p. 102]. Then $R(S)\approx(R(G))_m$, the ring of $m\times m$ matrices over the ring $R(G)$. Recall that $\Sigma(G)$ is a σ-set for $R(G)$, $\Sigma(G)_m$ is a σ-set for $(R(G))_m$, and $\Sigma(S)$ is a σ-set for $R(S)$ (since S is regular).

LEMMA. $\Sigma(G)_n=\Sigma(S)$.

Proof.

$$\begin{aligned} K\in\Sigma(G)_n &\Leftrightarrow K\supseteq T_n \text{ for some } T\in\Sigma(G)\\ &\Leftrightarrow K\supseteq (J(G))_n \text{ for some } J\in\Sigma\\ &\Leftrightarrow K\supseteq J(S) \text{ for some } J\in\Sigma \text{ since } J(S)=(J(G))_n\\ &\Leftrightarrow K\in\Sigma(S). \end{aligned}$$

THEOREM. *Let S be a finite Brandt semigroup, R a ring with 1, and Σ be a σ-set for R. If $Z_{\Sigma(S)}(R(S))=0$, then $Q_{\Sigma(S)}(R(S))\approx(Q_\Sigma(R))(S)$.*

Proof. First we show that $Z_{\Sigma(G)}(R(G))=0$. To this end let $r\in R(G)$ and $Lr=0$ for some $\Sigma(G)$-essential left ideal L of $R(G)$. Then L_n is a $\Sigma(G)_n$-essential left ideal of

$$(R(G))_n = R(S) \quad\text{and}\quad L_n\left(\sum_{i,j=1}^{n} re_{ij}\right)=0.$$

Since $\Sigma(G)_n=\Sigma(S)$,

$$\sum re_{ij}\in Z_{\Sigma(S)}(R(S))=0$$

so that

$$r=0 \quad\text{and}\quad Z_{\Sigma(G)}(R(G))=0.$$

To finish the proof of the theorem, we note that

$$\begin{aligned} (Q_\Sigma(R))(S) &\approx (Q_\Sigma(R)(G))_n\\ &\approx (Q_{\Sigma(G)}(R(G)))_n \text{ by the first part of the proof}\\ &\approx Q_{\Sigma(G)_n}((R(G))_n)\\ &\approx Q_{\Sigma(S)}(R(S)) \text{ by the lemma.} \end{aligned}$$

Now let S be a finite inverse semigroup (i.e. a regular semigroup in which idempotents commute). The semigroup ring $R(S)$ has an identity e [6, Theorem 2]. Let $S=S_0\supset S_1\supset\cdots\supset S_{n+1}$ be a principal series for S with $S_{n+1}=\{0\}$ if S has a zero and S_{n+1} empty otherwise. Then S_i/S_{i+1} is a Brandt semigroup for each $i=0,1,\ldots,n$ [1, Exercise 3, p. 103]. If $n=0$, then $S\approx S_0/S_1$ is a Brandt semigroup and $(Q_\Sigma(R))(S)\approx Q_{\Sigma(S)}(R(S))$ by the previous theorem.

Proceeding by induction, suppose that $(Q_\Sigma(R))(T)\approx Q_{\Sigma(T)}(R(T))$ for all finite inverse semigroups having a principal series of length less than n and which satisfy

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$Z_{\Sigma(T)}(R(T))=0$. Now S_n is a Brandt semigroup so $R(S_n)(\subseteq R(S))$ has an identity, say f. If $x \in R(S)$, then both $xf, fx \in R(S_n)$ so

$$xf = f(xf) = (fx)f = fx$$

so that f is central in $R(S)$. Thus

$$R(S) = R(S)(e-f) \oplus R(S)f,$$

a ring direct sum. Now $R(S)f \approx R(S_n)$ by the maps

$$\sum r(s_n)s_n \mapsto (\sum r(s_n)s_n)f \quad \text{and} \quad \sum r(s)sf \mapsto \sum r(s)sf$$

for $\sum r(s_n)s_n \in R(S_n)$ and $\sum r(s)s \in R(S)$. Also $R(S/S_n) \approx R(S)(e-f)$ by the maps

$$\sum r(s)s \mapsto (\sum r(s)s)(e-f) \quad \text{and} \quad (\sum r(s)s)(e-f) \mapsto \sum_{s \in S/S_n} r(s)s$$

for $\sum r(s)s \in R(S)$.

Recall that a σ-set Σ for a ring R is the neighborhood system of zero for a ring topology on R. Therefore we may consider bases for Σ.

Lemma 1. *$\{T(S)f \mid T \in \Sigma\}$ is a base for $\Sigma(S_n)$.*

Proof. Let $T(S)f \subseteq R(S)f$ where $T \in \Sigma$. If

$$t = \sum t(s_n)s_n \in T(S_n),$$

then

$$t \in R(S_n) \approx R(S)f.$$

Since f is the identity of $R(S_n)$, $tf = t$ so

$$T(S_n) = T(S_n)f \subseteq T(S)f.$$

Now choose $T \in \Sigma$ so that $T(S_n) \in \Sigma(S_n)$ and let $t = \sum t(s)s \in T(S)$. Then

$$f = \sum f(s_n)s_n \in R(S_n) \quad \text{and} \quad tf = \sum (t(s)f(s_n))ss_n.$$

For each $s_n \in S_n$ with $f(s_n) \neq 0$, let $T_n = Tf(s_n)^{-1} \in \Sigma$. Then $L = \bigcap T_n \in \Sigma$ since f has finite support and $L(S)f \subseteq T(S_n)$.

Lemma 2. *$\{T(S)(e-f) \mid T \in \Sigma\}$ is a base for $\Sigma(S/S_n)$.*

Proof. For $K \in \Sigma$, $K(S/S_n) \approx K(S)(e-f)$ using the isomorphism

$$R(S/S_n) \approx R(S)(e-f).$$

Remark. In Lemma 2, we have actually shown that the image of $\{T(S)(e-f) \mid T \in \Sigma\}$ under the isomorphism $R(S)(e-f) \approx R(S/S_n)$ is a base for $\Sigma(S/S_n)$. We identify $R(S)(e-f)$ with $R(S/S_n)$.

Lemma 3. $\Sigma(S) = \Sigma(S_n) \oplus \Sigma(S/S_n)$.

Proof. Let $K, J \in \Sigma$. Then

$$K(S)f \oplus J(S)(e-f) \in \Sigma(S_n) \oplus \Sigma(S/S_n).$$

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Letting $T=K\cap J$, we see that

$$T(S) \subseteq T(S)f \oplus T(S)(e-f) \subseteq K(S)f \oplus J(S)(e-f).$$

Thus $\Sigma(S_n)\oplus\Sigma(S/S_n)\subseteq\Sigma(S)$.

Now let $T\in\Sigma$. Then

$$K(S) \subseteq T(S)f^{-1} \quad\text{and}\quad J(S) \subseteq T(S)(e-f)^{-1},$$

where $K, J\in\Sigma$. Hence

$$K(S)f \oplus J(S)(e-f) \subseteq T(S)$$

so $\Sigma(S)\subseteq\Sigma(S_n)\oplus\Sigma(S/S_n)$ and the lemma follows.

LEMMA 4. $Z_{\Sigma(S)}(R(S))=0 \Rightarrow Z_{\Sigma(S_n)}(R/S_n))=0$ *and* $Z_{\Sigma(S/S_n)}(R(S/S_n))=0$.

Proof. We prove the result for $Z_{\Sigma(S_n)}(R(S_n))$; the result for S/S_n follows similarly. Recall that $R(S_n)\approx R(S)f$ and $\Sigma(S_n)$ has $\{T(S)f \mid T\in\Sigma\}$ as a base. Let $r=rf\in R(S_n)$ be such that $Lrf=0$ where L is a $\Sigma(S_n)$-essential left ideal of $R(S_n)$. Then $L\oplus R(S)(e-f)$ is a $\Sigma(S)(=\Sigma(S_n)\oplus(S/S_n))$-essential left ideal of $R(S)$, and $(L\oplus R(S)(e-f))(rf)=0$. Hence $rf\in Z_{\Sigma(S)}(R(S))=0$ so that $rf=r=0$ and $Z_{\Sigma(S_n)}(R(S_n))=0$.

We may now complete the proof of the following:

THEOREM. *Let S be a finite inverse semigroup, R be a ring with* 1, *and Σ be a σ-set for R. Then if $Z_{\Sigma(S)}(R(S))=0$, $Q_{\Sigma(S)}(R(S))\approx(Q_\Sigma(R))(S)$ (over $R(S)$).*

Proof. Recall that $S=S_0\supset S_1\supset\cdots\supset S_n\supset S_{n+1}$ is a principal series for S where $n>0$. Then S/S_n is a finite inverse semigroup having a principal series of length $n-1$, and S_n is a Brandt semigroup. By Lemma 4,

$$Z_{\Sigma(S_n)}(R(S_n)) = Z_{\Sigma(S/S_n)}(R(S/S_n)) = 0$$

so that

$$Q_\Sigma(R))(S_n) \approx Q_{\Sigma(S_n)}(R(S_n)) \quad\text{and}\quad (Q_\Sigma(R))(S/S_n) \approx Q_{\Sigma(S/S_n)}(R(S/S_n))$$

by the induction hypothesis. Then

$$\begin{aligned} Q_{\Sigma(S)}(R(S)) &\approx Q_{\Sigma(S_n)}(R(S_n)) \oplus Q_{\Sigma(S/S_n)}(R(S/S_n)) \quad \text{(by [3, Theorem 5.3])} \\ &\approx (Q_\Sigma(R))(S_n) \oplus (Q_\Sigma(R))(S/S_n) \\ &\approx (Q_\Sigma(R))(S)f \oplus (Q_\Sigma(R))(S)(e-f) \\ &= (Q_\Sigma(R))(S) \end{aligned}$$

since f and $(e-f)$ are central idempotents of $R(S)$. A simple calculation shows that the composition of these isomorphisms is the identity when restricted to $R(S)$; hence the result follows.

REMARK. If ${}_{R_n}R_n$ is Σ_n-injective iff ${}_RR$ is Σ-injective, then the above method of proof would show that $R(S)$ is $\Sigma(S)$-injective iff R is Σ-injective whenever S is a finite inverse semigroup. I suspect that both results are true; however, I have no proof. (The proof of Utumi [5, Theorem 8.3] does not seem to generalize to this situation.)

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