the values (5) of α belong to a certain class of quadratic surds for which

$$u_{v_n} = u_n + v_n .$$

We hope to publish elsewhere an account of this and other results connected with Beatty sequences.

The author would like to thank Dr. N.S. Mendelsohn for his advice and encouragement.

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SOME PROPERTIES OF BEATTY SEQUENCES I*

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1. Introduction. Two sequences of natural numbers are said to be complementary if they contain all the positive integers without repetition or omission. S. Beatty [1] observed that the sequences

(1) $$u_n = [n(1 + 1/\alpha)] , \quad n = 1, 2, 3, \ldots ,$$

(2) $$v_n = [n(1 + \alpha)] , \quad n = 1, 2, 3, \ldots ,$$

(where square brackets denote the integral part function) are complementary if and only if $\alpha > 0$ and α is irrational. We call the pair (1), (2) Beatty sequences of argument α .

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The particular values of α,

(3) $$A = \tfrac{1}{2}(k + \sqrt{k^2 + 4}), \quad k = 1, 2, 3, \ldots$$

give rise to Beatty sequences which are connected with a generalization of Wythoff's game. (See [2] where the proof of Theorem 1 may be used to prove Beatty's statement.) Since

$$1 + A = 1 + (1/A) + k,$$

(4) $$u_n = u_n + kn .$$

When $k = 1$, $A = \frac{1}{2}(1 + \sqrt{5})$ and $v_n = u_n + n$. The Beatty sequences of this argument are given in Table I which was constructed by the rules:

(i) $u_1 = 1$;

(ii) $v_n = u_n + n$;

(iii) u_{n+1} is the least positive integer distinct from the $2n$ integers $u_1, u_2, \ldots u_n, v_1, v_2, \ldots, v_n$.

n	u_n	v_n
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26
11	17	28
12	19	31
13	21	34

Table I

It was conjectured from this table that

(5) $$u_{v_n} = u_n + v_n .$$

This is proved below. The main object of this paper is to determine all $\alpha > 0$ for which the Beatty sequences of argument

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α obey (5) for every positive integer n.

2. If $0 < \alpha < 1$ then $u_1 > 1$, $v_1 = 1$ and

$$u_{v_1} = u_1 \neq u_1 + v_1 ,$$

so that (5) is violated for n = 1. Thus we assume $\alpha > 1$. Writing

$$(6) \qquad \xi = 1 + 1/\alpha \quad , \quad \eta = 1 + \alpha ,$$

(whence $1 < \xi < 2$), Beatty's theorem becomes: $[n\xi]$ and $[n\eta]$ for n = 1, 2, 3, ... are complementary if and only if $\xi > 1$, ξ is irrational and

$$(7) \qquad \xi^{-1} + \eta^{-1} = 1 .$$

Thus (5) is

$$(8) \qquad [[n\eta]\xi] = [n\xi] + [n\eta] ,$$

or $\qquad [n\xi] + [n\eta] < [n\eta]\xi < [n\xi] + [n\eta] + 1$

(strict inequalities since ξ is irrational).

Denote by $r(x) = x - [x]$ the fractional part of x and let

$$(9) \qquad r(n\xi) = \varepsilon \quad , \quad r(n\eta) = \delta .$$

Rewriting (8) we have,

$$n(\xi + \eta) - \varepsilon - \delta < n\eta\xi - \delta\xi < n(\xi + \eta) - \varepsilon - \delta + 1,$$

or since $\qquad \xi + \eta = \xi\eta ,$

$$-\varepsilon - \delta < -\delta\xi < -\varepsilon - \delta + 1 .$$

Now $-\delta\xi < -\varepsilon - \delta + 1$ is invariably true since $\xi > 1$, $\delta > 0$, and $\varepsilon < 1$. Thus (5) is equivalent to

$$-\varepsilon - \delta < -\delta\xi ,$$

or, since $\xi = 1 + 1/\alpha$,

$$(10) \qquad \delta < \alpha\varepsilon .$$

We require the following well known result which we state without proof.

LEMMA. If ξ, η and 1 are linearly independent the set of points with cartesian coordinates

$$(r(n\xi), r(n\eta)), \quad n = 1, 2, 3, \ldots$$

is uniformly distributed in the unit square. If one and only one relation

$$(11) \qquad c\eta = a\xi + b$$

for (not all zero) integers a, b, c holds, then the set is uniformly distributed on the segments within the unit square of the lines

$$cy = ax + \nu$$

where ν is any integer. The values of ν which give rise to segments within the unit square are (with $c > 0$):

$$\text{a positive:} \quad -a < \nu < c\,,$$

$$\text{a negative:} \quad 0 < \nu < c - a\,.$$

Our condition is $\delta < \alpha\varepsilon$. Thus, plotting the points (ε, δ) we see that if a point lies in region A (fig. 1) then (5) is false and if it lies in region B, (5) is true. Hence (5) cannot be true for every n if ξ, η and 1 are linearly independent, for region A contains infinitely many points. If there exists a relation (11), the possible situations are shown in figures 2-5. We disregard a = 0 and c = 0 which correspond to η and ξ rational, for by the additional condition (7) they would then both be rational and two relations of type (11) would exist. We take $c > 0$.

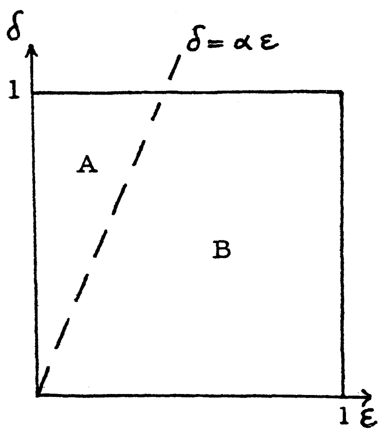

Fig. 1

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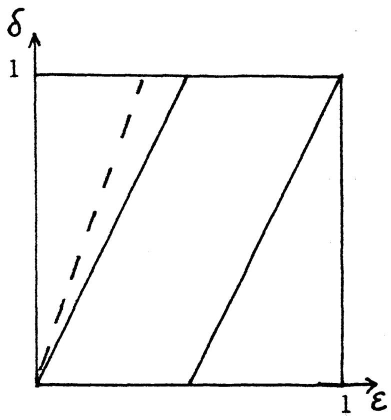
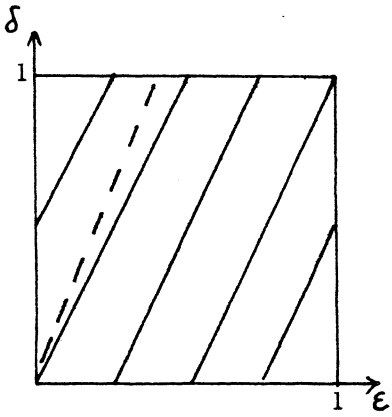


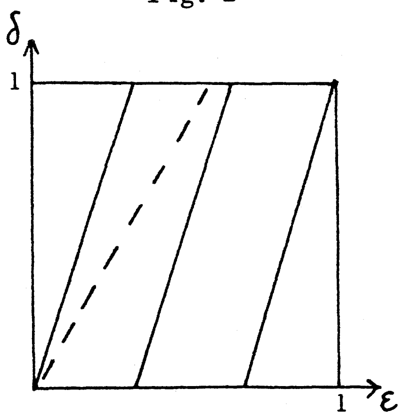
$c = 1, 0 < a < \alpha$

Fig. 2



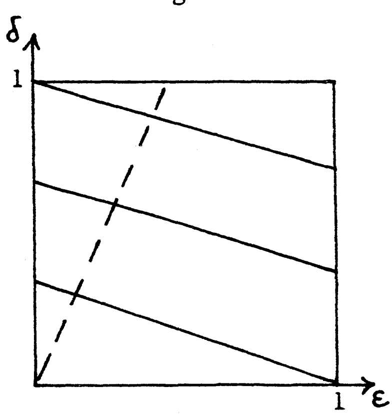
$c > 1, \; 0 < a/c < \alpha$

Fig. 3



$c > 0, \; a/c > \alpha$

Fig. 4



$c > 0 \; , \; a < 0$

Fig. 5

We see that $\delta < \alpha\varepsilon$ for all n only in the case $c = 1$, $0 < a < \alpha$. From (7) and (11) we calculate

$$(12) \qquad \alpha = \tfrac{1}{2}\left\{(a+b-1) + \sqrt{(a+b-1)^2+4a}\right\}.$$

The condition $\alpha > a$ gives

$$\sqrt{(a+b-1)^2 + 4a} > a-b+1 ,$$

or $4ab > 0$, and since $a > 0$, $b > 0$.

We notice that the values (12) of α are irrational; for let

$$(a + b-1)^2 + 4a = (a + b-1 + m)^2 ,$$

where m is a positive integer. Then

$$4a = 2m(a + b-1) + m^2$$

and $m = 2m_1$. Hence

$$a = m_1(a + b-1) + m_1^2 .$$

Since $b \geqslant 1$ and $m_1 > 0$,

$$a \geqslant m_1(a + m_1)$$

which is impossible for a a positive integer.

The equation $\eta = a\xi + b$ is equivalent to

(13) $$v_n = u_{an} + bn .$$

We have

THEOREM 1. If $\{u_n\}$ and $\{v_n\}$ are Beatty sequences of argument α,

(14) $$u_{v_n} = u_n + v_n,\ n = 1,2,3,\ldots$$

is true if and only if there exists a relation

(15) $$v_n = u_{an} + bn ,\ n = 1,2,3,\ldots$$

with fixed positive integers a and b. The only values of α for which this is true are

(16) $$\alpha = \tfrac{1}{2}\{(a + b-1) + \sqrt{(a + b-1)^2 + 4a}\},$$

$$a = 1,2,3,\ldots,\ b = 1,2,3,\ldots .$$

3. A restatement of Theorem 1.

THEOREM 2. The statement

(17) $$[[n(1 + \alpha)]/\alpha] = [n(1 + \alpha)/\alpha],\ n = 1,2,3,\ldots$$

is true for

(i) positive integers, $\alpha = 1,2,3,\ldots$

(ii) the irrational numbers

$$= \tfrac{1}{2}\{(a + b-1) + \sqrt{(a + b-1)^2 + 4a}\},$$

$$a = 1,2,3,\ldots,\quad b = 1,2,3,\ldots,$$

and is false for all other values of $\alpha > 0$. Written in order of magnitude the possible values of α are:

$$1, \tfrac{1}{2}(1+\sqrt{5}),\ 2,\ 1+\sqrt{2},\ 1+\sqrt{3},\ 3,\ \tfrac{1}{2}(3+\sqrt{13}),\ \tfrac{1}{2}(3+\sqrt{17}),\ \tfrac{1}{2}(3+\sqrt{21}),\ 4,\ldots$$

Proof: Writing (8) in terms of α,

$$[[n(1+\alpha)](1+1/\alpha)] = [n(1+1/\alpha)] + [n(1+\alpha)],$$

or
$$[[n(1+\alpha)]1/\alpha] = [n(1+1/\alpha)],$$

which is (17). By Theorem 1, (17) is true for irrational $\alpha > 0$ only for the values listed.

If $0 < \alpha < 1$, set $n = 1$. Then

$$[[n(1+\alpha)]/\alpha] = [1/\alpha],$$

and
$$[n(1+\alpha)/\alpha] = 1 + [1/\alpha],$$

so that (17) is violated for $n = 1$.

Clearly (17) is true if α is a positive integer. But it is false for any non-integral rational; for let

$$\alpha = p/q, \quad (p,q) = 1, \quad 1 < q < p,$$

and set $n = p$. Let

$$p^2 = rq + s, \quad 0 < s < q.$$

Then

$$[[n(1+\alpha)]/\alpha] = [(p+r)q/p] = q + [(p^2-s)/p]$$

$$= p+q + [-s/p] = p+q-1,$$

whereas,

$$[n(1+\alpha)/\alpha] = [p(1+p/q)q/p] = p + q,$$

and (17) is violated for $n = p$.

4. <u>Some asymptotic formulas</u>. Define $\lambda(n) = 1$ if (5) is true and $\lambda(n) = 0$ if (5) is false; set

$$\Lambda(N) = \sum_{n=1}^{N} \lambda(n). \tag{18}$$

Theorem 1 gives the values of α for which $\Lambda(N) = N$ for all N. In the other cases it is possible to evaluate $\Lambda(N)$ asymptotically.

If a relation (11) exists then by (7) ξ and η are quadratic surds and the points (ε, δ) are uniformly distributed on line segments in the unit square. Hence $\Lambda(N)/N$ is asymptotic to the ratio of the length of line segments in region B (fig. 1) to the total length of line segments in the unit square. However the expressions in the various cases are complicated.

If ξ, η and 1 are linearly independent, then, by the uniform distribution,

$$\lim_{N\to\infty} \Lambda(N)/N = \text{area of region B.}$$

That is,

THEOREM 3. If $\alpha > 1$ is irrational but not a quadratic surd,

$$\Lambda(N) \sim N(2\alpha - 1)/2\alpha \tag{19}$$

Using a theorem of W. Sierpinski [3] we can get a weaker result, but one which is valid for all irrational α.

THEOREM 4. If $\alpha > 1$ is irrational,

$$\Lambda(N) > N((\alpha - 1)/2\alpha) + o(1). \tag{20}$$

Proof. If (8) is true,

$$[n\eta]\xi < [n\xi] + [n\eta] + 1,$$

and if it is false,

$$[n\eta]\xi < [n\xi] + [n\eta].$$

In either case,

$$[n\eta]\xi < [n\xi] + [n\eta] + \lambda(n). \tag{21}$$

Sierpinski's result states that for x irrational,

$$\sum_{n=1}^{N} [nx] = \tfrac{1}{2}N(N+1)x - \tfrac{1}{2}N + o(1). \tag{22}$$

Adding inequalities (21) and applying (22),

$$\tfrac{1}{2}N(N+1)\eta\xi - \tfrac{1}{2}N\xi + o(1) < \tfrac{1}{2}N(N+1)(\xi + \eta) - N + \Lambda(N) + o(1),$$

i.e., $\Lambda(N) > \tfrac{1}{2}N(2 - \xi) + o(1)$,

which is the theorem.

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