Proceedings of the Edinburgh Mathematical Society (1983) **26**, 275–278 ©

A SHARPENING OF THE BERKSON–GLICKFELD THEOREM

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(Received 8th January 1982)

Introduction

It is known that if in a Banach*-algebra with unit the following holds:

$$\|\exp(ih)\| = 1 \qquad \text{if } h = h^*$$

then it is a C^*-algebra (see [**3**]).

We shall show that the above theorem can be sharpened in the following way: we replace the submultiplicativity of the norm by the weaker assumption

$$\|a^*a\| \leqq \|a^*\| \cdot \|a\| \qquad \text{for all } a.$$

Observe that under this assumption even the existence of $\exp(ih)$ is not at all obvious, but it will be proved to be true below. Our main result is Theorem 2 which depends on Theorem 1. Our last remark is the equivalent-norm-version of the statement.

Theorem 1. *Let $\mathscr{A}$ be a *-algebra with unit. Let p be a complete norm on it such that the following hold for a suitable positive constant D:*

(*i*) $p(a^*a) \leqq D \cdot p(a^*) \cdot p(a)$ *for all* $a \in \mathscr{A}$,
(*ii*) $p(\exp(ih)) \leqq D$ *if* $h = h^* \in \mathscr{A}$ *and* $\exp(ih)$ *exists.*

Then there is a norm $\|.\|_C$ on $\mathscr{A}$, equivalent to p and such that $(\mathscr{A}, \|.\|_C)$ is a C^*-algebra.

Proof. The following identity holds in each *-algebra:

$$4xy = (y + x^*)^*(y + x^*) - (y - x^*)^*(y - x^*)$$
$$+ i(y + ix^*)^*(y + ix^*) - i(y - ix^*)^*(y - ix^*). \tag{1}$$

Applying (i) we get from this

$$4p(xy) \leqq 4D \cdot (p(y^*) + p(x)) \cdot (p(y) + p(x^*)). \tag{2}$$

Writing $x = p(v^*)^{1/2} \cdot p(v)^{1/2} \cdot u$, $y = p(u^*)^{1/2} \cdot p(u)^{1/2} \cdot v$ in (2), we infer

$$p(uv) \leqq D \cdot (p(u^*)^{1/2} \cdot p(v^*)^{1/2} + p(u)^{1/2} \cdot p(v)^{1/2})^2. \tag{3}$$

Define a new norm by setting

$$\|a\| = 4D \cdot \max(p(a^*), p(a)). \tag{4}$$

Then we have, by (3),

$$\|ab\| \leqq \|a\| \cdot \|b\|; \ \|a^*\| = \|a\|; \ p(a) \leqq \frac{1}{4D} \cdot \|a\| \qquad \text{for all } a, b \in \mathscr{A}. \tag{5}$$

Let $\mathscr{B}$ be the completion of $(\mathscr{A}, \|.\|)$. Then by (5) the algebra operations and p have unique continuous extensions to $\mathscr{B}$. Thus $(\mathscr{B}, \|.\|)$ is a star-normed algebra, p is a continuous seminorm on it, and (i), (4) and (5) are valid in $\mathscr{B}$, too.

Since $(\mathscr{B}, \|.\|)$ is a Banach-algebra with unit, for any $a \in \mathscr{B}$ we can define $\exp_B a = \sum_{n=0}^{\infty} a^n/n!$, with respect to $\|.\|$. Let $a \in \mathscr{A}$, then $\sum_{n=0}^{\infty} a^n/n!$ is convergent in $(\mathscr{A}, \|.\|)$ and thus, by (5), in $(\mathscr{A}, p)$, too. But p is a complete norm on $\mathscr{A}$, and therefore there is a unique $\exp_A a = \sum_{n=0}^{\infty} a^n/n!$ in $\mathscr{A}$, with respect to p. We note that $p(\exp_A a - \exp_B a) = 0$ for all a because p is continuous with respect to $\|.\|$.

Thus we see from (ii) that

$$p(\exp_B(ih)) \leqq D \qquad \text{if } h = h^* \in \mathscr{A}. \tag{6}$$

Since the * is continuous with respect to $\|.\|$ thus $(\exp_B a)^* = \exp_B(a^*)$ for all $a \in \mathscr{B}$; in particular $(\exp_B(ih))^* = \exp_B(-ih)$, if $h = h^*$. Therefore by (6) and (4) we infer

$$\|\exp_B(ih)\| \leqq 4D^2 \qquad \text{if } h = h^* \in \mathscr{A}. \tag{7}$$

Since the self-adjoint part of $\mathscr{A}$ is dense in that of $\mathscr{B}$, thus (7) is true even if $h = h^* \in \mathscr{B}$. But this implies that $\|a\|_C = r(a^*a)^{1/2}$ defines a C^*-norm on $\mathscr{B}$, equivalent to $\|.\|$ (see [2]). Thus there are positive constants E, F such that

$$E \cdot \|a\|_C \leqq \|a\| \leqq F \cdot \|a\|_C \qquad \text{for all } a \in \mathscr{B}.$$

Writing $K = E(4D)^{-1}$, $L = F(4D)^{-1}$ we have by (4) that

$$p(a) \leqq L \cdot \|a\|_C \qquad \text{for all } a \in \mathscr{B} \quad \text{and}$$

$$p(h) \geqq K \cdot \|h\|_C \qquad \text{if } h = h^* \in \mathscr{B}. \tag{8}$$

Thus by (i) we have

$$K \cdot \|a\|_C^2 = K \cdot \|a^*a\|_C \leqq p(a^*a) \leqq D \cdot p(a^*) \cdot p(a).$$

This and (8) imply

$$p(a) \geqq K \cdot (DL)^{-1} \cdot \|a\|_C \qquad \text{for all } a \in \mathscr{B}.$$

Thus we have seen that p and $\|.\|_C$ are equivalent.

Note. The condition (i) in Theorem 1 implies that $\exp a$ exists in $\mathscr{A}$ for all $a \in \mathscr{A}$. We have seen it in the first part of the above proof.

Theorem 2. *If the assumptions of Theorem* 1 *hold with* $D=1$ *then* $p=\|.\|_C$, *that is* $(\mathscr{A}, p)$ *is a* C^**-algebra.*

Proof. Since $r(a)=\lim\|a^n\|_C^{1/n}$, we infer from Theorem 1 that

$$r(a)=\lim p(a^n)^{1/n} \qquad \text{for all } a \in \mathscr{A}. \tag{1}$$

Applying (i) to $a=h^{2^n}$, where h is self-adjoint, we infer by induction that $p(h^{2^n}) \leqq p(h)^{2^n}$ and thus by (1) we get

$$r(h) \leqq p(h) \qquad \text{if } h=h^* \in \mathscr{A}. \tag{2}$$

But $r(a^*a)=\|a\|_C^2$ for all a and thus

$$\|a\|_C^2 \leqq p(a^*a) \qquad \text{for all } a \in \mathscr{A}. \tag{3}$$

It is known that the unit ball of a C^*-algebra with unit is the closed convex hull of elements of the form $\exp(ih)$ where h is self-adjoint (see [1], p. 210).

On the other hand, from (ii) we see that $p(a) \leqq 1$ if a is convex combination of elements of the form $\exp(ih)$; further p is continuous with respect to $\|.\|_C$ and thus we get

$$p(a) \leqq \|a\|_C \qquad \text{for all } a \in \mathscr{A}. \tag{4}$$

Comparing (3), (4) and (i) we see that

$$\|a\|_C^2 \leqq p(a^*a) \leqq p(a^*) \cdot p(a) \leqq \|a^*\|_C \cdot \|a\|_C = \|a\|_C^2$$

that is $\|a\|_C^2 = p(a^*) \cdot p(a)$ for all $a \in \mathscr{A}$. This and (4) imply $p=\|\cdot\|_C$. Thus Theorem 2 is proved.

Remark. The completeness of the norm is not essential. Drop it and replace (ii) by this:

(iii) $\displaystyle \lim_{k \to \infty} p\left(\sum_{n=0}^{k} \frac{(ih)^n}{n!}\right) \leqq D$ if $h=h^* \in \mathscr{A}$ and the limit exists.

Then the conclusion has to be modified so that the completion of $(\mathscr{A}, p)$ is an equivalent C^*-algebra (resp. a C^*-algebra if $D=1$). The proof is the same.

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