Canad. Math. Bull. Vol. **57** (3), 2014 pp. 546–550
http://dx.doi.org/10.4153/CMB-2013-003-5

Compact Operators in Regular LCQ Groups

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Abstract. We show that a regular locally compact quantum group $\mathbb{G}$ is discrete if and only if $\mathcal{L}^\infty(\mathbb{G})$ contains non-zero compact operators on $\mathcal{L}^2(\mathbb{G})$. As a corollary we classify all discrete quantum groups among regular locally compact quantum groups $\mathbb{G}$ where $\mathcal{L}^1(\mathbb{G})$ has the Radon–Nikodym property.

It is known that for a locally compact group G, the following are equivalent:

(i) G is discrete,
(ii) $L^\infty(G)$ contains a compact operator on $L^2(G)$,
(iii) $L^1(G)$ has the Radon–Nikodym property,
(iv) the von Neumann algebra $L^\infty(G)$ is purely atomic (*cf.* [2] and [6]).

In the general setting of locally compact quantum groups $\mathbb{G}$, it is known that (i) implies other properties: (iii) and (iv) are equivalent, and there are examples of $\mathbb{G}$ that satisfy (iii), but not (i) (*cf.* [6] and [7]).

In this paper we investigate the relations between (ii) and other above properties. We prove that in the case of regular locally compact quantum groups, (ii) implies (i), whence (iii) and (iv). Moreover we classify regular locally compact quantum groups that satisfy (iv), and (or but not) (ii).

First, let us recall some definitions and preliminary results that we will be using in this paper. For more details on locally compact quantum groups we refer the reader to [4].

A *locally compact quantum group* $\mathbb{G}$ is a quadruple $\big(\mathcal{L}^\infty(\mathbb{G}), \Gamma, \varphi, \psi\big)$, where $\mathcal{L}^\infty(\mathbb{G})$ is a von Neumann algebra, $\Gamma\colon \mathcal{L}^\infty(\mathbb{G}) \to \mathcal{L}^\infty(\mathbb{G})\bar{\otimes}\mathcal{L}^\infty(\mathbb{G})$ is a co-associative co-multiplication, *i.e.*, a unital injective $*$-homomorphism, satisfying

$$(\Gamma \otimes \iota)\Gamma = (\iota \otimes \Gamma)\Gamma,$$

and φ and ψ are (normal faithful semi-finite) left and right invariant weights on $\mathcal{L}^\infty(\mathbb{G})$, that is

$$\varphi\big((\omega \otimes \iota)\Gamma(x)\big) = \varphi(x)\omega(1)$$

for all $\omega \in \mathcal{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathcal{L}^\infty(\mathbb{G})^+$ where $\varphi(x) < \infty$, and

$$\psi\big((\iota \otimes \omega)\Gamma(x)\big) = \psi(x)\omega(1)$$

for all $\omega \in \mathcal{L}^\infty(\mathbb{G})_*^+$ and $x \in \mathcal{L}^\infty(\mathbb{G})^+$ where $\psi(x) < \infty$. We denote by $\mathcal{L}^2(\mathbb{G})$ the GNS Hilbert space of φ. Then one obtains two distinguished unitary operators

Received by the editors December 5, 2012; revised January 25, 2013.
Published electronically March 20, 2013.
AMS subject classification: **46L89**.
Keywords: locally compact quantum groups, regularity, compact operators.

$W \in \mathcal{L}^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)$ and $V \in \mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)\overline{\otimes}\mathcal{L}^\infty(\mathbb{G})$, called the left and right *fundamental unitaries*, which satisfy the *pentagonal relation*, and such that the co-multiplication Γ on $\mathcal{L}^\infty(\mathbb{G})$ can be expressed as

$$\Gamma(x) = W^*(1\otimes x)W = V(x\otimes 1)V^*, \quad x \in \mathcal{L}^\infty(\mathbb{G}).$$

The *reduced quantum group C^*-algebra*,

$$\overline{\big\{(\iota\otimes\omega)W : \omega \in \mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)_*\big\}}^{\|\cdot\|} = \overline{\big\{(\omega\otimes\iota)V : \omega \in \mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)_*\big\}}^{\|\cdot\|},$$

is denoted by $\mathcal{C}_0(\mathbb{G})$ and is a weak* dense C^*-subalgebra of $\mathcal{L}^\infty(\mathbb{G})$. Let $\mathcal{M}(\mathbb{G})$ denote the dual space $\mathcal{C}_0(\mathbb{G})^*$. There exists a completely contractive multiplication on $\mathcal{M}(\mathbb{G})$ given by the convolution

$$\star\colon \mathcal{M}(\mathbb{G})\widehat{\otimes}\mathcal{M}(\mathbb{G}) \ni \mu\otimes\nu \longmapsto \mu\star\nu = \mu(\iota\otimes\nu)\Gamma = \nu(\mu\otimes\iota)\Gamma \in \mathcal{M}(\mathbb{G})$$

such that $\mathcal{M}(\mathbb{G})$ contains $\mathcal{L}^1(\mathbb{G}) := \mathcal{L}^\infty(\mathbb{G})_*$ as a norm closed two-sided ideal. Therefore, for each $\mu \in \mathcal{M}(\mathbb{G})$, we obtain a pair of completely bounded maps

$$f \longmapsto \mu\star f \quad \text{and} \quad f \longmapsto f\star\mu$$

on $\mathcal{L}^1(\mathbb{G})$ through the left and right convolution products of $\mathcal{M}(\mathbb{G})$. The adjoint maps give the convolution actions $x \mapsto \mu\star x$ and $x \mapsto x\star\mu$ that are normal completely bounded maps on $\mathcal{L}^\infty(\mathbb{G})$ (note that our notation for the convolution actions is opposite to the more commonly used (*e.g.*, [3]), where $\mu\star x$ is denoted by $x\star\mu$).

For a Hilbert space H, we denote by $\mathcal{B}(H)$, and $\mathcal{B}_0(H)$ the spaces of all bounded operators, and compact operators on H, respectively.

A locally compact quantum group $\mathbb{G}$ is said to be *regular* if the norm-closed linear span of $\big\{(\iota\otimes\omega)(\Sigma V) : \omega \in \mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)_*\big\}$ equals $\mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$, where Σ denotes the flip operator on $\mathcal{L}^2(\mathbb{G})\otimes\mathcal{L}^2(\mathbb{G})$. All Kac algebras, as well as discrete and compact quantum groups are regular [1].

It follows from [3, Theorem 3.1] that for $a \in \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$ and $\omega \in \mathcal{B}\big(\mathcal{L}^2(\mathbb{G})\big)_*$, we have

$$(\iota\otimes\omega)\big(W^*(1\otimes a)W\big) \in \mathcal{C}_0(\mathbb{G}) \quad \text{and} \quad (\omega\otimes\iota)\big(V(a\otimes 1)V^*\big) \in \mathcal{C}_0(\mathbb{G}),$$

and also it is proved in [3, Corollary 3.6] that if $\mathbb{G}$ is regular, then

$$(\omega\otimes\iota)\big(W^*(1\otimes a)W\big) \in \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \quad \text{and} \quad (\iota\otimes\omega)\big(V(a\otimes 1)V^*\big) \in \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big).$$

Therefore, if $\mathbb{G}$ is regular and $a \in \mathcal{L}^\infty(\mathbb{G})\cap\mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$, then

$$f\star a,\, a\star f \in \mathcal{C}_0(\mathbb{G})\cap\mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \tag{0.1}$$

for all $f \in \mathcal{L}^1(\mathbb{G})$.

The following is the main result of the paper.

https://doi.org/10.4153/CMB-2013-003-5 Published online by Cambridge University Press

Theorem 1 *Let $\mathbb{G}$ be a regular locally compact quantum group. If*

$$\mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \neq \{0\},$$

then $\mathbb{G}$ is discrete.

Note that the converse of this theorem is also true; in fact, it follows from the structure theory of discrete quantum groups (*cf.* [7]).

We break down the proof of the theorem into few lemmas that follow.

Lemma 2 *Let $0 \leq \mu \in \mathcal{M}(\mathbb{G})$ be non-zero. Then the convolution map $x \mapsto x \star \mu$ is faithful on $\mathcal{L}^\infty(\mathbb{G})^+$.*

Proof Let ψ be the right Haar weight of $\mathbb{G}$; then we have

$$\begin{aligned}\psi(x\star\mu) &= \psi(x\star\mu)\omega(1) = \psi\big((\iota\otimes\omega)\Gamma(x\star\mu)\big)\\ &= \psi\Big(\big(\iota\otimes(\omega\star\mu)\big)\,\Gamma(x)\Big) = \psi(x)\omega(1)\|\mu\|\end{aligned}$$

for all $x \in \mathcal{L}^\infty(\mathbb{G})^+$ and $\omega \in \mathcal{L}^1(\mathbb{G})^+$, and since ψ is faithful, the lemma follows. ■

Lemma 3 *The von Neumann algebra $\mathcal{L}^\infty(\mathbb{G})$ is purely atomic.*

Proof Since $\mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \neq \{0\}$, it follows that $\mathcal{L}^\infty(\mathbb{G})$ contains a non-zero minimal projection. Suppose that $\{e_\alpha\}$ is a maximal family of mutually orthogonal minimal projections in $\mathcal{L}^\infty(\mathbb{G})$. Set $e_0 = 1 - \sum_\alpha e_\alpha$, and let $a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$ be non-zero and positive. Moreover, suppose that $\omega_0 \in \mathcal{L}^1(\mathbb{G})^+$ is such that $\operatorname{supp}(\omega_0) \leq e_0$. Then, since e_0 does not dominate any non-zero minimal projection, it follows that $e_0\big(\mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)\big)\,e_0 = 0$, and therefore using (0.1) we obtain

$$\langle f, a\star\omega_0\rangle = \langle\omega_0, f\star a\rangle = \langle\omega_0, e_0(f\star a)e_0\rangle = 0$$

for all $f \in \mathcal{L}^1(\mathbb{G})$, and therefore $a\star\omega_0 = 0$. So, it follows from Lemma 2 that $\omega_0 = 0$. Hence, $e_0 = 0$ and $\mathcal{L}^\infty(\mathbb{G})$ is purely atomic. ■

So, by the previous lemma, we conclude that $\mathcal{L}^\infty(\mathbb{G})$ is a direct sum of type I factors:

$$\mathcal{L}^\infty(\mathbb{G}) = l^\infty - \bigoplus_{i\in\mathcal{I}} \mathcal{B}(H_i).$$

Then $\mathcal{L}^2(\mathbb{G})$, being the (unique) standard Hilbert space of $\mathcal{L}^\infty(\mathbb{G})$, can be identified with the Hilbert space $l^2 - \oplus_{i\in I}\mathcal{HS}(H_i)$, where $\mathcal{HS}(H_i)$ is the Hilbert–Schmidt space over H_i. Moreover, from the uniqueness of the standard representation, it follows that the representation of every summand $\mathcal{B}(H_i)$ on $\mathcal{L}^2(\mathbb{G})$ is equivalent to their representation on $H_i \otimes H_i$, mapping $a \in \mathcal{B}(H_i)$ to $a \otimes 1 \in \mathcal{B}(H_i \otimes H_i)$. In particular, if $a \in \mathcal{B}(H_i)$ is compact on $\mathcal{L}^2(\mathbb{G})$, then either $a = 0$ or H_i is finite dimensional.

We denote by $1_j \in \bigoplus_{i\in\mathcal{I}} \mathcal{B}(H_i)$ the projection onto H_j. Then 1_j is in the center of $\bigoplus_{i\in\mathcal{I}} \mathcal{B}(H_i)$, and $x = \sum_i 1_i x$ for all $x \in \bigoplus_{i\in\mathcal{I}} \mathcal{B}(H_i)$.

If $a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$, then $1_j a$ is a compact operator on $\mathcal{L}^2(\mathbb{G})$. Thus, if $1_j a \neq 0$, the Hilbert space H_j is finite dimensional.

https://doi.org/10.4153/CMB-2013-003-5 Published online by Cambridge University Press

Lemma 4 *For each $j \in \mathcal{J}$ there exists $a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$ such that $1_j a \neq 0$. In particular, $\dim(H_j) < \infty$ for all $j \in \mathcal{J}$.*

Proof Let $0 \leq a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$ be non-zero. We show that there exists $f \in \mathcal{L}^1(\mathbb{G})$ such that $1_j(a \star f) \neq 0$, and this yields the lemma by (0.1). So, suppose to the contrary that $1_j(a \star f) = 0$ for all $f \in \mathcal{L}^1(\mathbb{G})$, and choose $0 \leq \omega \in \mathcal{L}^1(\mathbb{G})$ such that $\omega(1_j) \neq 0$. Then we get

$$\langle f, (1_j\omega) \star a\rangle = \langle \omega, 1_j(a \star f)\rangle = 0$$

for all $f \in \mathcal{L}^1(\mathbb{G})$, which implies that $(1_j\omega) \star a = 0$. But, by Lemma 2 this contradicts our assumptions. ■

The proof of the following lemma is standard.

Lemma 5 *If H and K are Hilbert spaces, and $\Phi\colon \mathcal{B}(H) \to \mathcal{B}(K)$ is an injective $*$-homomorphism, then $\Phi(x) \in \mathcal{B}_0(K)$ implies $x \in \mathcal{B}_0(H)$.*

Lemma 6 *We have*

$$c_0 - \bigoplus_{i\in I} \mathcal{B}_0(H_i) = \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) = \mathcal{C}_0(\mathbb{G}).$$

Proof First equality follows from Lemmas 4 and 5. We first show that the first two spaces are included in $\mathcal{C}_0(\mathbb{G})$. Suppose that $\phi \in \mathcal{L}^\infty(\mathbb{G})^*$ is zero on $\mathcal{C}_0(\mathbb{G})$. Denote by ϕ_n and ϕ_s the normal and singular parts of ϕ, respectively. Then, by (0.1) we have $\phi(a \star f) = 0$ for all $a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$ and $f \in \mathcal{L}^1(\mathbb{G})$, and since ϕ_s is zero on any compact operator, it follows that $\langle \phi_n, a \star f\rangle = 0$. This implies that $\phi_n \star a = 0$ for all $a \in \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) = c_0 - \oplus_{i\in I}\mathcal{B}(H_i)$. Since the latter is weak* dense in $\mathcal{L}^\infty(\mathbb{G})$ and convolution map $x \mapsto \phi_n \star x$ is normal on $\mathcal{L}^\infty(\mathbb{G})$, it follows that $\phi_n = 0$. Hence, ϕ vanishes on $\mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big)$, and therefore the inclusion follows.

For the reverse inclusion, suppose that $\mu \in \mathcal{M}(\mathbb{G})$ is zero on $c_0 - \bigoplus_{i\in I} \mathcal{B}(H_i) = \mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \subseteq \mathcal{C}_0(\mathbb{G})$, then similar to the above we get $\mu \star a = 0$ for all $a \in c_0 - \bigoplus_{i\in I} \mathcal{B}(H_i)$, and since convolution action by μ is normal on $\mathcal{L}^\infty(\mathbb{G})$, we get $\mu = 0$. This completes the proof. ■

Proof of Theorem 1 From Lemma 6 we have $\mathcal{C}_0(\mathbb{G}) = c_0 - \bigoplus_i \mathcal{B}(H_i)$ is an ideal in $\mathcal{C}_0(\mathbb{G})^{**} = l^\infty - \bigoplus_i \mathcal{B}(H_i)$. Hence $\mathbb{G}$ is discrete by [5, Theorem 4.4]. ■

Corollary 7 *Let $\mathbb{G}$ be a regular locally compact quantum group such that*

$$\mathcal{L}^\infty(\mathbb{G}) = l^\infty - \bigoplus_{i\in\mathcal{J}} \mathcal{B}(H_i).$$

Then the following are equivalent:

(i) $\dim(H_i) < \infty$ *for all $i \in \mathcal{J}$;*
(ii) $\dim(H_i) < \infty$ *for some $i \in \mathcal{J}$;*
(iii) $\dim(H_i) = 1$ *for some $i \in \mathcal{J}$;*
(iv) *$\mathbb{G}$ is discrete.*

Proof (i) $\Rightarrow$ (ii) is trivial. From the structure theory of discrete quantum groups [7], (iv) implies the other statements. The implication (iii) $\Rightarrow$ (iv) was proved in [5, Proposition 4.1] (without the regularity condition). (ii) implies that $\mathcal{L}^\infty(\mathbb{G}) \cap \mathcal{B}_0\big(\mathcal{L}^2(\mathbb{G})\big) \neq \{0\}$, and hence by Theorem 1 yields (iv). ■

Remark By [6, Theorem 3.5], $\mathcal{L}^1(\mathbb{G})$ has the Radon–Nikodym property (RNP) if and only if the von Neumann algebra $\mathcal{L}^\infty(\mathbb{G})$ is purely atomic, *i.e.*, an l^∞-direct sum of type I factors. So, under the regularity condition, Corollary 7 gives a distinction between discreteness of $\mathbb{G}$ and the RNP of $\mathcal{L}^1(\mathbb{G})$, based on the dimension of direct summands of $\mathcal{L}^\infty(\mathbb{G})$.

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