Canad. Math. Bull. Vol. **48** (2), 2005 pp. 175–179

Weighted Convolution Operators on ℓ_p

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Abstract. The main results deal with conditions for the validity of the weighted convolution inequality $\sum_{n\in\mathbb{Z}} \left|b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k\right|^p \le C^p \sum_{k\in\mathbb{Z}} |x_k|^p$ when $p \ge 1$.

1 Introduction and Main Result

We suppose throughout that

$$1 \le p \le \infty,\ \frac{1}{p} + \frac{1}{q} = 1; \quad 1 \le r \le \infty,\ \frac{1}{r} + \frac{1}{s} = 1,$$

and observe the convention that $q = \infty$ when $p = 1$.

Given a two-sided complex sequence $x = (x_n)_{n\in\mathbb{Z}}$, we define

$$\|x\|_p := \Bigl(\sum_{k\in\mathbb{Z}} |x_k|^p\Bigr)^{1/p} \text{ for } 1 \le p < \infty, \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{Z}} |x_n|,$$

and we say that $x \in \ell_p$ if $\|x\|_p < \infty$. Given a two-sided complex sequence $a = (a_n)$ and a two-sided complex sequence $b = (b_n)$ of weights, we define the weighted convolution linear transformation $y = (y_n) = \lambda x$ by

$$y_n := (\lambda x)_n := b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k,$$

and aim to obtain sufficient conditions for λ to be a bounded operator on ℓ_p. In other words, our objective is to establish conditions under which there is a positive constant C such that for all $x \in \ell_p$,

$$\|y\|_p \le C\|x\|_p, \tag{1}$$

in which case the operator norm of λ, defined as $\|\lambda\|_p := \sup_{\|x\|_p\le 1} \|\lambda x\|_p \le C$. When $1 \le p < \infty$, (1) amounts to

$$\sum_{n\in\mathbb{Z}} \Bigl| b_n \sum_{k\in\mathbb{Z}} a_{n-k}x_k \Bigr|^p \le C^p \sum_{k\in\mathbb{Z}} |x_k|^p. \tag{2}$$

Our main result is the following:

Received by the editors June 3, 2003.

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

AMS subject classification: 40G10; 40E05.

Keywords: convolution operators on ℓ_p.

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https://doi.org/10.4153/CMB-2005-015-x Published online by Cambridge University Press

Theorem 1 *If $1 \le p \le \infty$, $1 \le r \le q$, $a \in \ell_r$, $b \in \ell_s$, then* (1) *holds for all $x \in \ell_p$ with $C = \|a\|_r \|b\|_s$.*

Note that all the above concerns two-sided sequences. The situation is very different when one-sided sequences are considered. This amounts to having $a_n = b_n = x_n = 0$ for all $n < 0$. In this case (2) reduces to

$$\sum_{n=0}^{\infty} \Big| b_n \sum_{k=0}^{n} a_{n-k} x_k \Big|^p \le C^p \sum_{k=0}^{\infty} |x_k|^p,$$

and when $a_n \ge 0$, $A_n := a_0 + a_1 + \cdots + a_n > 0$ for $n \ge 0$, and $b_n := 1/A_n$ for $n \ge 0$, we get the following known proposition about the Nörlund transform (see [1, Theorem 2] or [2, Theorem 1]).

Proposition 1 *If $1 < p < \infty$ and $na_n = O(A_n)$ as $n \to \infty$, then there is a positive constant C such that*

$$\sum_{n=0}^{\infty} \Big| \frac{1}{A_n} \sum_{k=0}^{n} a_{n-k} x_k \Big|^p \le C^p \sum_{k=0}^{\infty} |x_k|^p.$$

2 Lemmas

We prove two lemmas.

Lemma 1 *If $1 < p < \infty$ and $\sum_{k\in\mathbb{Z}} c_k x_k$ is convergent whenever $\sum_{k\in\mathbb{Z}} |x_k|^p < \infty$, then $\sum_{k\in\mathbb{Z}} |c_k|^q < \infty$.*

Proof This result is certainly known. It is essentially Example 2 in [3, p. 117] where the proof involves the Banach–Steinhaus theorem and knowledge of the form of the general continuous linear functional on ℓ_p. For completeness, we offer the following elementary non-functional analytic proof.

The hypothesis is equivalent to the pair of statements:

$$\sum_{k=0}^{\infty} c_k x_k \text{ is convergent whenever } \sum_{k=0}^{\infty} |x_k|^p < \infty,$$

and

$$\sum_{k=1}^{\infty} c_{-k} x_{-k} \text{ is convergent whenever } \sum_{k=1}^{\infty} |x_{-k}|^p < \infty.$$

Suppose $\sum_{k=0}^{\infty} |c_k|^q = \infty$. Let $D_n := \sum_{k=0}^{n} |c_k|^q$. Assume without loss in generality that $D_0 > 0$, and take

$$x_k := \begin{cases} \frac{|c_k|^{q-1}}{D_k} \frac{|c_k|}{c_k} & \text{when } c_k \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

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Then by the Abel–Dini theorem,

$$\sum_{k=0}^{\infty} c_k x_k = \sum_{k=0}^{\infty} \frac{|c_k|^q}{D_k} = \infty, \quad \text{while} \quad \sum_{k=0}^{\infty} |x_k|^p = \sum_{k=0}^{\infty} \frac{|c_k|^q}{D_k^p} < \infty,$$

contrary to hypothesis. Thus we must have $\sum_{k=0}^{\infty} |c_k|^q < \infty$, and likewise $\sum_{k=1}^{\infty} |c_{-k}|^q < \infty$. ■

Lemma 2 *If $1 \le p < \infty$, $1 < r \le q$, and some finite $t \ge 1$ is such that*

$$\sum_{n\in\mathbb{Z}} \Big| b_n \sum_{k\in\mathbb{Z}} a_{n-k} x_k \Big|^p < \infty$$

whenever $a \in \ell_r$, $b \in \ell_t$, $x \in \ell_p$, then $t \le s$.

Proof Suppose, to the contrary, that $t > s$, and let $3\varepsilon := \frac{1}{s} - \frac{1}{t}$. Let

$$a_n := \begin{cases} (n+1)^{-\frac{1}{r}-\varepsilon} & \text{for } n \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$x_n := \begin{cases} (n+1)^{-\frac{1}{p}-\varepsilon} & \text{for } n \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$b_n := \begin{cases} (n+1)^{-\frac{1}{t}-\varepsilon} & \text{for } n \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $a \in \ell_r$ $x \in \ell_p$, $b \in \ell_t$, but

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \Big| b_n \sum_{k\in\mathbb{Z}} a_{n-k} x_k \Big|^p &= \sum_{n=0}^{\infty} \Big((n+1)^{-\frac{1}{t}-\varepsilon} \sum_{k=0}^{n} (n+1-k)^{-\frac{1}{r}-\varepsilon} (k+1)^{-\frac{1}{p}-\varepsilon} \Big)^p \\ &\ge \sum_{n=0}^{\infty} \Big((n+1)^{-\frac{1}{t}-\varepsilon} (n+1)(n+1)^{-\frac{1}{r}-\varepsilon} (n+1)^{-\frac{1}{p}-\varepsilon} \Big)^p \\ &= \sum_{n=0}^{\infty} (n+1)^{-1} = \infty. \end{aligned}$$

■

3 Proof of the Theorem

Case 1: $1 < p < \infty$. For inequality (2) to be meaningful and non-trivial, observe that for any n for which $b_n \ne 0$, $\sum_{k\in\mathbb{Z}} a_{n-k} x_k$ has to be convergent whenever $\sum_{k\in\mathbb{Z}} |x_k|^p < \infty$. It thus follows from Lemma 1 that we must have $\sum_{k\in\mathbb{Z}} |a_{n-k}|^q =$

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$\sum_{k\in\mathbb{Z}} |a_k|^q < \infty$. This explains why we make the restriction $1 \le r \le q$ in the hypothesis, and Lemma 2 shows why it is not sufficient to require $b \in \ell_t$ for any $t > s$.

An application of Hölder's inequality yields

$$\Big|\sum_{k\in\mathbb{Z}} a_{n-k}x_k\Big|^p \le \|a\|_r^{r(p-1)} \sum_{k\in\mathbb{Z}} |a_{n-k}|^{(q-r)(p-1)}|x_k|^p,$$

and hence that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\Big|b_n\sum_{k\in\mathbb{Z}} a_{n-k}x_k\Big|^p &\le \|a\|_r^{r(p-1)}\|x\|_p^p \sum_{n\in\mathbb{Z}} |b_n|^p|a_{n-k}|^{(q-r)(p-1)}\\ &\le \|a\|_r^{r(p-1)}\|x\|_p^p \cdot \|a\|_r^{(q-r)(p-1)}\|b\|_s^p\\ &= \|a\|_r^p\|b\|_s^p\|x\|_p^p,\end{aligned}$$

since $\|x\|_p^p = \sum_{k\in\mathbb{Z}} |x_k|^p < \infty$ and $\|b\|_s^s = \sum_{n\in\mathbb{Z}} |b_n|^s < \infty$, and this establishes (1) with $C = \|a\|_r\|b\|_s$. Note that Hölder's inequality with $\tilde{r} = \frac{r}{(q-r)(p-1)}$, $\tilde{s} = \dfrac{s}{p}$ is used in the penultimate step above.

Case 2: $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$. When $p = 1$ the result follows by changing the order of summation in (2) and then applying Hölder's inequality, and when $p = \infty$ the desired conclusion is even more immediate. ∎

We have shown that if $1 \le p < \infty$, $1 < r \le q$, $a \in \ell_r$, then (2) holds for all $x \in \ell_p$ provided $b \in \ell_s$, but may fail to hold if $b \in \ell_t$ with a finite $t > s$. In the following section we show by means of an example that if $1 < p < \infty$, then (2) may hold for all $x \in \ell_p$ when $b \notin \ell_t$ for any finite $t > 1$.

4 Example

Suppose $1 < p < \infty$. Let $A_n := a_0 + a_1 + \cdots + a_n$ for $n \ge 0$, where

$$a_n := \begin{cases} \frac{1}{n+1} & \text{for } n \ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

let

$$b_n := \begin{cases} \frac{1}{A_n} & \text{for } n \ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

and let

$$y_n := \Big|b_n\sum_{k\in\mathbb{Z}} a_{n-k}x_k\Big| = \Big|b_n\sum_{k=0}^{\infty} a_kx_{n-k}\Big| \le y_{1,n} + y_{2,n},$$

where

$$y_{1,n} := \Big|\frac{1}{A_n}\sum_{k=0}^{n} a_kx_{n-k}\Big| \text{ and } y_{2,n} := \Big|\frac{1}{A_n}\sum_{k=n+1}^{\infty} a_kx_{n-k}\Big|.$$

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Note that $\sum_{k\in\mathbb{Z}} |a_k| = \infty$ and $\|a\|_r^r = \sum_{k\in\mathbb{Z}} |a_k|^r < \infty$ for all $r > 1$. Suppose that the sequence $x = (x_n) \in \ell_p$. Since

$$A_n \sim \log n \text{ and } \frac{na_n}{A_n} \sim \frac{1}{\log n} = O(1) \text{ as } n \to \infty,$$

it follows from the Proposition that

$$\sum_{n=0}^{\infty} y_{1,n}^p \le C_1 \sum_{k=0}^{\infty} |x_k|^p \le C_1 \|x\|_p^p.$$

Further, by Hölder's inequality,

$$\begin{aligned}\sum_{n=0}^{\infty} y_{2,n}^p &\le \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p}\Big(\sum_{k=n+1}^{\infty} a_k^q\Big)^{p-1} \le \|x\|_p^p \sum_{n=0}^{\infty} \frac{1}{A_n^p}\Big(\int_{n+1}^{\infty} \frac{dt}{t^q}\Big)^{p-1} \\ &= (q-1)^{1-p}\|x\|_p^p \sum_{n=0}^{\infty} \frac{(n+1)^{(q-1)(1-p)}}{A_n^p} = (q-1)^{1-p}\|x\|_p^p \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} \\ &= C_2 \|x\|_p^p,\end{aligned}$$

where $C_2 = (q-1)^{1-p} \sum_{n=0}^{\infty} \frac{a_n}{A_n^p} < \infty$. Hence

$$\sum_{n\in\mathbb{Z}} y_n^p = \sum_{n=0}^{\infty} y_n^p \le 2^p \sum_{n=0}^{\infty} (y_{n,1}^p + y_{n,2}^p) \le 2^p (C_1 + C_2) \|x\|_p^p.$$

Thus (2) is satisfied but $b \notin \ell_t$ for any finite $t > 1$, since $\|b\|_t^t = \sum_{n=0}^{\infty} \frac{1}{A_n^t} = \infty$.

A similar but slightly more complicated argument can be used to show that we could get the same result by taking for any real α,

$$a_n := \begin{cases} \frac{\log^{\alpha}(n+1)}{n+1} & \text{for } n \ge 0, \\ 0 & \text{otherwise,} \end{cases}$$

in the example.

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