Canad. Math. Bull. Vol. 37 (2), 1994 pp. 213–218

IMAGINARY VERMA MODULES FOR AFFINE LIE ALGEBRAS

V. M. FUTORNY

ABSTRACT. We study a class of irreducible modules for Affine Lie algebras which possess weight spaces of both finite and infinite dimensions. These modules appear as the quotients of "imaginary Verma modules" induced from the "imaginary Borel subalgebra".

Introduction. Let $A = (a_{ij})$ $1 \leq i, j \leq n$ be an Affine Cartan matrix, $\mathcal{L} = \mathcal{L}(A)$ be a corresponding Affine Lie algebra with Cartan subalgebra $\mathcal{H}$, root system Δ and one-dimensional centre $C = \mathbb{C}c$.

It is well known that irreducible $\mathcal{L}$-modules with highest weight are unique irreducible quotients of Verma modules associated with the set of positive roots for some choice of the base of Δ. It is less widely known that we can obtain a number of new irreducible $\mathcal{L}$-modules as the irreducible quotients of the Verma modules associated with arbitrary closed subsets $P \subset \Delta$ (*i.e.*, $\alpha, \beta \in P$, $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in P$) with $P \cap (-P) = \emptyset$, $P \cup (-P) = \Delta$ [1–4].

Let W be the Weyl group. There are finitely many but never one W-inequivalent classes of the subsets P described above (in contrast with the finitely-dimensional case, when all subsets P satisfying the condition above are W-equivalent). The classification of all $W \times \{\pm 1\}$-inequivalent classes was obtained by H. P. Jakobsen and V. G. Kac [1, 2] and independently by the present author [5, 6].

Let $\delta = \sum_{i=0}^{n} k_i \alpha_i$ be a minimal positive imaginary root in Δ where $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is the set of simple roots of Δ, indexed in such a way that $k_0 = 1$ and either $-\alpha_0 + \delta \in \Delta$ or $\frac{1}{2}(-\alpha_0 + \delta) \in \Delta$.

Set $\psi_i = \sum_{i=1}^{n} \alpha_i^* - (\sum_{i=1}^{n} k_i)\alpha_0^*$ where $\alpha_i^* \in \mathcal{H}$, $\alpha_i^*(\alpha_j) = \delta_{ij}$. Consider a set $N = \{\alpha \in \Delta \mid \psi(\alpha) > 0\} \cup \{k\delta \mid k \in \mathbb{Z}_+\} \subset \Delta$. One can see that $N \cup (-N) = \Delta$, $N \cap (-N) = \emptyset$ and N is not $W \times \{\pm 1\}$-equivalent to the set of positive roots of Δ.

For any $F \subset \Delta$ set $\mathcal{L}_F = \sum_{\alpha \in F} \mathcal{L}_\alpha$. Consider $B_N = \mathcal{L}_N \oplus \mathcal{H}$ which is a solvable subalgebra of $\mathcal{L}$. We shall call B_N the *imaginary Borel subalgebra*.

The purpose of the present article is to study Verma modules associated with N, *i.e.*, the modules induced from the imaginary Borel subalgebra B_N, which we shall call *imaginary Verma modules* (IVM).

The irreducible quotients of these modules are very different from the irreducible quotients of the usual Verma modules associated with the set of positive roots. In fact

---

Supported by the Natural Sciences and Engineering Research Council of Canada.
Received by the editors June 5, 1992.
AMS subject classification: 17B67, 17B10.

they always (except the trivial case) have both finite and infinite-dimensional weight subspaces with any scalar action of C. The irreducible quotients of imaginary Verma modules remain irreducible when viewed as $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$-modules.

Imaginary Verma $\mathcal{L}'$-modules were also considered in [1, 2] provided the action of C is trivial. In particular all unitarizable modules among their irreducible quotients were described.

We establish criteria for the irreducibility of IVM and study some families of their irreducible subquotients. In fact, we obtain a class of irreducible subquotients which are not the unique quotients of IVM and which are no longer irreducible as $\mathcal{L}'$-modules.

Imaginary Verma modules. We have the following decomposition of $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_{-N} \oplus B_N.$$

DEFINITION. Let $\lambda \in \mathcal{H}^*$ and V be a $\mathcal{L}$-module. A non-zero vector $v \in V$ is called an *imaginary highest vector* of weight λ if $\mathcal{L}_N v = 0$ and $hv = \lambda(h)v$ for all $h \in \mathcal{H}$.

Denote by $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$. Let $\lambda \in \mathcal{H}^*$. Consider $\mathbb{C}$ as a one-dimensional B_N-module under the action $(h+x)1 = \lambda(h).1$ for any $h \in \mathcal{H}, x \in \mathcal{L}_N$.

Define an imaginary Verma $\mathcal{L}$-module

$$\bar{M}(\lambda) = \mathcal{U}(\mathcal{L}) \bigotimes_{\mathcal{U}(B_N)} \mathbb{C}$$

associated with N and λ.

The following proposition can be proved analogously to that of the "standard" Verma modules [7].

PROPOSITION 1. *(i) $\bar{M}(\lambda)$ is a $\mathcal{U}(\mathcal{L}_{-N})$-free module generated by the imaginary highest vector, $1 \otimes 1$, of weight λ.*

(ii) $\dim \bar{M}(\lambda)_\lambda = 1$*;* $0 < \dim \bar{M}(\lambda)_{\lambda - k\delta} < \infty$ *for any integer $k > 0$; if $\bar{M}(\lambda)_\mu \neq 0$, $\mu \neq \lambda - k\delta$ for all integers $k \geq 0$ then* $\dim \bar{M}(\lambda)_\mu = \infty$.

(iii) Let V be some $\mathcal{L}$-module generated by an imaginary highest vector v of weight λ. Then there exists a unique surjective homomorphism $f\colon \bar{M}(\lambda) \longrightarrow V$ such that $f(1 \otimes 1) = v$.

(iv) The module $\bar{M}(\lambda)$ has a unique maximal submodule.

(v) Let λ, $\mu \in \mathcal{H}^*$. Any non-zero element of $\operatorname{Hom}_{\mathcal{L}}\big(\bar{M}(\lambda), \bar{M}(\mu)\big)$ *is injective.* ∎

Let $\bar{L}(\lambda)$ denote the unique irreducible quotient of $\bar{M}(\lambda)$.

Consider a set $P_N = \{P \subset \Delta \mid P \supset N, P \neq N, P \text{ is closed}\}$. Let $T \subset \{1, 2, \ldots, n\} = J$ and $f_T = \sum_{i \in J \setminus T} \alpha_i^* - (\sum_{i \in J \setminus T} k_i)\alpha_0^*$ if $T \neq J$ and $f_J = 0$. For each $T \subset J$ define a set $P(T) = \{\alpha \in \Delta \mid f_T(\alpha) \geq 0\}$. Here $P(T) \cap \big(-P(T)\big) = (\sum_{i \in T} \mathbb{Z}\alpha_i + \mathbb{Z}\delta) \cap \Delta$.

PROPOSITION 2. $P_N = \{P(T) \mid T \subset J\}$.

PROOF. This is essentially Theorem 2.8 of [6]. ∎

Choose $h_i \in \mathcal{H}$ such that $\alpha_j(h_i) = a_{ij}$, $1 \leq i \leq n$, $0 \leq j \leq n$.

https://doi.org/10.4153/CMB-1994-031-9 Published online by Cambridge University Press

Denote by $\mathcal{H}_T$ the subspace of $\mathcal{H}$ spanned by h_i, $i \in T$. For each T consider the subspace $\mathcal{L}_{P(T)}$ and the subalgebra $\mathcal{L}_{-\bar{P}(T)}$ where $\bar{P}(T) = P(T) \setminus \big(-P(T)\big)$. Then we have the following decomposition of $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_{-\bar{P}(T)} \oplus \mathcal{H} \oplus \mathcal{L}_{P(T)}.$$

Let $\lambda \in \mathcal{H}^*$ and $\mathcal{H}_T \oplus C \subset \operatorname{Ker} \lambda$. Then we can define a structure of a one-dimensional $\mathcal{H} \oplus \mathcal{L}_{P(T)}$-module on $\mathbb{C}$ under the action:

$$(h + x)1 = \lambda(h).1 \quad \text{for any } h \in \mathcal{H} \text{ and } x \in \mathcal{L}_{P(T)}.$$

We can now construct a $\mathcal{L}$-module

$$M(\lambda, T) = \mathcal{U}(\mathcal{L}) \bigotimes_{\mathcal{U}(\mathcal{H} \oplus \mathcal{L}_{P(T)})} \mathbb{C}$$

associated with N, T, λ.

PROPOSITION 3. *(i) $M(\lambda, T)$ is a $\mathcal{U}(\mathcal{L}_{-\bar{P}(T)})$-free module generated by an imaginary highest vector* $1 \otimes 1$.

(ii) If $M(\lambda, T)_\mu \neq 0$ then $\dim M(\lambda, T)_\mu = 1$ *for $\mu = \lambda - \sum_{j \in T} n_j \alpha_j - \alpha_i + k\delta$, where $\mu - \lambda \in \Delta$, $i \in J \setminus T$, $k, n_j \in \mathbb{Z}$, $n_j \geq 0$, and* $\dim M(\lambda, T)_\mu = \infty$ *in other cases.*

(iii) The $\mathcal{L}$-module $M(\lambda, T)$ has a unique maximal submodule.

(iv) Let $\lambda \in \mathcal{H}^*$, $\mathcal{H}_T \oplus C \subset$ $\ker \lambda$, *$T' \subset T$. Then there exists a chain of surjective homomorphisms*

$$\bar{M}(\lambda) \longrightarrow M(\lambda, T') \longrightarrow M(\lambda, T).$$

(v) Let $\lambda, \mu \in \mathcal{H}^*$. Any non-zero element of $\operatorname{Hom}_{\mathcal{L}}\big(M(\lambda, T), M(\mu, T)\big)$ *is injective.*

PROOF. A proof of (i), (iii) and (iv) can be given along the lines of Proposition 1. Whereas (ii) follows immediately from the definition of $M(\lambda, T)$ and (iv) follows from (i). ■

Denote by $L(\lambda, T)$ the unique irreducible quotient of $M(\lambda, T)$. Proposition 3, (iv) implies that if $\mathcal{H}_T \oplus C \subset \operatorname{Ker} \lambda$ then $\bar{L}(\lambda) \simeq L(\lambda, J)$. Note that $M(\lambda, J) = L(\lambda, J)$ is a trivial one-dimensional module.

Consider the restriction of the $\mathcal{L}$-modules $\bar{L}(\lambda)$ and $L(\mu, T)$ to the subalgebra $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$.

PROPOSITION 4. *The modules $\bar{L}(\lambda)$ and $L(\mu, T)$ are irreducible $\mathcal{L}'$-modules.*

PROOF. The statement follows from the existence of the imaginary highest vectors in both $\bar{L}(\lambda)$ and $L(\mu, T)$. ■

https://doi.org/10.4153/CMB-1994-031-9 Published online by Cambridge University Press

The irreducible quotients of IVM. Our main result is the following theorem which describes the irreducible quotients of $\bar{M}(\lambda)$. The statement (i) was announced in [4]. For $\mathcal{L} = A_1^{(1)}$, the result was obtained in [3]. A proof of the statement (ii), alternative to the one given below, follows from Proposition 4 and [1, Proposition 6.2].

THEOREM 1. *(i) $\bar{M}(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$.*

(ii) Let $\lambda \in \mathcal{H}^*$, $T \subset J$, $\mathcal{H}_T \oplus C \subset \operatorname{Ker}\lambda$. The module $M(\lambda, T)$ is irreducible if and only if $\mathcal{H}_{T'} \not\subset \operatorname{Ker}\lambda$ for any $T' \underset{\neq}{\supset} T$.

To prove the theorem we will need the following

LEMMA 1. *Let $0 \neq v \in \bar{M}(\lambda)$ and $\bar{M} := \sum_{k=0}^{\infty} \bar{M}(\lambda)_{\lambda-k\delta}$ then $\mathcal{U}(\mathcal{L})v \cap \bar{M} \neq 0$.*

PROOF. Let $v \in \bar{M}(\lambda)_\mu$ and $\mu = \lambda - \tau$, $\tau = \sum_{i=1}^n n_i\alpha_i + k\delta$, $n_i, k \in \mathbb{Z}$, $n_i \geq 0$, $1 \leq i \leq n$. Let $|\tau| = \sum_{i=1}^n n_i$. If $|\tau| = 0$ then $\mu = \lambda - k\delta$ for some non-negative integer k. Assume that $|\tau| > 0$. Then one can prove that there exists $\varphi \in \Delta$ and $m \in \mathbb{Z}_+$ such that $|\tau + \varphi| < |\tau|$ and $\mathcal{L}_{\varphi - m\delta} v \neq 0$. Now we can use induction on $|\tau|$ and conclude that $\mathcal{U}(\mathcal{L})v \cap \bar{M} \neq 0$. ∎

Consider a Heisenberg subalgebra

$$G = C \oplus \sum_{k \in \mathbb{Z} - \{0\}} \mathcal{L}_{k\delta}.$$

Then the subspace $\bar{M}$ can be viewed as a Verma G-module. One can easily prove

LEMMA 2. *The G-module $\bar{M}$ is irreducible if and only if $\lambda(c) \neq 0$.*

PROOF OF THEOREM 1. (i) Let $\lambda(c) = 0$. Then the G-module $\bar{M}$ is reducible by Lemma 2 and, therefore, there exists a non-zero imaginary highest vector of weight $\lambda - k\delta$ for some $k > 0$ which generates a proper submodule of $\bar{M}(\lambda)$. It shows that if $\bar{M}(\lambda)$ is irreducible then $\lambda(c) \neq 0$.

Assume now that $\lambda(c) \neq 0$. Then Lemmas 1 and 2 imply that $\mathcal{U}(\mathcal{L})v = \bar{M}(\lambda)$ for any non-zero $v \in \bar{M}(\lambda)$. Thus, $\bar{M}(\lambda)$ is irreducible. This completes the proof of the statement (i).

(ii) If $T = J$ then $M(\lambda, J) = L(\lambda, J)$ is a trivial one-dimensional module. Assume that $T \neq J$. Let $\lambda(h_i) \neq 0$ for $i \in J \setminus T$ and $0 \neq v \in M(\lambda, T)$. Denote by $\Sigma(T)$ the closed subset of Δ generated by $\{-\alpha_j + k\delta \mid k \in \mathbb{Z}, j \in J \setminus T\} \cap \Delta$. Then one can find an element $u \in \mathcal{U}(\mathcal{L})$ such that $0 \neq uv \in \mathcal{U}(\mathcal{L}_{\Sigma(T)}) \otimes \mathbb{C}$. If $|J \setminus T| = 1$ then the subalgebra of $\mathcal{L}$ generated by $\mathcal{L}_{\Sigma(T)} \oplus \mathcal{L}_{-\Sigma(T)}$ is the algebra of type $A_1^{(1)}$ and the statement (ii) follows from [3]. If $|J \setminus T| > 1$ then one can show that there always exist $u' \in \mathcal{U}(\mathcal{L})$ such that $0 \neq u'u\, v \in \mathcal{U}(\mathcal{L}_{\Sigma(T')}) \otimes \mathbb{C}$ where $T' \underset{\neq}{\supset} T$. Induction on $|J \setminus T|$ implies that $M(\lambda, T)$ is irreducible. If $M(\lambda, T)$ is irreducible then $\lambda(h_i) \neq 0$ for $i \in J \setminus T$ by Proposition 3, (iv). This completes the proof the statement(ii). ∎

As an immediate consequence of Theorem 1, (ii) and Proposition 3, (iv) we have:

COROLLARY 1. *Let $\lambda \in \mathcal{H}^*$, $T \subset J$, $\mathcal{H}_T \oplus C \subset \operatorname{Ker}\lambda$ and $\mathcal{H}_{T'} \not\subset \operatorname{Ker}\lambda$ for any $T' \underset{\neq}{\supset} T$. Then $\bar{L}(\lambda) \simeq M(\lambda, T) = L(\lambda, T) \simeq L(\lambda, T'')$ for any $T'' \subset T$.* ∎

https://doi.org/10.4153/CMB-1994-031-9 Published online by Cambridge University Press

The irreducible subquotients of IVM. *Denote by $T(\lambda)$ the subspace of all imaginary highest vectors of $\bar{M}(\lambda)$. Then Lemma 1 implies that $T(\lambda) \subset \bar{M}$ and $K \cap T(\lambda) \neq 0$ for any $\mathcal{L}$-submodule $K \subset \bar{M}(\lambda)$. Let $[K] = K \cap \bar{M}$. Then $[K] \neq 0$ for all non-zero K.*

Assume now that $\lambda \in \mathcal{H}^*$ and $\lambda(c) = 0$, $\lambda(h_i) \neq 0$ for $1 \leq 1 \leq n$. In this case $T(\lambda) = \bar{M}$. Theorem 1, (ii) implies that the maximal submodule M_1 of $\bar{M}(\lambda)$ is generated by $[M_1] = \sum_{k=1}^{\infty} \bar{M}(\lambda)_{\lambda - k\delta}$. Moreover, one can show that any submodule K is generated by $[K]$ and if K' is a G-submodule of $T(\lambda)$ then $[\mathcal{U}(\mathcal{L})K'] = K'$.

The above discussion implies

THEOREM 2. *Let $\lambda \in \mathcal{H}^*$, $\lambda(c) = 0$, $\lambda(h_i) \neq 0$ for $1 \leq i \leq n$. Then*

(i) $\bar{M}(\lambda)$ has an infinite composition series.

(ii) The modules $M(\lambda - k\delta, \emptyset)$ with multiplicities $m_k = \dim \bar{M}(\lambda)_{\lambda - k\delta}$ for non-negative integers k exhaust all irreducible subquotients of $\bar{M}(\lambda)$.

(iii) $\mathrm{Hom}_{\mathcal{L}}\big(\bar{M}(\mu), \bar{M}(\lambda)\big) \neq 0$ if and only if $\mu = \lambda - k\delta$ for some non-negative integer k and $\dim \mathrm{Hom}_{\mathcal{L}}\big(\bar{M}(\lambda - k\delta), \bar{M}(\lambda)\big) = m_k$. ∎

Now, suppose that $\emptyset \neq T \subset J$, $\mathcal{H}_T \oplus C \subset \operatorname{Ker} \lambda$ and $\mathcal{H}_{T'} \not\subset \operatorname{Ker} \lambda$ for any $T' \underset{\neq}{\supset} T$. In this case $\bar{L}(\lambda) \simeq M(\lambda, T)$. It is clear that the modules $M(\lambda - k\delta, T)$ for non-negative integers k are irreducible subquotients of $\bar{M}(\lambda)$ with multiplicities m_k. But they do not exhaust all irreducible subquotients of $\bar{M}(\lambda)$.

Let $i \in T$, $\alpha = \alpha_i$. For each root $-\alpha + m\delta$, $m \in \mathbb{Z}$, and for each $k \in \mathbb{Z} \setminus \{0\}$ one can choose a non-zero element $X_{-\alpha+m\delta} \in \mathcal{L}_{-\alpha+m\delta}$ and a basis $X^1_{k\delta}, \ldots, X^{s(k)}_{k\delta}$ of $\mathcal{L}_{k\delta}$, $s(k) = \dim \mathcal{L}_{k\delta}$, such that $[X_{-\alpha+m\delta}, X^i_{k\delta}] = 2\delta_{i1} X_{-\alpha+(m+k)\delta}$ if $-\alpha + (m+k)\delta \in \Delta$ and $[X_{-\alpha+m\delta}, X^i_{k\delta}] = 0$ for all i, otherwise.

Consider the vector space $V = \sum_{i \in p\mathbb{Z}} \mathbb{C} v_i$ where $p = 1$ if α is a short root and $p = 2$ (resp. $p = 3$) if α is a long root and $\alpha + 2\delta \in \Delta$ (resp. $\alpha + 2\delta \notin \Delta$). Denote $\bar{N} = N \setminus \{k\delta \mid k \in Z\}$ and $D = \mathcal{L}_{\bar{N}} + \mathcal{H} + G$. Then V can be viewed as a D-module under the action $\mathcal{L}_{\bar{N}} v_j = 0$, $X^i_{k\delta} v_j = -2\delta_{i1} v_{j+k}$, if $k \in p\mathbb{Z}$, $X^i_{k\delta} v_j = 0$ if $k \notin p\mathbb{Z}$, $h v_j = (\lambda - \alpha + j\delta)(h) v_j$, $h \in \mathcal{H}, j \in p\mathbb{Z}$.

Consider a $\mathcal{L}$-module

$$B(\lambda, \alpha) = \mathcal{U}(\mathcal{L}) \bigotimes_{\mathcal{U}(D)} V.$$

PROPOSITION 5. *(i) $B(\lambda, \alpha)$ has a unique maximal submodule B.*

(ii) There exists an irreducible subquotient of $\bar{M}(\lambda)$ isomorphic to $B(\lambda, \alpha) / B$.

(iii) The module $B(\lambda, \alpha) / B$ is no longer irreducible when viewed as an $\mathcal{L}'$-module.

PROOF. The module $B(\lambda, \alpha)$ is induced from the irreducible $\mathcal{H} + G$-module V and therefore has a unique maximal submodule. Let M_1 be the maximal submodule of $\bar{M}(\lambda)$. Then from Theorem 1, (ii) we conclude that $\mathcal{U}(\mathcal{L})[M_1] \neq M_1$. Consider a quotient $M_2 = M_1 / \mathcal{U}(\mathcal{L})[M_1]$. One can show that M_2 has a submodule generated by a D-module isomorphic to V. Therefore $B(\lambda, \alpha) / B$ is isomorphic to the irreducible subquotient of $\bar{M}(\lambda)$. As a $(\mathcal{H} \cap \mathcal{L}') + G$-module, V has a non-trivial proper submodule. This implies that

https://doi.org/10.4153/CMB-1994-031-9 Published online by Cambridge University Press

$B(\lambda,\alpha)/B$ has a non-trivial proper $\mathcal{L}'$-submodule. This completes the proof of Proposition 5. ∎

EXAMPLES. Let $\mathcal{L} = A_1^{(1)}$, $\lambda \in \mathcal{H}^*$, $T = J = \{1\}$, $\mathcal{H}_T \oplus C \subset \ker\lambda$, $\alpha = \alpha_1$.

1) $\bar{N} = \{\alpha \pm k\delta \mid k \in \mathbb{Z}\}$, $\mathcal{L}_{\bar{N}}$ is commutative and $[B_N, B_N] = \mathcal{L}_{\bar{N}}$.

2) The weight spaces of the module $B(\lambda,\alpha)/B$ are at most one-dimensional. Moreover, one can describe the action of $\mathcal{L}$ on $B(\lambda,\alpha)/B$ [8].

3) Consider a vector space $V = \sum_{i\in 2\mathbb{Z}} \mathbb{C}v_i$ as a D-module under the action $\mathcal{L}_{\bar{N}}v_i = 0$, $X_{(2k+1)\delta}v_i = 0$, $X_{2k\delta}v_i = -4v_{i+2k}$, $k \in \mathbb{Z}$, $hv_i = (\lambda - 2\alpha + i\delta)(h)v_i$, $h \in \mathcal{H}$, $i \in 2\mathbb{Z}$.

Let $E(\lambda) = \mathcal{U}(\mathcal{L}) \otimes_{\mathcal{U}(D)} V$. The module $E(\lambda)$ has a unique maximal submodule E. One can show that the irreducible module $E(\lambda)/E$ appears as a subquotient of $\bar{M}(\lambda)$ and has all finite-dimensional weight spaces. In particular, $\dim\big(E(\lambda)/E\big)_\mu = 1$ if and only if $\mu = \lambda - 2\alpha + 2k\delta$ or $\mu = \lambda - 3\alpha + k\delta$ or $\mu = \lambda - 4\alpha + (2k+1)\delta$, $k \in \mathbb{Z}$.

ACKNOWLEDGEMENTS. I am grateful to Professor A. J. Coleman for the comments which led to improvements in the paper.

REFERENCES

1. H. P. Jakobsen and V. G. Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras*, Lecture Notes in Physics, **226**(1985), 1–20.

2. ———, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras, II*, J. Funct. Anal. **82**(1989), 69–90.

3. V. M. Futorny, *On imaginary Verma modules over the Affine Lie algebra* $A_1^{(1)}$, Oslo Univ. **9**(1991), preprint.

4. ———, *Modules of Verma type for Affine Lie algebras*, Funktsional. Anal. i Prilozhen., to appear.

5. ———, *Root systems, representations and geometries*, Ac. Sci. Ukrain. Math. **8**(1990), 30–39.

6. ———, *The parabolic subsets of root systems and corresponding representations of Affine Lie algebras*, Contemp. Math (2) **131**(1992), 45–52.

7. V. G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, third edition, 1990.

8. S. A. Spirin, *$\mathbb{Z}^2$-graded modules with one-dimensional components over the Lie algebra* $A_1^{(1)}$, Funktsional. Anal. i Prilozhen. **21**(1987), 84–85.

Department of Mathematics
Queen's University
Kingston, Ontario
K7L 3N6